Module 2.4: Method of Separation of Variables

Problems:

1. Solve $y^3 \frac{\partial z}{\partial x} + x^2 \frac{\partial z}{\partial y} = 0$ using the method of separation of variables.

Sol: Let $Z = X(x) \cdot Y(y)$ be the solution of the given p.d.e. $y^3 \frac{\partial z}{\partial x} + x^2 \frac{\partial z}{\partial y} = 0 \longrightarrow (1)$

Then, we get $\frac{\partial z}{\partial x} = \frac{\partial X}{\partial x} \cdot Y$ and $\frac{\partial z}{\partial y} = X \cdot \frac{\partial Y}{\partial y}$

Therefore, we have (from (1))

$$y^3 \frac{\partial X}{\partial x} Y + x^2 X \cdot \frac{\partial Y}{\partial y} = 0$$

i.e., $y^3 X' \cdot Y = -x^2 X \cdot Y'$

or, $\boxed{\frac{X'}{x^2 X} = \frac{-Y'}{y^3 \cdot Y} = \lambda}$ (say). $\longrightarrow (2)$

From (2), we have

$$\frac{X'}{x^2 X} = \lambda \quad \text{and} \quad \frac{-Y'}{y^3 Y} = \lambda$$

or, $X' = \lambda x^2 X$ and $Y' = -\lambda y^3 Y$.

Let us solve $X' = \lambda x^2 X$.

i.e., $\frac{dX}{dx} - \lambda x^2 X = 0$ or, $\frac{dX}{X} = \lambda x^2 dx$

Integrating, we get

$$\log X = \lambda \frac{x^3}{3} + c_1$$

or, $X = A e^{\lambda x^3/3}$.

Let us solve $Y' = -\lambda y^3 Y$.

or, $\frac{Y'}{Y} = -\lambda y^3$ or, $\frac{dY}{Y} = -\lambda y^3 dy$

Integrating, we get $\log Y = -\lambda \frac{y^4}{4} + c_2$

or, $Y = B e^{-\lambda y^4/4}$.

Hence, the solution of (1) is

$$Z = X(x) \cdot Y(y)$$
$$= A e^{\lambda x^3/3} \cdot B e^{-\lambda y^4/4}$$

i.e., $Z = c\, e^{\lambda\left(\frac{x^3}{3} - \frac{y^4}{4}\right)}$, where c is an arbitrary constant.

2. Solve $\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u$, where $u(x,0) = 6e^{-3x}$

Sol: we have to find $u(x,t)$ such that

$$\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u \longrightarrow (1)$$

subject to the condition $u(x,0) = 6e^{-3x}$ $\longrightarrow (2)$

Let $u(x,t) = X(x) \cdot T(t)$ be the solution of (1).

Then, we get $\frac{\partial u}{\partial x} = X'(x) \cdot T(t)$

and $\frac{\partial u}{\partial t} = X(x) \cdot T'(t)$.

Therefore from (1), we have

$$X'.T = 2X.T' + X.T$$

or, $$\frac{X'}{X} = \frac{2T'+T}{T} = \lambda \text{ (say)}.$$

Now,

$$\frac{X'}{X} = \lambda \Rightarrow \frac{dX}{dx} = X\lambda$$

$$\Rightarrow \frac{dX}{X} = \lambda dx$$

Integrating, we get $\log X = \lambda x + c_1$

$$\Rightarrow X = e^{\lambda x}.A.$$

and $$\frac{2T'+T}{T} = \lambda \Rightarrow 2T' = \lambda T - T$$

$$\Rightarrow \frac{dT}{dt} = \frac{(-1+\lambda)}{2} T$$

$$\Rightarrow \frac{dT}{T} = \frac{(\lambda-1)}{2} dt$$

Integrating, we get $\log T = \frac{(\lambda-1)}{2} t + C_2$

or, $T = B.e^{\left(\frac{\lambda-1}{2}\right)t}$.

Therefore, $$u(x,t) = Ae^{\lambda x}.Be^{\left(\frac{\lambda-1}{2}\right)t}$$

$$= Ce^{\lambda x}.e^{\left(\frac{\lambda-1}{2}\right)t}$$

Using condition (2), $u(x,0) = Ce^{\lambda x}$

$$\Rightarrow 6e^{-3x} = Ce^{\lambda x}$$

$$\Rightarrow C = 6 \text{ and } \lambda = -3$$

Hence the required solution is

$$u(x,t) = 6e^{-3x}.e^{-2t} = 6e^{-(3x+2t)} \quad //$$

3. Solve $4u_x + u_y = 3u$ with $u(0,y) = 3e^{-y} - e^{-5y}$

Sol: we have to solve $4\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 3u \longrightarrow (1)$

subject to the condition $u(0,y) = 3e^{-y} - e^{-5y} \longrightarrow (2)$

Let $u(x,y) = X(x) \cdot Y(y)$ be the solution of (1).

Then, we get $4X'Y + XY' = 3XY$

$\Rightarrow 4\frac{X'}{X} = \frac{3Y - Y'}{Y} = \lambda$ (say).

Now, $4\frac{X'}{X} = \lambda \Rightarrow X = Ae^{\lambda x/4}$

and $\frac{3Y - Y'}{Y} = \lambda \Rightarrow Y = Be^{(3-\lambda)y}$

From (2), we have $3 - \lambda = -1$ or $3 - \lambda = -5$

i.e., $\lambda = 4$ or, $\lambda = 8$.

Taking $\lambda = 4$, we get $X = Ae^{x}$ and $Y = Be^{-y}$

$\therefore u(x,y) = Ce^{x-y}$

Taking $\lambda = 8$, we get $X = Ae^{2x}$ and $Y = Be^{-5y}$

$\therefore u(x,y) = De^{2x-5y}$.

Hence, we have $u(x,y) = Ce^{x-y} + De^{2x-5y}$.

From (2), we have $u(0,y) = 3e^{-y} - e^{-5y}$

$\Rightarrow Ce^{-y} + De^{-5y} = 3e^{-y} - e^{-5y}$

$\Rightarrow C = 3, D = -1$

Therefore the required solution of (1) is

$u(x,y) = 3e^{x-y} - e^{2x-5y}$

4. Solve $3u_x + 2u_y = 0$ with $u(x,0) = 4e^{-x}$

Answer: $u(x,y) = 4e^{-(2x-3y)/2}$

5. Solve $u_x - 4u_y = 0$ with $u(0,y) = 8e^{-3y}$

Answer: $u(x,y) = 8e^{-3(4x+y)}$