Kernal (Null space) and Range of a linear transformation:-

Kernal of L.T:-

Let $T: V \rightarrow W$ be a linear transformation. The kernal or null space of T is the set of all vectors in V, that maps to 0, under the transformation T.

Image or Range of T:-

The range of T, denoted by $R(T)$, is the set of all vectors in W that are images of at least one vector in V under T.

(i) $\operatorname{Ker} T$ is a subspace of V

(ii) Range T is a subspace of W.

Rank and Nullity of linear transformation:-

Let $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$.

Rank(T) = dim. of Range T

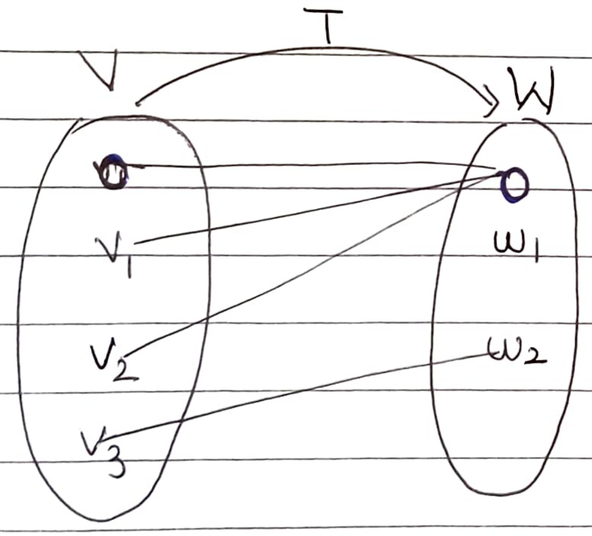
Nullity (T) = dim of Null space ie Ker(T)

Dimension theo. for linear transformation.

$T: V \longrightarrow W$ is a linear transformation

$$\boxed{\text{rank}(T) + \text{nullity}(T) = \dim V}$$

Ex



$\text{Ker}(T) = \{0, v_1, v_2\}$

$\text{Range}(T) = \{0, w_2\}$

Q Find basis and dim of $\ker T$ and range T for linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$ defined by.

$$T(x_1, x_2, x_3) = (x_1 - x_2 + 2x_3,\ 2x_1 + x_2,\ -x_1 - 2x_2 + 2x_3)$$

<u>Sol</u>

Sol^n $\ker(T)$ is the set of all element v s.t $T(v) = 0$

Let $v = (x_1, x_2, x_3) \in \mathbb{R}^3$

s.t $T(v) = T(x_1, x_2, x_3) = (0, 0, 0)$

ie $(x_1 - x_2 + 2x_3,\ 2x_1 + x_2,\ -x_1 - 2x_2 + 2x_3) = (0, 0, 0)$

$$\left.\begin{aligned} x_1 - x_2 + 2x_3 &= 0 \\ 2x_1 + x_2 &= 0 \\ -x_1 - 2x_2 + 2x_3 &= 0 \end{aligned}\right\} \text{homo. sys. of eq}^{n}\text{s}$$

$$\begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 0 \\ -1 & -2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0. \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 0 \\ -1 & -2 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & -4 \\ 0 & -3 & 4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & -4 \\ 0 & 0 & 0 \end{bmatrix}$$

$\text{Rank}(A) = 2$

i.e non trivial sol^n

$3x_2 - 4x_3 = 0$

$x_1 - x_2 + 2x_3 = 0$

Let $x_2 = K$, $x_1 = x_2 - 2x_3$

$x_3 = \frac{3}{4}K$, $x_1 = K - 2 \times \frac{3}{4}K$

$x_1 = -\frac{1}{2}K$

ie $(x_1, x_2, x_3) = \left(-\frac{1}{2}K, K, \frac{3}{4}K\right)$

$(x_1, x_2, x_3) = K(-2, 4, 3)$

ie dim of $Ker(T) = 1$

Basis of $Ker(T) = \{(-2, 4, 3)\}$

Nullity(T) = 1

Range T :-

range(T) is set of all element of ~~range~~ codomain, that are image of at least one vector of element of domain.

ie $R(T) = \{w \mid u = T(w)\}$

take standerd basis of $\mathbb{R}^3$

$(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$

then $T(1,0,0) = (1, 2, -1)$

$T(0,1,0) = (-1, 1, -2)$

$T(0,0,1) = (2, 0, 2)$

Now. $v = c_1 e_1 + c_2 e_2 + c_3 e_3$

$T(v) = c_1 T(e_1) + c_2 T(e_2) + c_3 T(e_3)$

ie to check, whether $T(e_1), T(e_2), T(e_3)$ are L.I or not

$$\begin{bmatrix} 1 & 2 & -1 \\ -1 & 1 & -2 \\ 2 & 0 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & -3 \\ 0 & -4 & 4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

$(1, 2, -1)$ and $(0, 1, -1)$ are basis for Range of T,

$\dim(\text{Range } T) = 2.$

ie Rank$(T) = 2$
Nullity$(T) = 1$.

dim theo.

$\text{Ran}(T) + \text{Null}(T) = \dim V$

$2 + 1 = 3$

verified

Q:- Let $T : \mathbb{R}^4 \longrightarrow \mathbb{R}^3$ be the L.T, given by

$$T(x_1, x_2, x_3, x_4) = (4x_1 + x_2 - 2x_3 - 3x_4,\; 2x_1 + x_2 + x_3 - 4x_4,\; 6x_1 - 9x_3 + 9x_4)$$

(a) Find a basis for Ker(T).

(b) Find a basis for R(T).

(c) verify the dimension theorem

Sol^n^ ~~Ker T is {u | T~~

$$\text{Ker}(T) = \{ (x_1, x_2, x_3, x_4) \mid T(x_1, x_2, x_3, x_4) = 0 \}$$

$(0,0,0)$

ie ~~Ker(T)~~

$$(4x_1 + x_2 - 2x_3 - 3x_4,\; 2x_1 + x_2 + x_3 - 4x_4,\; 6x_1 - 9x_3 + 9x_4) = (0,0,0)$$

ie Ker(T) is solution space of homo. sys. of eq^n^

$$4x_1 + x_2 - 2x_3 - 3x_4 = 0$$
$$2x_1 + x_2 + x_3 - 4x_4 = 0$$
$$6x_1 - 9x_3 + 9x_4 = 0$$

$$\begin{bmatrix} 4 & 1 & -2 & -3 \\ 2 & 1 & 1 & -4 \\ 6 & 0 & -9 & 9 \end{bmatrix}$$

$$\sim \begin{bmatrix} 4 & 1 & -2 & -3 \\ 0 & 1 & 4 & -5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rank = 3

no. of un = 4

1 free var

ie $x_4 = 0$

$x_2 + 4x_3 - 5x_4 = 0$

$4x_1 + x_2 - 2x_3 - 3x_4 = 0$

$x_4 = 0$ say $x_3 = t$

$x_2 = -4t$

$x_1 = \frac{3}{2}t$

ie $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3/2\, t \\ -4t \\ t \\ 0 \end{bmatrix}$

$= t \begin{bmatrix} 3/2 \\ -4 \\ 1 \\ 0 \end{bmatrix}$

$\mathrm{Ker}(T) = \left\{ t \begin{bmatrix} 3/2 \\ -4 \\ 1 \\ 0 \end{bmatrix},\ t \text{ is free} \right\}$

ie ~~basis for~~

basis for $\mathrm{Ker}(T) = \left\{ \begin{bmatrix} +3/2 \\ -4 \\ 1 \\ 0 \end{bmatrix} \right\}$

$\dim(\mathrm{Ker}(T)) = 1$

ie nullity $= 1$

(ii) Range $T = \{ (x, y, z) \in \mathbb{R}^3 \mid T(u) = (x, y, z) \}$
$u \in \mathbb{R}^4$

Let $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$, e_1, e_2, e_3, e_4 are standard basis of $\mathbb{R}^4$.

$(x_1, x_2, x_3, x_4) = c_1 e_1 + c_2 e_2 + c_3 e_3 + c_4 e_4$

$T(x_1, x_2, x_3, x_4) = c_1 T(e_1) + c_2 T(e_2) + c_3 T(e_3) + c_4 T(e_4)$

$T(1, 0, 0, 0) = (4, 2, 6)$
$T(0, 1, 0, 0) = (1, 1, 0)$
$T(0, 0, 1, 0) = (-2, 1, -9)$
$T(0, 0, 0, 1) = (-3, -4, 9)$

ie $T(x_1, x_2, x_3, x_4) = c_1 (4, 2, 6) + c_2 (1, 1, 0) + c_3 (-2, 1, -9) + (-3, -4, 9)$

Range:
ie $T(x_1, x_2, x_3, x_4)$ can be expressed as Linear combination of $(4, 2, 6)$, $(1, 1, 0)$, $(-2, 1, -9)$, $(-3, -4, 9)$

ie

To check whether these vector as L.I or not

$$\begin{bmatrix} 4 & 2 & 6 \\ 1 & 1 & 0 \\ -2 & 1 & -9 \\ -3 & -4 & 9 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 4 & 2 & 6 \\ -2 & 1 & -9 \\ -3 & -4 & 9 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & 6 \\ 0 & 3 & -9 \\ 0 & -1 & 9 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -3 \\ 0 & 1 & -3 \\ 0 & 1 & -9 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \end{bmatrix}$$

Basis for Range $(T) = \{(1,1,0), (0,1,-3), (0,0,6)\}$

$\dim(R(T)) = 3$

ie $\text{rank}(T) = 3$

dimension theo.

$\text{rank}(T) + \text{Nullity}(T) = \dim \mathbb{R}^4$

$3 + 1 = 4$

Verified.

Q Find basis and dimension of Range(T) and Nullspace (T) for the L.T $T: \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$T(x_1, x_2, x_3) = (x_1 - x_2 + 2x_3,\ 2x_1 + x_2,\ -x_1 - 2x_2 + 2x_3)$

Ans Basis for Range(T) = $\{(1, 2, 1), (0, 1, 1)\}$
$\text{rank}(T) = 2$.

Basis Nullspace (T) = $\{(2, 4, 3)\}$

Nullity $(T) = 1$

Q:- If V is the vector space of 2×2 matrices

if $M = \begin{pmatrix} 1 & -1 \\ -2 & 2 \end{pmatrix}$ and $T : V \longrightarrow V$ be

linear transformation defined by

$T(A) = MA$

Find basis and dimension of $\mathrm{Ker}(T)$

and $\mathrm{Range}(T)$.

Sol^n:- Let $A = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$

ie L.T is defined by:

$$T\left(\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}\right) = \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 - x_3 & x_2 - x_4 \\ -2x_1 + 2x_3 & -2x_2 + 2x_4 \end{bmatrix}$$

For $\mathrm{Ker}(T)$

$T(A) = 0$

$$T\left(\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$x_1 - x_3 = 0$

$x_2 - x_4 = 0$

$-2x_1 + 2x_3 = 0$

$-2x_2 + 2x_4 = 0$

Basis for $\ker(T) = \left\{ \begin{bmatrix} 1 & 0 \\ 4 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \right\}$

For Range (T)

Let $e_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $e_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$,

$e_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ be standard basis for V

Find

$T(e_1) = \begin{bmatrix} 1 & 0 \\ -2 & 0 \end{bmatrix}$ $\quad T(e_3) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

$T(e_2) = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}$ $\quad T(e_4) = \begin{bmatrix} 0 & -1 \\ 0 & 2 \end{bmatrix}$

To ~~check~~ find no. of L.I vectors among these images

write these as row vectors

$$\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\dim(\text{Range}(T)) = 2.$$

$$\text{Basis} = \left\{ \begin{bmatrix} 1 & 0 \\ -2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} \right\}$$

Q:- Let W be vectorspace of all symmetric 2×2 matrices and $T: W \to P_2$ be the L.T defined by $T\begin{bmatrix} a & b \\ c & d \end{bmatrix} = (a-b) + (b-c)x + (c-a)x^2$

find Rank and nullity of T

Ans: $\text{Rank}(T) = 2$

$\text{Nullity}(T) = 1$, Basis $\text{Ker}(T) = \left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\}$

Q:- Let $P: P_2 \rightarrow P_3$, be the Linear transformation defined by $T(P(x)) = x\,P(x)$

(1) find basis for Ker(T)

(2) find " " Range(T)

Soln (1) Let $P(x) \in P_2$

$P(x) = a_0 + a_1 x + a_2 x^2$

For Kernel, set of all $P(x) \in P_2$ s.t

$T(P(x)) = 0.$

$x(a_0 + a_1 x + a_2 x^2) = 0$

$a_0 = 0, \quad a_1 = 0, \quad a_2 = 0$

ie $\text{Ker}(T) = \{0\}$

ie there is no basis for Ker(T)

ie $\dim(\text{Ker}(T)) = 0$

Nullity $= 0$.

(2) For Range (T).

Let $\underset{e_1}{1}, \underset{e_2}{x}, \underset{e_3}{x^2}$ be basis for P_2

$T(e_1) = T(1) = x$

$T(e_2) = T(x) = x^2$

$T(e_3) = T(x^2) = x^3$

Now $T(e_1), T(e_2), T(e_3)$ are L.I

ie form a basis for Range (T).

basis for $R(T) = \{x, x^2, x^3\}$

$\dim(R(T)) = 3$

ie $\boxed{\text{Rank}(T) = 3}$

Q: Let $T: P_2 \rightarrow \mathbb{R}^2$ be Linear transformation s.t

$$T(a_0 + a_1 x + a_2 x^2) = (a_0 - a_1, a_1 + a_2)$$

(1) Find basis for Ker(T)

(2) " " " Range(T)

soln (1) Kernel is the set of all elements of P_2 s.t $T(P_2(x)) = 0$.

i.e $T(a_0 + a_1 x + a_2 x^2) = 0$

i.e $(a_0 - a_1, a_1 + a_2) = (0, 0)$

$$a_0 - a_1 = 0$$
$$a_1 + a_2 = 0$$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Let $a_2 = K$

$a_1 = -K$

$a_0 = -K$

i.e $\text{Kernel}(T) = \{ -K - Kx + Kx^2 \mid K \in \mathbb{R} \}$

$\text{Ker}(T) = \{ K(-1 - x + x^2) \mid K \in \mathbb{R} \}$

basis for $\text{Ker}(T) = \{ (-1 - x + x^2) \}$

$$\boxed{\text{Nullity}(T) = 1}$$

(2) For range (T) :- $v = c_1 e_1 + c_2 e_2 + c_3 e_3$

$T(v) = c_1 T(e_1) + c_2 T(e_2) + c_3 T(e_3)$

Let $e_1 = 1$, $e_2 = x$, $e_3 = x^2$ be basis of P_2

$T(e_1) = T(1) = (1, 0)$

$T(e_2) = T(x) = (-1, 1)$

$T(e_3) = T(x^2) = (0, 1)$

To check now L.I of $T(e_1)$, $T(e_2)$, $T(e_3)$

$$\begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

ie basis Range(T) $= \{(1,0), (0,1)\}$

dim. Range(T) $= 2$

Rank(T) $= 2$

as Range(T) has dim 2 ie

$$\boxed{\text{Range}(T) = \mathbb{R}^2}$$

Q: Let $T : M_{2,2} \longrightarrow M_{2,2}$.

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a+b & b+c \\ a+d & b+d \end{bmatrix}$$

find Nullity(T) and R(T).

Soln:- for Ker(T)

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = 0$$

ie $$\begin{bmatrix} a+b & b+c \\ a+d & b+d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$a+b=0$$
$$b+c=0$$
$$a+d=0$$
$$b+d=0$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$d=0$
$b=0$
$a=0$
$c=0$

ie $\mathrm{Ker}(T) = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$

nullity $= 0$.

ie Ker(T) has no basis.

(ii) for Range (T)

take standard basis

$e_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $e_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$,

$e_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$T(e_1) = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ $T(e_2) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $T(e_3) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

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$$T(e_4) = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$$

To check L.I of $T(e_1), T(e_2), T(e_3), T(e_4)$

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{array}{l} \leftarrow \text{first matrix } T(e_1) \\ \leftarrow T(e_2) \\ \leftarrow T(e_3) \\ \leftarrow T(e_4) \end{array}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

all four vector (matrices) are L.I

ie basis for Range(T)

$$= \left\{ \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \right\}$$

$$\boxed{\text{Rank}(T) = 4}$$