Module 2.2

1. Singular Integral [Singular solution]:

Let $f(x, y, z, p, q) = 0 \longrightarrow (1)$

be the partial differential equation whose complete integral is $\phi(x, y, z, a, b) = 0$, $\longrightarrow (2)$

where a and b are arbitrary constants.

Differentiating (2) partially w.r.t. a and b, we obtain $\frac{\partial \phi}{\partial a} = 0 \longrightarrow (3)$

and $\frac{\partial \phi}{\partial b} = 0 \longrightarrow (4)$

The eliminant of a and b from the equations (2), (3) and (4), when it exists, is called the Singular Integral of (1).

2. Solutions of standard types of first order partial differential equations:

The first order partial differential equation can be written as $f(x, y, z, p, q) = 0$, where $p = \frac{\partial z}{\partial x}$ and $q = \frac{\partial z}{\partial y}$. In this section, we shall solve some standard forms of equations by special methods.

Standard Type I: $f(p, q) = 0$ i.e., equations containing p and q only.

Let the required solution be

$$z = ax + by + c$$

Then, we have $\frac{\partial z}{\partial x} = a$ and $\frac{\partial z}{\partial y} = b$

Therefore $f(a, b) = 0$.

From this we can obtain b in terms of a. Let $b = \phi(a)$, then the required solution is $\boxed{z = ax + \phi(a)y + c}$

Problems:

1. Solve $pq = 1$.

Sol: Given $pq = 1 \longrightarrow (1)$

Let the complete solution be

$$z = ax + by + c$$

Then $p = \frac{\partial z}{\partial x} = a$ and $q = \frac{\partial z}{\partial y} = b$

$\therefore$ From (1), we have $ab = 1$

$$\Longrightarrow b = \frac{1}{a}$$

and hence $\boxed{z = ax + \frac{y}{a} + c}$ which is the required solution.

2. Solve $p^2+q^2=n^2$

Sol: Let $z = ax+by+c$ be the required solution of $p^2+q^2=n^2 \longrightarrow (1)$

Then, we have $p = \frac{\partial z}{\partial x} = a$ and $\frac{\partial z}{\partial y} = b$.

$\therefore$ From (1), we get

$a^2+b^2=n^2$ or, $\cancel{(a^2+b^2)}$ $b = \pm\sqrt{n^2-a^2}$

and hence $z = ax \pm \left(\sqrt{n^2-a^2}\right)y + c$ is the required solution of (1).

3. Solve $p^2+q^2=pq$ Answer: $z = ax + \left[\frac{a \pm \sqrt{3}\,ai}{2}\right]y + c$.

4. $pq = p+q$ Answer: $(a-1)z = a(a-1)x + ay + c$

5. $p^2+q^2=9$ Answer: $z = ax + \sqrt{9-a^2}\,y + c$

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Standard Type II: Equations of the form $f(z,p,q)=0$ $\longrightarrow$ ①

Let $z=\phi(u)$, where $u=x+ay$, be the solution of ①.

Then, $p=\frac{\partial z}{\partial x}=\frac{dz}{du}\cdot\frac{\partial u}{\partial x}=\frac{dz}{du}$,

and $q=\frac{\partial z}{\partial y}=\frac{dz}{du}\cdot\frac{\partial u}{\partial y}=a\frac{dz}{du}$.

Substituting the values of p and q in $f(z,p,q)=0$,

we get $f\left(x,\frac{dz}{du},a\frac{dz}{du}\right)=0 \longrightarrow$ ②

By solving eqn ②, we get the general solution of ①.

Problems:

1. Solve $p^2+pq=z^2$.

Sol: The given equation $p^2+pq=z^2$ is $\longrightarrow$ ① of the form $f(z,p,q)=0$.

Let $z=\phi(u)$, where $u=x+ay$, be the solution of ①.

Then $p=\frac{dz}{du}$ and $q=a\frac{dz}{du}$.

Therefore, from ①, we have

$$\left(\frac{dz}{du}\right)^2+a\left(\frac{dz}{du}\right)^2=z^2$$

or, $(1+a)\left(\frac{dz}{du}\right)^2 = z^2$

$\Rightarrow \frac{dz}{du} = \frac{z}{\sqrt{1+a}}$

$\Rightarrow \frac{dz}{z} = \frac{1}{\sqrt{1+a}} du$

Integrating, ~~og~~ we get

$\int \frac{dz}{z} = \frac{1}{\sqrt{1+a}} \int du$

$\Rightarrow \log z = \frac{1}{\sqrt{1+a}} u + c$

Hence the required general solution of (1) is $\log z = \frac{1}{\sqrt{1+a}}(x+ay)+c$

(2) solve: $q^2 = z^2 p^2(1-p^2)$.

Sol: Given $q^2 = z^2 p^2(1-p^2) \longrightarrow (1)$

let $z = \phi(u)$, where $u = x+ay$, be the solution of (1).

Then $p = \frac{dz}{du}$ and $q = a\frac{dz}{du}$.

Therefore, from (1), we get

$\left(a\frac{dz}{du}\right)^2 = z^2\left(\frac{dz}{du}\right)^2\left(1-\left(\frac{dz}{du}\right)^2\right)$

$\Rightarrow a^2 = z\left(1-\left(\frac{dz}{du}\right)^2\right)$

$$\Rightarrow \frac{a^2}{z^2} = 1-\left(\frac{dz}{du}\right)^2$$

$$\Rightarrow \left(\frac{dz}{du}\right)^2 = \frac{z^2-a^2}{z^2}$$

$$\Rightarrow \frac{dz}{du} = \frac{\sqrt{z^2-a^2}}{z}$$

$$\Rightarrow \frac{z}{\sqrt{z^2-a^2}}\,dz = du$$

Integrating, we have

$$\frac{1}{2}\int \frac{2z}{\sqrt{z^2-a^2}}\,dz = \int du$$

$$\Rightarrow \frac{1}{2}\left[2\sqrt{z^2-a^2}\right] = u+c$$

$$\Rightarrow \sqrt{z^2-a^2} = u+c$$

$$\Rightarrow z^2 = a^2+(u+c)^2$$

$\therefore$ The general solution of (1) is

$$z^2 = a^2+(u+c)^2$$

3. Solve $p^2 = qz$ Ans: $2\sqrt{z} = \sqrt{a}\,u + c \Rightarrow 2\sqrt{z} = \sqrt{a}\,(x+ay)+c$

4. solve $z^2 = 1+p^2+q^2$ Ans: $\cosh^{-1} z = \frac{x+ay}{\sqrt{1+a^2}} + c$

5. solve $p(1+q) = qz$ Ans: $\log\left(z-\frac{1}{a}\right) = x+ay+c$

Standard Type III :

Equations of the form $f(x,p) = g(y,q)$.

Let $f(x,p) = a$ and $g(y,q) = a$.

Solving for p and q, we get

$$p = F(x,a) \text{ and } q = G(y,a).$$

Since z is a function of x and y, we have

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

$$= p\,dx + q\,dy$$

$$= F(x,a)dx + G(y,a)dy$$

Integrating, we get

$$\int dz = \int F(x,a)\,dx + \int G(y,a)\,dy$$

$$\Rightarrow z = \int F(x,a)\,dx + \int G(y,a)\,dy + b,$$

which is the required complete solution.

Problems :

1. Solve $p - q = x^2 + y^2$

Sol: Given $p - q = x^2 + y^2 \longrightarrow (1)$

This can be written as

$$p - x^2 = q + y^2.$$

Let $p - x^2 = a$ and $q + y^2 = a$ say.

Then, $p = a + x^2 = F(x, a)$ and $q = a - y^2 = G(y, a)$

Therefore, we have $dz = F(x,a)dx + G(y,a)dy$

$$\Rightarrow dz = (a + x^2)\,dx + (a - y^2)\,dy$$

Integrating, we get

$$z = ax + \frac{x^3}{3} + ay - \frac{y^3}{3} + b$$

$$= a(x+y) + \left(\frac{x^3 - y^3}{3}\right) + b.$$

which is the required solution.

2. Solve $pq + qx = y$

Sol: Given equation can be written as

$$(p+x)q = y \quad \text{or,} \quad p + x = \frac{y}{q}$$

let $p + x = a$ and $\frac{y}{q} = a$.

Then $p = a - x$ and $q = \frac{y}{a}$.

Therefore, we have $dz = p\,dx + q\,dy$

$$\Rightarrow dz = (a - x)\,dx + \frac{y}{a}\,dy$$

Integrating, we get

$$z = ax - \frac{x^2}{2} + \frac{y^2}{2a} + b$$

which is the required complete solution.

3. Solve $q^2 - p = y - x$ $\underline{\text{Ans}}$: $z = \frac{(a+x)^2}{2} + \frac{2}{3}(a+y)^{3/2} + b$

4. Solve $xp - yq = y^2 - x^2$

$\underline{\text{Ans}}$: $z = a\log(xy) - \frac{1}{2}(x^2+y^2) + b$

<u>Standard Type IV [Clairaut's Form]</u>

Equations of the form $z = px + qy + f(p,q)$.

The complete solution is $z = ax + by + f(a,b)$.

Let $z = ax + by + c$ be the required solution.

Then $p = \frac{\partial z}{\partial x} = a$ and $q = \frac{\partial z}{\partial y} = b$.

Therefore, we have

$$z = px + qy + f(p,q)$$

$$\Rightarrow z = ax + by + f(a,b)$$

<u>Problems</u>:

1. Solve $z = px + qy + \sqrt{p^2+q^2+1}$.

<u>Sol</u>: A complete solution is

$$z = ax + by + \sqrt{a^2+b^2+1}.$$

2. Solve $(p+q)(z - px - qy) = 1$

Ans: $z = ax + by + \frac{a^3}{b} + \frac{b^3}{a}$.