Matrix of linear Transformation

Let $T: V \to W$ be a linear transformation from n dimensional vector space to m dim vector space.

Let $\alpha = \{v_1, v_2, \dots, v_n\}$ and $\beta = \{w_1, w_2, \dots w_m\}$ be the basis of V and W resp.

Then the linear transformation $T: V \to W$ can be represented by a matrix of order $m \times n$ ie $A_{m\times n}$

$$A = \left[[T(v_1)]_\beta \quad [T(v_2)]_\beta \quad \cdots\cdots \quad [T(v_n)]_\beta \right]$$

matrix of Linear transformation is also denoted by $[T]_\alpha^\beta$ or $[T: \alpha, \beta]$ or Matrix of T w.r.to basis α and β

ie $[T]_\alpha^\beta = \left[[T(v_1)]_\beta \quad [T(v_2)]_\beta \quad \cdots \quad [T(v_n)]_\beta \right]$

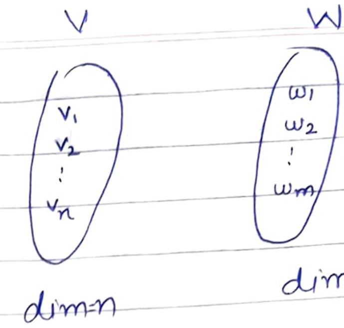
Note:- $T: V \to W$
ie $T(v) = Av$
$\uparrow$
matrix of L.I

in coordinate vector form

$$\boxed{[T(v)]_\beta = [T]_\alpha^\beta \, [v]_\alpha}$$

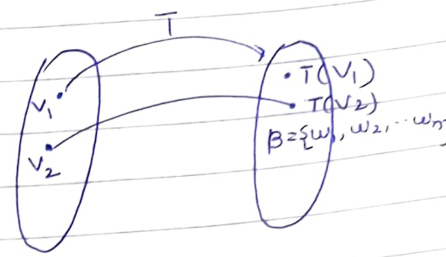
relation b/w matrix of linear transformation $[T]_\alpha^\beta$, $[v]_\alpha$ and $[T(v)]_\beta$.

Finding $[T]_\alpha^\beta$:-



$\alpha = \{v_1, v_2, \cdots v_n\}$ are ~~be~~ basis of V

$\beta = \{w_1\ w_2, \cdots w_m\}$ are basis for W.



Now
for $v_1 \in V$

find $T(v_1) \in W$

$\beta = \{w_1, w_2, - w_m\}$ are basis of W

ie $T(v_1) = a_{11} w_1 + a_{21} w_2 + a_{31} w_3 + \cdots + a_{m1} w_m$ is coordinate vector of $T(v_1)$ relative to basis β is

$$[T(v_1)]_\beta = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ \vdots \\ a_{m1} \end{bmatrix}$$

~~So~~ for v_2, $T(v_2) \in W$

$$T(v_2) = a_{12} w_1 + a_{22} w_2 + \cdots + a_{m2} w_m$$

$$[T(v_2)]_\beta = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}$$

Similary for each basis of V, ie for $v_1, v_2, \cdots v_n$ find $T(v_1), T(v_2) \cdots T(v_n)$ and find their coordinate vectors relative to basis of W ie β

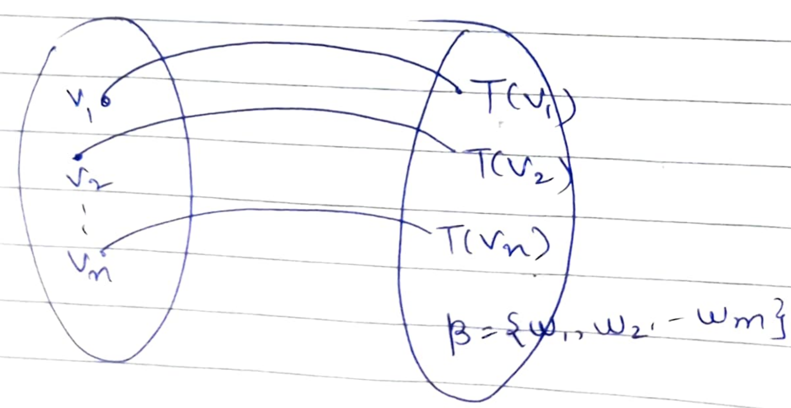
ie $[T(v_1)]_\beta, [T(v_2)]_\beta, \cdots [T(v_n)]_\beta$

Matrix of linear transformation w.r.to basis α and β ~~as~~ is

$$[T]_{\alpha}^{\beta} = \left[[T(v_1)]_{\beta} \quad [T(v_2)]_{\beta} \quad \cdots \quad [T(v_n)]_{\beta} \right]$$

$$= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix}_{m \times n}$$

$V = n$ (dim) $\qquad W = m$ (dim)



Ex-1 If $T: \mathbb{R}^3 \to \mathbb{R}^2$ is the linear transformation defined by

$$T(x, y, z) = (2x + y - z,\ 3x - 2y + 4z)$$

Find the matrix of linear transformation relative to the basis e and f, where

$$e = \{e_1, e_2, e_3\}, \quad f = \{f_1, f_2\}$$

$e_1 = (1, 1, 1)$, $e_2 = (1, 1, 0)$, $e_3 = (1, 0, 0)$

$f_1 = (1, 3)$, $f_2 = (1, 4)$.

also verify that

$$\boxed{[T(v)]_f = [T]_e^f [v]_e.}$$

Solⁿ:- $e = \{e_1, e_2, e_3\}$ are basis of $\mathbb{R}^3$

$T(e_1) = T(1, 1, 1) = (2, 5)$

write $(2, 5) = c_1(1, 3) + c_2(1, 4)$

which gives $c_1 + c_2 = 2$

$3c_1 + 4c_2 = 5$.

$c_1 = 3$ & $c_2 = -1$

ie $[T(e_1)]_f = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ —— (1)

$T(e_2) = T(1, 1, 0) = (3, 1)$

$(3, 1) = c_1(1, 3) + c_2(1, 4)$

$c_1 = 11$, $c_2 = -8$

ie $[T(e_2)]_f = \begin{bmatrix} 11 \\ -8 \end{bmatrix}$ —— (2)

Now $T(e_3) = T(1,0,0) = (2, 3)$

$(2,3) = c_1(1,3) + c_2(1,4)$

$c_1 = 5, \quad c_2 = -3$

$[T(e_3)]_f = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$

Matrix of given L.T w.r. to basis e and f is $[T]_e^f = [[T(e_1)]_f \; [T(e_2)]_f \; [T(e_3)]_f]$

$[T]_e^f = \begin{bmatrix} 3 & 11 & 5 \\ -1 & -8 & -3 \end{bmatrix}$

Now

To check

$[T(v)]_f = [T]_e^f [v]_e.$

$[V]_e$ is coordinate of a vector $V \in \mathbb{R}^3$ w.r. to basis e.

Let $v = (x, y, z) \in \mathbb{R}^3$

ie $(x, y, z) = c_1(1,1,1) + c_2(1,1,0) + c_3(1,0,0)$

$c_1 = z$

$c_2 = y - z$

$c_3 = x - y.$

ie $[v]_e = \begin{bmatrix} z \\ y-z \\ x-y \end{bmatrix}$

To find $[T(v)]_f$.

<u>Now</u>

$$T(v_x) = T(x, y, z) = [(2x+y-z), (3x-2y+4z)]$$

$$(2x+y-z, 3x-2y+4z) = c_1(1, 3) + c_2(1, 4)$$

$$c_1 + c_2 = 2x+y-z$$
$$3c_1 + 4c_2 = 3x-2y+4z$$

$$c_1 = 5x+6y-8z, \quad c_2 = -3x-5y+7z.$$

ie $$[T(v)]_f = \begin{bmatrix} 5x+6y-8z \\ -3x-5y+7z \end{bmatrix}$$

To check
$$[T(v)]_f = [T]_e^f [v]_e.$$

$$[T]_e^f [v]_e = \begin{bmatrix} 3 & 11 & 5 \\ -1 & -8 & -3 \end{bmatrix} \begin{bmatrix} z \\ y-z \\ x-y \end{bmatrix}$$

$$= \begin{bmatrix} 3z + 11(y-z) + 5(x-y) \\ -z - 8(y-z) - 3(x-y) \end{bmatrix}$$

$$= \begin{bmatrix} 5x + 6y - 8z \\ -3x - 5y + 7z \end{bmatrix}$$

$$= [T(v)]_f$$

~~[illegible]~~. verified.

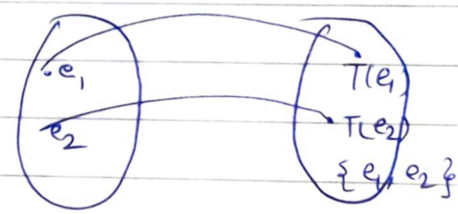
Ex 2 T is the linear transformation defined by

$$T(x_1, x_2) = (-x_2, x_1), \text{ find}$$

matrix of T w.r. to basis

$$e = \{ e_1 = (1, 2),\ e_2 = (1, -1) \}.$$

Sol^n $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$



$e_1 = (1, 2), \quad e_2 = (1, -1)$

$T(e_1) = T(1, 2) = (-2, 1)$

$(-2, 1) = c_1 (1, 2) + c_2 (1, -1)$

$(-2, 1) = (c_1 + c_2,\ 2c_1 - c_2)$

$c_1 + c_2 = -2, \quad 2c_1 - c_2 = 1$

$c_1 = -1/3, \quad c_2 = -5/3$

ie $T(e_1) = -\frac{1}{3}(1, 2) - \frac{5}{3}(1, -1)$

ie $[T(e_1)]_e = \begin{bmatrix} -1/3 \\ -5/3 \end{bmatrix}$

$T(e_2) = T(1,-1) = (1,1)$

$T(e_2) = c_1 e_1 + c_2 e_2$

$(1,1) = c_1(1,2) + c_2(1,-1)$.

$c_1 = 2/3 \qquad c_2 = 1/3$

$$[T(e_2)]_e = \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix}$$

Matrix of T, w.r. to e

$$[T]_e = \begin{bmatrix} -1/3 & 2/3 \\ -5/3 & 1/3 \end{bmatrix}$$

x ——— x ——— x

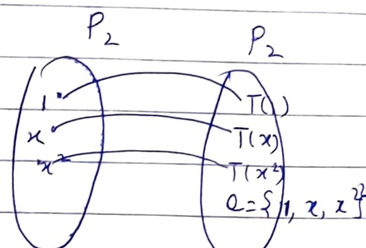
Ex:- $T : P_2 \to P_2$ be L.T defined by

$$T(P(x)) = P(2x+1)$$

ie $T(a_0 + a_1 x + a_2 x^2) = a_0 + a_1(2x+1) + a_2(2x+1)^2$.

(i) Find $[T]_s$ w.r. to basis $\{1, x, x^2\}$
(ii) compute $T(2-3x+4x^2)$.

Soln say $e = \{1, x, x^2\}$



$T(1) = 1$

$T(x) = 2x+1$

$T(x^2) = (2x+1)^2 = 1 + 4x + 4x^2$.

$T(1) = 1 = c_1 e_1 + c_2 e_2 + c_3 e_3$

$1 = c_1 + 0 + 0$ $\qquad [T(1)]_e = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\boxed{c_1 = 1}$

$T(x) = 2x + 1 = c_1(1) + c_2(x) + c_3(x^2)$

$c_1 = 1, \quad c_2 = 2 \quad c_3 = 0$

$$[T(x)]_e = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

$T(x^2) = 1 + 4x + 4x^2 = c_1(1) + c_2(x) + c_3(x^2)$

$c_1 = 1, \quad c_2 = 4, \quad c_3 = 4$

$$[T(x^2)]_e = \begin{bmatrix} 1 \\ 4 \\ 4 \end{bmatrix}$$

ie

$$[T]_e = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 4 \\ 0 & 0 & 4 \end{bmatrix}$$

(ii) Now coordinate of $v = 2-3x+4x^2$ relative of e

$$[V]_e = \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix}$$

$$[T(v)]_e = [T]_e \, [V]_e$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 4 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 10 \\ 16 \end{bmatrix}$$

$$T(2-3x+4x^2) = 3+10x+16x^2$$

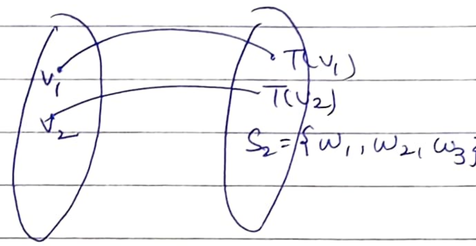
Q:- Let $T : P_1 \longrightarrow P_2$ be L.T defined by

$$T(P(x)) = x\,P(x)$$

(i) Find the matrix T w.r.to basis $S_1 = \{v_1, v_2\}$ and $S_2 = \{w_1, w_2, w_3\}$

$v_1 = 1$, $v_2 = x$, ~~$v_3 = ?$~~ $w_1 = x+1$, $w_2 = x-1$, $w_3 = x^2$

solⁿ:-



$v_1 = 1$, $v_2 = x$

$T(v_1) = x$

write $x = c_1(x+1) + c_2(x-1) + c_3 x^2$

$c_1 + c_2 = 1$, $c_1 - c_2 = 0$, $c_3 = 0$.

$c_1 = 1/2$ $c_2 = 1/2$ $c_3 = 0$

$c_1 + c_2 = 1$
$c_1 - c_2 = 0$

ie $x = \frac{1}{2}(x+1) + \frac{1}{2}(x-1) + 0 \cdot x^2$

ie $[T(v_1)]_{S_2} = \begin{bmatrix} 1/2 \\ 1/2 \\ 0 \end{bmatrix}$

Now $T(v_2) = T(x) = x^2$

$x^2 = c_1(x+1) + c_2(x-1) + c_3 x^2$

$c_1 + c_2 = 0$ $c_1 - c_2 = 0$ $c_3 = 1$

$c_1 = 0$, $c_2 = 0$, $c_3 = 1$

ie $[T(v_2)]_{S_2} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$$[T]_{S_1}^{S_2} = \left[[T(v_1)]_{S_2} \quad [T(v_2)]_{S_2} \right]$$

$$[T]_{S_1}^{S_2} = \begin{bmatrix} 1/2 & 0 \\ 1/2 & 0 \\ 0 & 1 \end{bmatrix}$$

matrix of L.T.

classmate

Ex-6 If $T(x,y) = (2x-3y,\ x+y)$, find $[T]_e$ where $e = \{e_1, e_2\}$

$e_1 = (1, 2),\quad e_2 = (2, 3)$

also verify that

$[T]_e [v]_e = [T(v)]_e$ for any $v \in \mathbb{R}^2$.

Soln $T(e_1) = T(1,2) = (-4, 3)$

$(-4, 3) = c_1 e_1 + c_2 e_2$

$(-4, 3) = c_1 (1, 2) + c_2 (2, 3)$

$$\left.\begin{aligned} c_1 + 2c_2 &= -4 \\ 2c_1 + 3c_2 &= 3 \end{aligned}\right\} \quad \begin{aligned} c_1 &= 18 \\ c_2 &= -11 \end{aligned}$$

$$[T(e_1)]_e = \begin{bmatrix} 18 \\ -11 \end{bmatrix}$$

$T(e_2) = T(2, 3) = (-5, 5)$

$(-5, 5) = c_1 e_1 + c_2 e_2$

$= c_1(1, 2) + c_2(2, 3)$

$$\left.\begin{aligned} c_1 + 2c_2 &= -5 \\ 2c_1 + 3c_2 &= 5 \end{aligned}\right\} \quad c_1 = 25,\ c_2 = -15$$

$$[T(e_2)]_e = \begin{bmatrix} 25 \\ -15 \end{bmatrix}$$

Matrix of L.T w.r. to basis e is

$$[T]_e = \begin{bmatrix} 18 & 25 \\ -11 & -15 \end{bmatrix}$$

To verify

$$[T]_e [v]_e = [T(v)]_e$$

Let $v = (x, y) \in \mathbb{R}^2$

$v = c_1 e_1 + c_2 e_2$

$v = c_1 (1, 2) + c_2 (2, 3)$

$(x, y) = (c_1 + 2c_2, 2c_1 + 3c_2)$

$c_1 + 2c_2 = x$

$2c_1 + 3c_2 = y$

$c_1 = -3x + 2y, \quad c_2 = 2x - y$

ie

$$[v]_e = \begin{bmatrix} -3x + 2y \\ 2x - y \end{bmatrix}$$

$T(v) = (2x - 3y, x + y)$

Now

$(2x - 3y, x + y) = c_1 e_1 + c_2 e_2$

$= c_1 (1, 2) + c_2 (2, 3)$

$c_1 = -4x + 11y, \quad c_2 = 3x - 7y$

$$[T(v)]_e = \begin{bmatrix} -4x + 11y \\ 3x - 7y \end{bmatrix}$$

Q

$\alpha =$

Ans

$$[T]_e[v]_e = \begin{bmatrix} 18 & 25 \\ -11 & -15 \end{bmatrix} \begin{bmatrix} -3x+2y \\ 2x-y \end{bmatrix}$$

$$= (-4x+11y,\ 3x-7y)$$

it is clear that

$$[T(v)]_e = [T]_e[v]_e$$

verified

Q. $T: \mathbb{R}^3 \to \mathbb{R}^3$, be Linear Transformation defined by

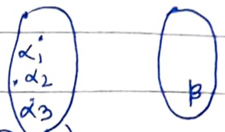
$$T(x,y,z) = (x+2y+z,\ 2x-y,\ 2y+z)$$

where α is standard basis and

$\beta = \{(1,0,1),\ (0,1,1),\ (0,0,1)\}$ be basis of $\mathbb{R}^3$.

compute $[T]_\alpha^\beta$ and $[T]_\beta^\alpha$.

Sol^n $\alpha_1 = (1,0,0)$, $\alpha_2 = (0,1,0)$ $\alpha_3 = (0,0,1)$



$T(\alpha_1) = (1,2,0)$, $T(\alpha_2) = (2,-1,2)$, $T(\alpha_3) = (1,0,1)$

$T(\alpha_1) = c_1\beta_1 + c_2\beta_2 + c_3\beta_3$

$(1,2,0) = c_1(1,0,1) + c_2(0,1,1) + c_3(0,0,1)$

$c_1 = 1$, $c_2 = 2$ $c_1 + c_2 + c_3 = 0$

$c_3 = -3$

ie $[T(\alpha_1)]_\beta = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$ —— ①

Now $T(\alpha_2) = c_1\beta_1 + c_2\beta_2 + c_3\beta_3$

$(2,-1,+2) = c_1\beta_1 + c_2\beta_2 + c_3\beta_3$

ie $c_1 = 2$, $c_2 = -1$, $c_1 + c_2 + c_3 = -2$

$c_3 = 1$

$[T(\alpha_2)]_\beta = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ —— ②

$T(\alpha_3) = c_1\beta_1 + c_2\beta_2 + c_3\beta_3$

$(1,0,1) = (c_1,\ c_2,\ c_1+c_2+c_3)$

$c_1 = 1$, $c_2 = 0$, $c_1 + c_2 + c_3 = 1$

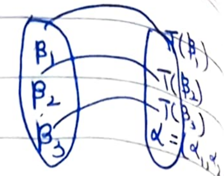
$c_3 = 0$

$[T(\alpha_3)]_\beta = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ —— ③

ie $[T]_\alpha^\beta = \left[[T(\alpha_1)]_\beta \quad [T(\alpha_2)]_\beta \quad [T(\alpha_3)]_\beta \right]$

$$[T]_\alpha^\beta = \begin{bmatrix} 1 & 2 & 1 \\ 2 & -1 & 0 \\ -3 & 1 & 0 \end{bmatrix}$$

(b) To find $[T]_{\beta}^{\alpha}$.



$T(\beta_1) = T(1,0,1) = (2,2,1)$
$T(\beta_2) = T(0,1,1) = (3,-1,3)$
$T(\beta_3) = T(0,0,1) = (1,0,1)$

$T(\beta_1) = c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3$
$(2,2,1) = c_1(1,0,0) + c_2(0,1,0) + c_3(0,0,1)$
$(2,2,1) = (c_1, c_2, c_3)$

$c_1 = 2, \quad c_2 = 2, \quad c_3 = 1$

$$[T(\beta_1)]_{\alpha} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \quad \text{—— } 1$$

$T(\beta_2) = (3,1,-3)$
~~$T(\beta_2) = c_1\alpha_1 + c_2\alpha_2 +$~~
$(3,1,-3) = (c_1, c_2, c_3)$

$c_1 = 3, \quad c_2 = 1, \quad c_3 = -3$

$$[T(\beta_2)]_{\alpha} = \begin{bmatrix} 3 \\ 1 \\ -3 \end{bmatrix} \quad \text{—— } 2$$

$T(\beta_3) = (1,0,1)$
$T(\beta_3) = c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3$
$(1,0,1) = (c_1, c_2, c_3)$

$c_1 = 1, \quad c_2 = 0, \quad c_3 = 1$

$$[T(\beta_3)]_{\alpha} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \text{—— } 3$$

ie
$$[T]_{\beta}^{\alpha} = \begin{bmatrix} 2 & 3 & 1 \\ 2 & 1 & 0 \\ 1 & -3 & 1 \end{bmatrix}$$

Q. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear operator defined by

$$T(x_1, x_2, x_3) = (3x_1 + x_3, -2x_1 + x_2, -x_1 + 2x_2 + 4x_3)$$

find matrix of T w.r.to basis

$$S = \{(1,0,1), (-1,2,1), (2,1,1)\}$$

Q. $T : \mathbb{R}^2 \to \mathbb{R}^2$

$$T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 + x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

$S_1 = \{v_1, v_2\}$

$S_2 = \{w_1, w_2\}$

$v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $v_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

$w_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ $w_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

(1) find matrix of T w.r.to basis S_1.

(2) find matrix of T w.r.to " S_2.

Finding L-T if matrix of L.T is given

Let matrix A of L-T is given then we can find the L.T $T: V \to W$.

Q:- Let $A = \begin{bmatrix} 1/2 & 1 \\ 2/3 & 4 \end{bmatrix}$ be the matrix of L-T. Determine the linear transformation T on $\mathbb{R}^2$ w.r.t the basis

$B = \{(1,0), (1,1)\}$

Soln:- To find $T: \mathbb{R}^2 \to \mathbb{R}^2$

Let $(x, y) \in \mathbb{R}^2$.

$(x, y) = c_1(1, 0) + c_2(\cancel{0}; 1)$

$c_1 + c_2 = x, \quad c_2 = y$

$c_1 + y = x$

$\boxed{c_1 = x - y} \quad \boxed{c_2 = y}$

$(x, y) = c_1(1, 0) + c_2(1, 1)$

$(x, y) = (x - y)(1, 0) + y(1, 1)$

$T(x, y) = (x - y)\,T(1, 0) + y\,T(1, 1) \longrightarrow (1)$

from $[T]_B = \begin{bmatrix} 1/2 & 1 \\ 2/3 & 4 \end{bmatrix}$

we have.

$[T(e_1)]_B = \begin{bmatrix} 1/2 \\ 2/3 \end{bmatrix}$, $[T(e_2)]_B = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$

ie

$\cancel{T\left(\frac{1}{2}\right)}$ $T(1,0) = \frac{1}{2}(1,0) + \frac{2}{3}(1,1)$

$= \left(\frac{7}{6}, \frac{2}{3}\right)$

slly $T(1,1) = 1(1,0) + 4(1,1)$

$= (5,4)$

using (1)

$T(x,y) = (x-y)\,T(1,0) + y\,T(1,1)$

$= (x-y)\left(\frac{7}{6}, \frac{2}{3}\right) + y\cdot(5,4)$

$= \cancel{\frac{7}{6}x} + \left(\frac{7}{6}(x-y) + 5y,\ \frac{2}{3}(x-y) + 4y\right)$

$$\boxed{T(x,y) = \left(\frac{7x+23y}{6},\ \frac{2x+10y}{3}\right)}$$

Q Let $T: \mathbb{R}^3 \to \mathbb{R}^2$ be the L.T w.r.t the basis $S_1 = \{v_1, v_2, v_3\}$ and $S_2 = \{w_1, w_2\}$ and matrix of L.T is given by $A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 1 & 0 \end{bmatrix}$.

Find the L.T $T(x_1, x_2, x_3)$

$v_1 = (-1, 1, 0)$, $v_2 = (0, 1, 1)$, $v_3 = (1, 0, 0)$

$w_1 = (1, 2)$, $w_2 = (1, -1)$

Solⁿ: matrix of L.T is given by

$$A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 1 & 0 \end{bmatrix}$$

ie $[T(v_1)]_{S_2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $[T(v_2)]_{S_2} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$[T(v_3)]_{S_2} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Let $(x_1, x_2, x_3) \in \mathbb{R}^3$

then

$$(x_1, x_2, x_3) = c_1(-1, 1, 0) + c_2(0, 1, 1) + c_3(1, 0, 0) \quad \text{--- (1)}$$

$c_3 - c_1 = x_1$, $c_1 + c_2 = x_2$, $c_2 = x_3$

ie $\cancel{(x_1, x_2, x_3) = x_3(-1, 1, 0) + (x_2 + x_1)}$

$c_1 = x_2 - x_3, \quad c_2 = x_3, \quad c_3 = x_1 + x_2 - x_3$

ie

$(x_1, x_2, x_3) = (x_2 - x_3)\, v_1 + x_3\, v_2 + (x_1 + x_2 - x_3)\, v_3$

ie

$$\boxed{T(x_1, x_2, x_3) = (x_2 - x_3)\, T(v_1) + x_3\, T(v_2) + (x_1 + x_2 - x_3)\, T(v_3)} \quad \text{—— (2)}$$

Now

$$[T(v_1)]_{S_2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

ie $T(v_1) = 1(1, 2) + (-1)\cdot(1, -1)$

$$\boxed{T(v_1) = (0, 3)}$$

$$[T(v_2)]_{S_2} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$T(v_2) = 2(1, 2) + 1(1, -1)$

$$\boxed{T(v_2) = (3, 3)}$$

$$[T(v_3)]_{S_2} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$T(v_3) = 1(1, 2) + 0(1, -1)$

$$\boxed{T(v_3) = (1, 2)}$$

ie $T(x_1, x_2, x_3) = (x_2 - x_3)(0, 3)$

$+ \; x_3(3, 3) + (x_1 + x_2 - x_3)$
$(1, 2)$

$$\boxed{T(x_1, x_2, x_3) = (x_1 + x_2 + 2x_3, \; 2x_1 + 5x_2 - 2x_3)}$$

Example 5. *Given the matrix* $\begin{bmatrix} 1/2 & 1 \\ 2/3 & 4 \end{bmatrix}$, *determine the corresponding linear operator T on* R^2 *w.r.t. the basis* $B = \{(1, 0), (1, 1)\}$. (G.N.D.U. 1985 S)

Solution. We have : $[T : B] = \begin{bmatrix} 1/2 & 1 \\ 2/3 & 4 \end{bmatrix}$.

$$\therefore \quad \text{Coefficient matrix} = \begin{bmatrix} 1/2 & 2/3 \\ 1 & 4 \end{bmatrix}$$

$$\therefore T(1, 0) = \frac{1}{2}(1, 0) + \frac{2}{3}(1, 1) = \left(\frac{7}{6}, \frac{2}{3}\right)$$

$$T(1, 1) = 1(1, 0) + 4(1, 1) = (5, 4).$$

Let $(a, b) = \alpha(1, 0) + \beta(1, 1) = (\alpha + \beta, \beta)$

$\therefore \alpha + \beta = a, \beta = b \therefore \alpha = a - b.$

$\therefore \quad (a, b) = (a - b)(1, 0) + b(1, 1)$

$$\therefore T(a, b) = T[(a - b)(1, 0) + b(1, 1)]$$

$$= (a - b)T(1, 0) + bT(1, 1) \qquad [\because T \text{ is a linear}]$$

$$= (a - b)\left(\frac{7}{6}, \frac{2}{3}\right) + b(5, 4)$$

$$= \left(\frac{7}{6}(a - b) + 5b, \frac{2}{3}(a - b) + 4b\right)$$

Hence $T(a, b) = \left(\dfrac{7a + 23b}{6}, \dfrac{2a + 10b}{3}\right)$.

Example 6. *Let $V(F)$ be a vector space of all polynomials in x of degree atmost n on a real field and a differentiation transformation D is defined on V as follows : $D : P_n \to P_n$*

$$\text{s.t. } D[p(x)] = \frac{d}{dx}[p(x)] \ \forall\, p(x) \in V(F).$$

Find the matrix of operator D w.r.t. a basis of $V(F)$.

Solution : The basis set is $B = \{1, x, x^2, \ldots, x^n\}$.

Since D is a linear operator on V,

$\therefore$ we shall find $D(x^p)$, $p = 0, 1, 2, \ldots, n$ *i.e.* for each basis number.

$D(1) = 0 = 0.1 + 0x + 0x^2 + \ldots + 0x^n$

$D(x) = 1 = 1.1 + 0x + 0x^2 + \ldots + 0x^n$

$D(x^2) = 2x = 0.1 + 2x + 0x^2 + \ldots + 0x^n$

....

$D(x^n) = x^{n-1} = 0.1 + 0x + 0x^2 + \ldots + nx^{n-1} + 0x^n$

$$\therefore \text{Co-eff. matrix is } \begin{bmatrix} 0 & 0 & 0 \ldots & 0 \\ 1 & 0 & 0 \ldots & 0 \\ 0 & 2 & 0 \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 \ldots n & 0 \end{bmatrix}$$

$\therefore$ Matrix of operator D = Transpose of co-eff. matrix

$$= \begin{bmatrix} 0 & 1 & 0 \ldots & 0 \\ 0 & 0 & 2 \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 \ldots & n \\ 0 & 0 & 0 \ldots & 0 \end{bmatrix}.$$

Example 7. *If $V(F)$ is a vector space of polynomial in t of degree atmost 3 and D be the differentiation transformation on V. Then basis for $V(F)$ is $B = \{1, t, t^2, t^3\}$. Verify that $[D : B]\,[x : B] = [D(x) : B]$.*

Solution. Let $x = p(t) = a + bt + ct^2 + dt^3 \in V$.

[As in Ex 6]

$$\text{Then} \quad [x : B] = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{and} \quad [x : B] = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

Since $D(p(t)) = b + 2ct + 3dt^2 + 0t^3$,

$$\therefore [D(p)(t) : B] = [D(x) : B] = \begin{bmatrix} b \\ 2c \\ 3d \\ 0 \end{bmatrix}$$

$$\text{And} \quad [D : B]\,[x : B] = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

$$= \begin{bmatrix} b \\ 2c \\ 3d \\ 0 \end{bmatrix} = [D(x) : B]$$

Hence $\quad [D : B]\,[x : B] = [D(x) : B]$

Example 8. *If the matrix of a linear transformation T on $\mathbb{R}^3$ relative to the basis*

$$B = \{(1, 0, 0), (0, 1, 0) (0, 0, 1)\} \text{ is } \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & -1 & 1 \end{bmatrix}, \text{ then what is}$$

the matrix of T relative to the basis

$B_1 = \{(0, 1, -1), (1, -1, 1), (-1, 1, 0)\}$. (G.N.D.U. 1988)

Solution. The basis is given as

$B = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$.

First we define T when

$$[T : B] = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & -1 & 1 \end{bmatrix} \quad ...(1)$$

Now $T((1, 0, 0)) = 0 (1, 0, 0) + 1 (0, 1, 0) + (-1) (0, 0, 1)$

$= (0, 1, -1)$ $[\because \alpha = 0, \beta = 0, \gamma = -1 \text{ by } (1)]$

$T((0, 1, 0)) = 1 (1, 0, 0) + 0 (0, 1, 0) + (-1) (0, 0, 1) = (1, 0, -1)$

$[\because \alpha = 1, \beta = 0, \gamma = 1 \text{ by } (1)]$

$T((0, 0, 1)) = 1 (1, 0, 0) + (-1) (0, 1, 0) + 0 (0, 0, 1) = (1, -1, 0)$

$[\because \alpha = 1, \beta = -1, \gamma = 0 \text{ by } (1)]$

In general, we have

$T((a, b, c)) = T(a (1, 0, 0) + b (0, 1, 0) + c (0, 0, 1))$

[$\because$ *each* $x = (a, b, c) \in V$ *can be expressed as l.c. of elements of B*]

$= aT((1, 0, 0)) + bT((0, 1, 0)) + cT((0, 0, 1))$

$= a (0, 1, -1) + b (1, 0, -1) + c (1, -1, 0)$

$= (b + c, a - c, -a - b)$ ...(2)

which defines the linear transformation T on V for each (a, b, c).

Now basis is given as

$B_1 = \{(0, 1, -1), (1, -1, 1), (-1, 1, 0)\}$

Then by (2), we have

$T((0, 1, -1)) = (0, 1, -1)$ [*By putting* $a = 1, b = 0, c = 0$ *in* (2)]

$= 1 (0, 1, -1) + 0 (1, -1, 1) + 0 (-1, 1, 0)$

$T((1, -1, 1)) = (0, 0, 0) = 0(0, 1, -1) + 0 (1, -1, 1) + 0 (-1, 1, 0)$

$T((-1, 1, 0)) = (1, -1, 0) = (0, 1, -1) + 0 (1, -1, 1) + (-1) (-1, 1, 0)$

Thus by def.

$$[T : B_1] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Example 9. *Each of the sets*

(i) $B_1\ \{e^t, e^{2t}, te^{2t}\}$ *(ii)* $B_2 = \{\sin t, \cos t\}$

is a basis of a vector space V of function $f : R \to R$. *Let D be the differential operator on V i.e.* $D(f) = \frac{df}{dt}$. *Find the matrix of D in the given basis.*

Solution. (i) $D(e^t) = e^t = 1e^t + 0e^{2t} + 0te^{2t}$

$D(e^{2t}) = 2e^{2t} = 0e^t + 2e^{2t} + 0te^{2t}$

$D(te^{2t}) = t.2e^{2t} + e^{2t} = 0e^t + 1e^{2t} + 2te^{2t}$

$$\therefore \quad [D : B_1] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$

(ii) $D(\sin t) = \cos t = 0.\sin t + 1.\cos t$

$D(\cos t) = -\sin t = -1.\sin t + 0.\cos t$

$$\therefore \quad [D : B_2] = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Example 10. *If the matrix of the linear transformation T on* R^2 *relative to usual basis of R is* $\begin{bmatrix} 2 & -3 \\ 1 & 1 \end{bmatrix}$.

Then find the matrix of T relative to the basis $B_1 = \{(1, 1), (1, -1)\}$.

(G.N.D.U. 1993)

Solution. The usual basis of R^2 is $B = \{(1, 0), (0, 1)\}$.

Also we have : $[T : B] = \begin{bmatrix} 2 & -3 \\ 1 & 1 \end{bmatrix}$.

$\therefore T(1, 0) = 2(1, 0) + 1(0, 1) = (2, 1)$

$T(0, 1) = -3(1, 0) + 1(0, 1) = (-3, 1)$

$\therefore T(a, b) = T(a(1, 0) + b(0, 1)) = aT(1, 0) + bT(0, 1)$

$= a(2, 1) + b(-3, 1) = (2a - 3b, a + b).$

Thus defines T on R^2.

The basis of R^2 is $B_1 = \{(1, 1), (1, -1)\}$

and $\quad T(a, b) \quad = (2a - 3b, a + b).$

We have : $T(1, 1) = (-1, 2) = \frac{1}{2}(1, 1) - 3(1, -1)$

and $T(1, -1) = (5, 0) = \frac{5}{2}(1, 1) + \frac{5}{2}(1, -1)$.

Hence, by def. $[T : B_1] = \begin{bmatrix} 1/2 & 5/2 \\ -3/2 & 5/2 \end{bmatrix}$.

Example 11. *Consider the vector space $V(F)$ of all 2×2 matrices and let T be a linear transformation on $V(F)$ such that*

$$T(x) = Mx, \text{ where } x \in V(F) \text{ and } M = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Find the matrix of T relative to basis B

$$= \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\} \text{ of } V(F).$$

Solution. Given $T : V \to V : T(x) = Mx \ \forall x \in V$ ...(I)

Now $T(x_1) = T\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ [By (I)]

$$= \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = 1\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 0\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 1\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 0\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

$T(x_2) = T\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$ [By (I)]

$$= 0\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 1\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 0\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

$T(x_3) = T\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$

$$= 1\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 0\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 1\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 0\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

$$T(\alpha_4) = T\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

$$= 0\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 1\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 0\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 1\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Then $[T : B] = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$

EXERCISE 4 (a)

1. Let $T : R^2 \to R^2$ defined by $T(x, y) = (2x - 3y, x + y)$.
Compute the matrix of T relative to the basis $B = \{(1, 2), (2, 3)\}$.

2. Find the matrix representation of each of the following operators T on R^2 relative to the basis :

(a) $B_1 = \{(1, 3), (2, 5)\}$ (b) $B_2 = \{(1, 0), (0, 1)\}$

(i) $T(x, y) = (2y, 3x - y)$ (ii) $T(x, y) = (3x - 4y, x + 5y)$.

3. Find the matrix representation of linear operators on $T : R^3 \to R^2$ defined as

$T(x, y, z) = (2x - 3y, x + 4y)$

relative to the basis $B_1 = \{(1, 0), (0, 1)\}$.

$B_2 = \{(1, 3), (2, 5)\}$. *(G.N.D.U. 1987 S)*

4. Find the matrix representation of each of the following linear operators relative to given basis of R^3 :

(a) $T : R^3 \to R^3$ is defined by
$T(x, y, z) = (2x, x - 2y, x + 2y)$
and basis $B_1 = \{(1, 2, 1), (1, 1, 1), (1, 1, 0)\}$.

(b) $T : R^3 \to R^3$ is defined by
$T(x, y, z) = (3x + z, -2x + y, -x + 2y + 4z)$
and basis is $B_2 = \{(1, 0, 1), (-1, 2, 1), (2, 1, 1)\}$.

5. Let $T : R^3 \to R^3$ defined by $T(a, b, c) = (a + 3b, 2c - b, a + c)$ be a linear operator.

Find the matrix of T w.r.t. the basis
$B = \{(1, 1, 1), (1, 1, 0), (1, 0, 1)\}$. *(G.N.D.U. 1991)*

17. If the matrix of a linear transformation T on R^3 relative to the basis

$B = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ is

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix},$$

then what is the matrix of relative to the basis

$B_1 = \{(0, 1, -1), (1, -1, 1), (-1, 1, 0)\}$.

(*Pbi. U. 1997*)

18. If the matrix of a linear operator T on R^3 relative to usual basis is

$$\begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix},$$

then find the matrix of T relative to the basis

$B_1 = \{(1, 2, 2), (1, 1, 2), (1, 2, 1)\}$.

(*P.U. 1989*)

19. Let A be any $n \times n$ square matrix over a field F, then A denotes a linear operator T on F^n by the mapping $T(v) = Av$ $v \in F^n$, where v is written as a column vector. Show that the matrix of T relative to usual ordered basis of F^n is the matrix A itself.

20. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Let T be the linear operator on R^2 defined by $T(v) = Av$, where v is written as a column vector. Find the matrix of T in each of the following bases :

(*i*) $B_1 = \{(1, 0), (0, 1)\}$ (*ii*) $B_2 = \{(1, 3), (2, 5)\}$.

21. Each of the sets

(*i*) $B_1 = \{1, t, e^t, te^t\}$ (*ii*) $B_2 = \{e^{3t}, te^{3t}, t^2e^{3t}\}$

(*iii*) $B_3 = \{e^{5t}, te^{5t}, t^2e^{5t}\}$ (*iv*) $B_4 = \{1, t, \sin 3t, \cos 3t\}$

is a basis of a vector space V of function $f : R \to R$. Let D be the differential operator V *i.e.* $D(f) = \frac{df}{dt}$. Find the matrix D in the given basis.