Module:4 Application of Multivariable Calculus

- ✓ Taylor’s expansion for two variables
- ✓ Maxima and minima
- ✓ Constrained maxima and minima
- ✓ Lagrange’s multiplier method.

Taylor’s expansion of f(x) is

$$f(x+h) = f(x) + \frac{1}{1!} f'(x)\,\mathrm{h} + \frac{1}{2!} f''(x)\,\mathrm{h}^2 + \frac{1}{3!} f'''(x)\,\mathrm{h}^3 + \ldots$$

Taylor’s expansion of f(x) about x=a is

$$f(x) = f(a) + \frac{1}{1!} f'(\mathrm{a})(x-a) + \frac{1}{2!} f''(\mathrm{a})(x-a)^2 + \frac{1}{3!} f'''(\mathrm{a})(x-a)^3 + \ldots$$

Notations

$$f_x = \frac{\partial f}{\partial x}; \qquad f_x(a,b) = \frac{\partial f}{\partial x} \, at \ \text{(a,b)}$$

$$f_y = \frac{\partial f}{\partial y}; \qquad f_y(a,b) = \frac{\partial f}{\partial y} \, at \ \text{(a,b)}$$

$$f_{xx} = \frac{\partial^2 f}{\partial x^2}; \qquad f_{xx}(a,b) = \frac{\partial^2 f}{\partial x^2} \, at \ \text{(a,b)}$$

$$f_{xy} = \frac{\partial^2 f}{\partial x \partial y}; \qquad f_{xy}(a,b) = \frac{\partial^2 f}{\partial x \partial y} \, at \ \text{(a,b)}$$

$$f_{yy} = \frac{\partial^2 f}{\partial y^2}; \qquad f_{yy}(a,b) = \frac{\partial^2 f}{\partial y^2} \, at \ \text{(a,b)}$$

Notations:

$$f_{xxx} = \frac{\partial^3 f}{\partial x^3}; \qquad f_{xxx}(a,b) = \frac{\partial^3 f}{\partial x^3} at \text{ (a,b)}$$

$$f_{xyy} = \frac{\partial^3 f}{\partial x \partial y^2}; \qquad f_{xyy}(a,b) = \frac{\partial^3 f}{\partial x \partial y^2} at \text{ (a,b)}$$

$$f_{xxy} = \frac{\partial^3 f}{\partial x^2 \partial y}; \qquad f_{xxy}(a,b) = \frac{\partial^3 f}{\partial x^2 \partial y} at \text{ (a,b)}$$

$$f_{yyy} = \frac{\partial^3 f}{\partial y^3}; \qquad f_{yyy}(a,b) = \frac{\partial^3 f}{\partial y^3} at \text{ (a,b)}$$

Example:

$$f = x^2 + y^2 \ ;$$

$$f_x = 2x; \ \ f_x(1,1) = 2$$

$$f_y = 2y; \ f_y(1,1) = 2$$

Taylor’s expansion of $f(x, y)$ in powers of (x-a) and (y-b) upto the terms of third degree is given below:.

$$f(x,y) = f(a,b) + \frac{1}{1!}\Big(f_x(a,b)(x-a) + f_y(a,b)(y-b)\Big) +$$

$$\frac{1}{2!}\left(\begin{aligned} f_{xx}(a,b)(x-a)^2 + 2 f_{xy}(a,b)(x-a)(y-b) \\ + f_{yy}(a,b)(y-b)^2 \end{aligned} \right)$$

$$+ \frac{1}{3!}\left(\begin{aligned} f_{xxx}(a,b)(x-a)^3 + 3 f_{xxy}(a,b)(x-a)^2(y-b) + \\ 3 f_{xyy}(a,b)(x-a)(y-b)^2 + f_{yyy}(a,b)(y-b)^3 \end{aligned} \right)$$

Find the Talyor's expansion of $f(x,y)=e^{y}\log(1+x)$ in powers of x and y upto the terms of third degree.

Now,

Taylor's series of $f(x,y)$ in powers of (x-a) and (y-b) upto the terms of third degree is given as:

$$f(x,y)=f(a,b)+\frac{1}{1!}\Big(f_x(a,b)(x-a)+f_y(a,b)(y-b)\Big)+$$

$$\frac{1}{2!}\begin{pmatrix} f_{xx}(a,b)(x-a)^2+2f_{xy}(a,b)(x-a)(y-b) \\ \qquad\qquad +f_{yy}(a,b)(y-b)^2 \end{pmatrix}$$

$$+\frac{1}{3!}\begin{pmatrix} f_{xxx}(a,b)(x-a)^3+3f_{xxy}(a,b)(x-a)^2(y-b)+ \\ 3f_{xyy}(a,b)(x-a)(y-b)^2+f_{yyy}(a,b)(y-b)^3 \end{pmatrix}$$

Here, $f(x,y)=e^{y}\log(1+x)$ and a = b=0.

Now, $f(0,0)=0$

Now, $f_x=e^{y}\dfrac{1}{1+x}$; $f_x(0,0)=1.$

$f_y=e^{y}\log(1+x)$; $f_y(0,0)=0.$

Now, $f_{xx} = -e^{y}\dfrac{1}{(1+x)^{2}};\ f_{xx}(0,0) = -1.$

$$f_{yy} = e^{y}\log(1+x); \qquad f_{yy}(0,0) = 0.$$

$$f_{xy} = e^{y}\frac{1}{1+x}; \qquad f_{xy}(0,0) = 1.$$

Now, $f_{xxx} = 2e^{y} \dfrac{1}{(1+x)^{3}}$; $f_{xxx}(0,0) = 2.$

$$f_{xxy} = -e^{y} \frac{1}{(1+x)^{2}};\ f_{xxy}(0,0) = -1.$$

$$f_{yyy} = e^{y} \log(1+x); \qquad f_{yyy}(0,0) = 0.$$

$$f_{xyy} = e^{y} \frac{1}{1+x};\ f_{xyy}(0,0) = 1.$$

Now, the Taylor's expansion of $e^y \log(1+x)$ in powers of x and y upto the terms of third degree .

$$e^y \log(1+x) = f(0,0) + \frac{1}{1!}\Big(f_x(0,0)x + f_y(0,0)y \Big) +$$

$$\frac{1}{2!}\begin{pmatrix} f_{xx}(0,0)x^2 + 2f_{xy}(0,0)xy \\ \qquad\qquad\qquad + f_{yy}(0,0)y^2 \end{pmatrix}$$

$$+\frac{1}{3!}\begin{pmatrix} f_{xxx}(0,0)x^3 + 3f_{xxy}(0,0)x^2y + \\ 3f_{xyy}(0,0)xy^2 + f_{yyy}(0,0)y^3 \end{pmatrix}$$

That is,

$$e^{y}\log(1+x) = 0 + x + 0 + \frac{1}{2}\left(-x^{2} + 2xy + 0\right) + \frac{1}{6}\left(2x^{3} - 3x^{2}y + 3xy^{2} + 0\right)$$

Therefore,

$$\boxed{\begin{aligned} e^{y}\log(1+x) = x + \frac{1}{2}\left(-x^{2} + 2xy\right) + \frac{1}{6}\left(2x^{3} - 3x^{2}y + 3xy^{2}\right) \end{aligned}}$$

Find the Taylor's series of $f(x,y)=\tan^{-1}\left(\dfrac{y}{x}\right)$ in powers of (x-1) and (y-1) up to the terms of second degree.

Now

Taylor's series of $f(x,y)$ in powers of (x-a) and (y-b) upto the terms of second degree is given as:

$$f(x,y)=f(a,b)+\frac{1}{1!}\Big(f_x(a,b)(x-a)+f_y(a,b)(y-b)\Big)+$$

$$\frac{1}{2!}\left(\begin{array}{r} f_{xx}(a,b)(x-a)^2+2f_{xy}(a,b)(x-a)(y-b) \\ +f_{yy}(a,b)(y-b)^2 \end{array}\right)$$

Here, $f(x,y) = \tan^{-1}\left(\frac{y}{x}\right)$ and a = 1; b= 1.

Now, $f(1,1) = \tan^{-1}(1) = \frac{\pi}{4}$.

Now, $f_x = \frac{1}{1+(y/x)^2}(-\frac{y}{x^2}) = \frac{-y}{x^2+y^2}$; $f_x(1,1) = -\frac{1}{2}$.

$f_y = \frac{1}{1+(y/x)^2}(\frac{1}{x}) = \frac{x}{x^2+y^2}$; $f_y(1,1) = \frac{1}{2}$.

Now,

$$f_{xx} = (-y)(-1)(x^2 + y^2)^{-2}(2x) = \frac{2xy}{(x^2 + y^2)^2};$$

$$f_{xx}(1,1) = \frac{1}{2}.$$

$$f_{yy} = (x)(-1)(x^2 + y^2)^{-2}(2y) = \frac{-2xy}{(x^2 + y^2)^2};$$

$$f_{yy}(1,1) = -\frac{1}{2}.$$

$$f_{xy} = \frac{(x^2 + y^2) - (x)(2x)}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2};$$

$$f_{xy}(1,1) = 0.$$

Now, the Taylor's expansion of $f(x,y)=\tan^{-1}(y/x)$ in powers of (x-1) and (y-1) upto the terms of second degree .

$$\tan^{-1}(y/x) = f(1,1)+\frac{1}{1!}\Big(f_x(1,1)(x-1)+ f_y(1,1)(y-1)\Big)+$$

$$\frac{1}{2!}\left(\begin{array}{r} f_{xx}(1,1)(x-1)^2+2f_{xy}(1,1)(x-1)(y-1) \\ + f_{yy}(1,1)(y-1)^2 \end{array}\right)$$

$$=\frac{\pi}{4}+\left((-\frac{1}{2})(x-1)+(\frac{1}{2})(y-1)\right)+$$

$$\frac{1}{2}\left((\frac{1}{2})(x-1)^2+0+(-\frac{1}{2})(y-1)^2\right).$$

Therefore,

$$\boxed{\tan^{-1}\left(\frac{y}{x}\right) = \frac{\pi}{4}+\frac{1}{2}\Big(-(x-1)+(y-1)\Big)+\frac{1}{4}\Big((x-1)^2-(y-1)^2\Big).}$$

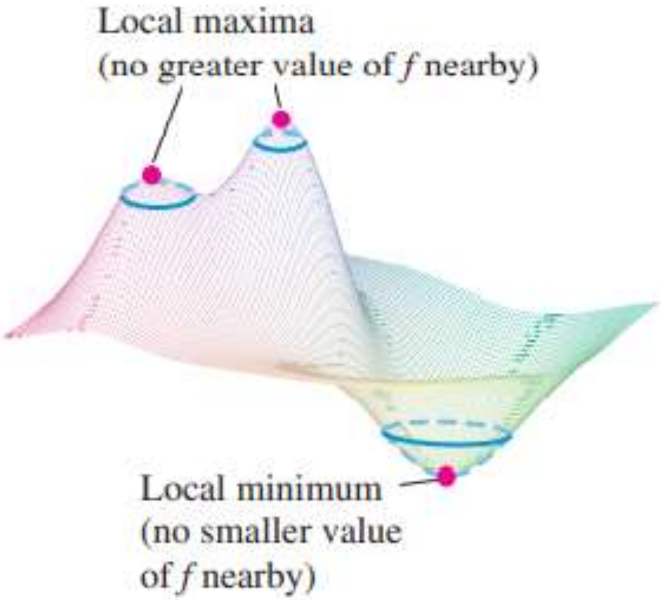
Practice Problems:

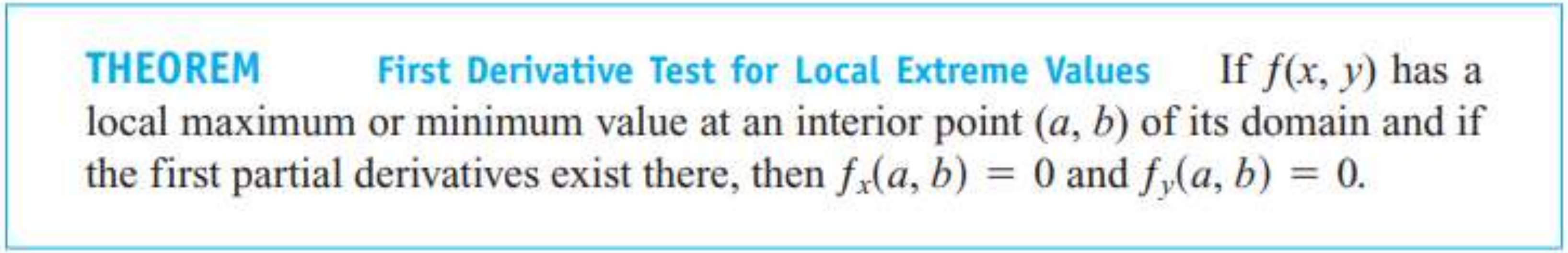
1. Find the Taylor's series of $f(x,y)=e^{x}\sin y$ in powers of x and y up to the terms of third degree.

2. Find the Taylor's expansion of $f(x,y)=\sin(xy)$ in powers of (x-1) and $(y-\frac{\pi}{2})$ up to the terms of second degree.

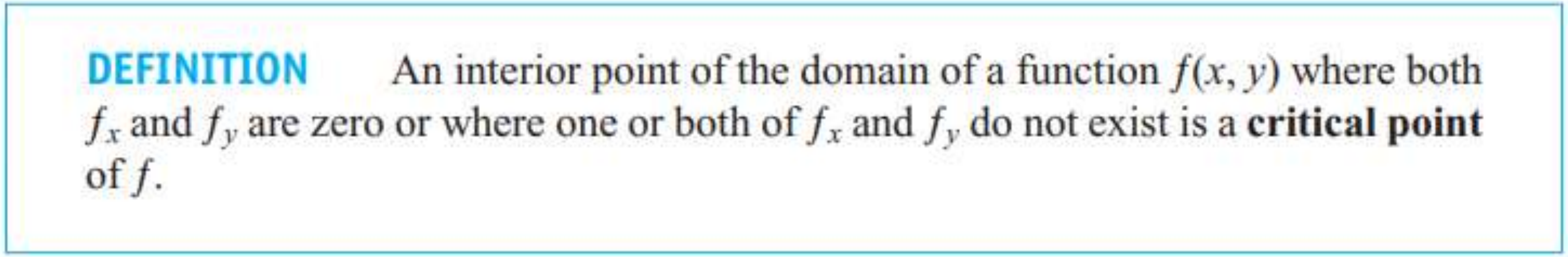
DEFINITIONS Let $f(x, y)$ be defined on a region R containing the point (a, b). Then

1. $f(a, b)$ is a **local maximum** value of f if $f(a, b) \geq f(x, y)$ for all domain points (x, y) in an open disk centered at (a, b).
2. $f(a, b)$ is a **local minimum** value of f if $f(a, b) \leq f(x, y)$ for all domain points (x, y) in an open disk centered at (a, b).

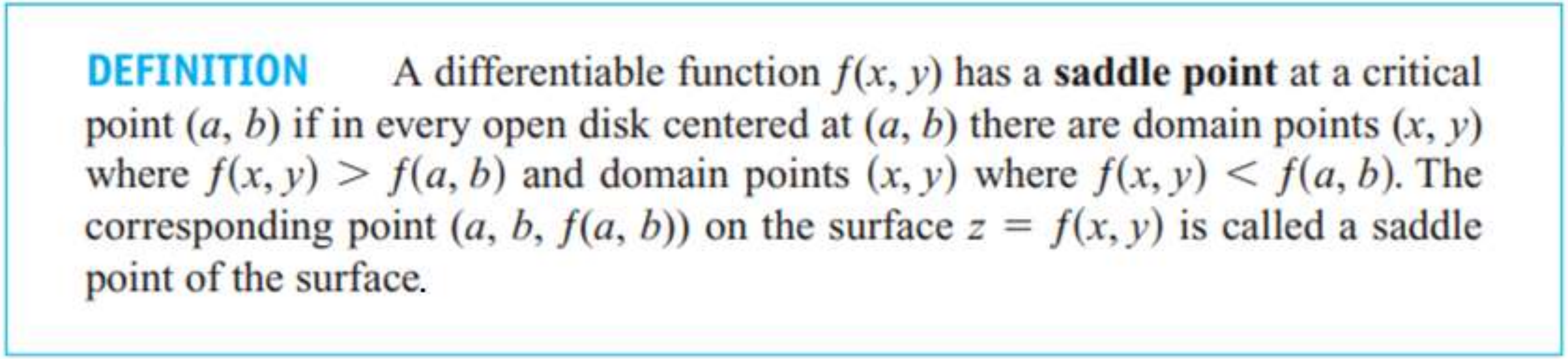




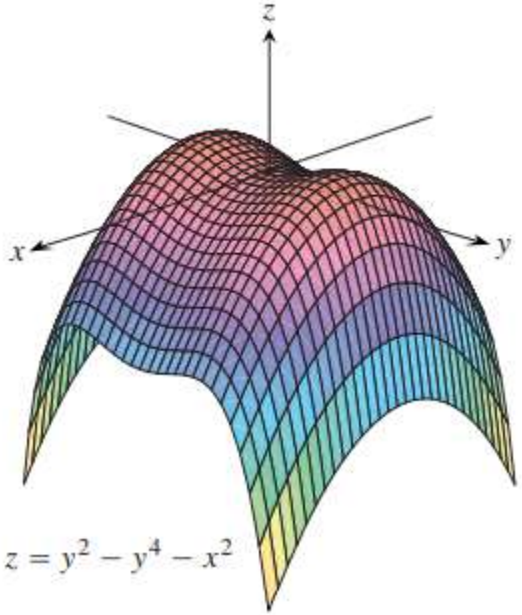
THEOREM **First Derivative Test for Local Extreme Values** If $f(x, y)$ has a local maximum or minimum value at an interior point (a, b) of its domain and if the first partial derivatives exist there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.



DEFINITION An interior point of the domain of a function $f(x, y)$ where both f_x and f_y are zero or where one or both of f_x and f_y do not exist is a **critical point** of f.



DEFINITION A differentiable function $f(x, y)$ has a **saddle point** at a critical point (a, b) if in every open disk centered at (a, b) there are domain points (x, y) where $f(x, y) > f(a, b)$ and domain points (x, y) where $f(x, y) < f(a, b)$. The corresponding point $(a, b, f(a, b))$ on the surface $z = f(x, y)$ is called a saddle point of the surface.



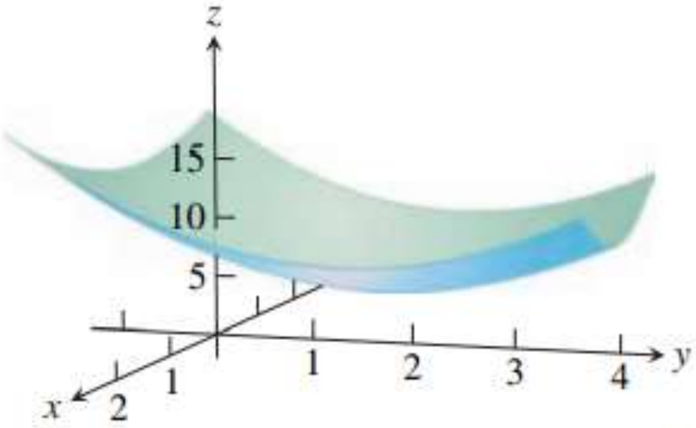
Saddle point at origin

EXAMPLE Find the local extreme values of $f(x, y) = x^2 + y^2 - 4y + 9$.

Solution The domain of f is the entire plane (so there are no boundary points) and the partial derivatives $f_x = 2x$ and $f_y = 2y - 4$ exist everywhere. Therefore, local extreme values can occur only where

$$f_x = 2x = 0 \qquad \text{and} \qquad f_y = 2y - 4 = 0.$$

The only possibility is the point (0, 2), where the value of f is 5. Since $f(x, y) = x^2 + (y - 2)^2 + 5$ is never less than 5, we see that the critical point (0, 2) gives a local minimum.



The graph of the function $f(x, y) = x^2 + y^2 - 4y + 9$ is a paraboloid which has a local minimum value of 5 at the point (0, 2)

THEOREM —Second Derivative Test for Local Extreme Values Suppose that $f(x, y)$ and its first and second partial derivatives are continuous throughout a disk centered at (a, b) and that $f_x(a, b) = f_y(a, b) = 0$. Then

- **i)** f has a **local maximum** at (a, b) if $f_{xx} < 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a, b).
- **ii)** f has a **local minimum** at (a, b) if $f_{xx} > 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a, b).
- **iii)** f has a **saddle point** at (a, b) if $f_{xx}f_{yy} - f_{xy}^2 < 0$ at (a, b).
- **iv)** **the test is inconclusive** at (a, b) if $f_{xx}f_{yy} - f_{xy}^2 = 0$ at (a, b). In this case, we must find some other way to determine the behavior of f at (a, b).

The expression $f_{xx}f_{yy} - f_{xy}^2$ is called the **discriminant** or **Hessian** of f. It is sometimes easier to remember it in determinant form,

$$f_{xx}f_{yy} - f_{xy}^2 = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}.$$

EXAMPLE Find the local extreme values of the function

$$f(x, y) = xy - x^2 - y^2 - 2x - 2y + 4.$$

Solution The function is defined and differentiable for all x and y and its domain has no boundary points. The function therefore has extreme values only at the points where f_x and f_y are simultaneously zero. This leads to

$$f_x = y - 2x - 2 = 0, \qquad f_y = x - 2y - 2 = 0,$$

or

$$x = y = -2.$$

Therefore, the point $(-2, -2)$ is the only point where f may take on an extreme value. To see if it does so, we calculate

$$f_{xx} = -2, \qquad f_{yy} = -2, \qquad f_{xy} = 1.$$

The discriminant of f at $(a, b) = (-2, -2)$ is

$$f_{xx}f_{yy} - f_{xy}^2 = (-2)(-2) - (1)^2 = 4 - 1 = 3.$$

The combination $f_{xx} < 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$

tells us that f has a local maximum at $(-2, -2)$. The value of f at this point is $f(-2, -2) = 8$.