5) <u>Transformation $w = z^2$</u> :-

$$w = u + iv$$
$$u + iv = (x + iy)^2$$
$$u + iv = x^2 - y^2 + 2ixy$$
$$u = x^2 - y^2, \quad v = 2xy$$

I(a) :- any ~~xxx~~ line $\parallel^{el}$ to x axis i.e $y = c$ maps into

$$u = x^2 - c^2, \quad v = 2cx$$

$$u = \frac{v^2}{4c^2} - c^2$$

$[v^2 = 4c^2(u + c^2)]$, which is a <u>parabola</u>.

(b) any real line parallel to y axis i.e $x = b$ maps into

$$u = b^2 - y^2, \quad v = 2by$$

$$u = b^2 - \frac{v^2}{4b^2} \Rightarrow \frac{v^2}{4b^2} = b^2 - u$$

$$\Rightarrow v^2 = 4b^2(b^2 - u)$$

$$\Rightarrow v^2 = -4b^2(u - b^2)$$

<u>Parabola</u>

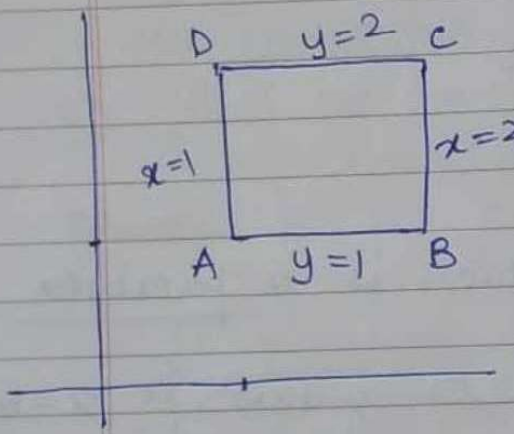
(c) The rectangular region, say $x=1, x=2, y=1, y=$ maps into region bounded by parabolas.

$x=1$ will map to $\Rightarrow$ $v^2 = -4(u-1)$

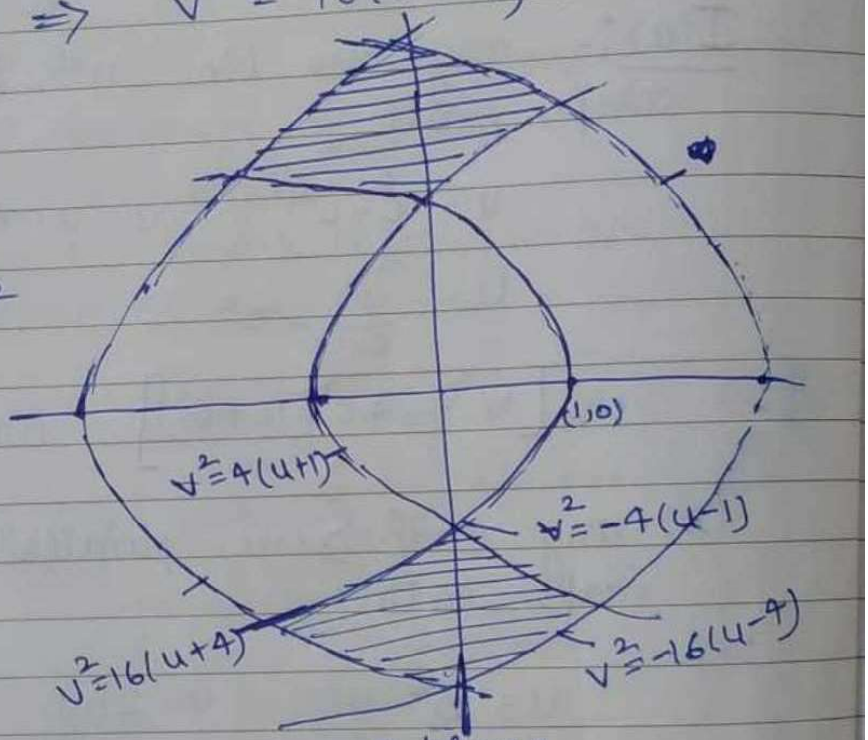
$x=2$ " " " $\Rightarrow$ $v^2 = -16(u-4)$

$y=1$ " " " $\Rightarrow$ $v^2 = 4(u+1)$

$y=2$ " " " $\Rightarrow$ $v^2 = 16(u+4)$



z plane



w plane

rectangle in z plane mapped to the region bounded by parabolas in w-plane.

Q:- Determine the region of w plane into which the following regions are mapped by the transformation $w = z^2$.

(i) first quardant of z-plane.

(ii) the region $1 \le x \le 2$ and $1 \le y \le 2$.

Sol^n:-

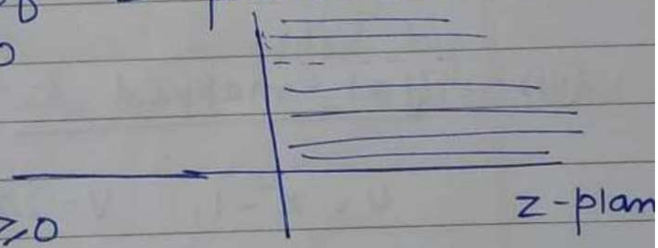
$$w = z^2$$

$$u + iv = (x+iy)^2$$

$$u + iv = (x^2 - y^2) + 2ixy.$$

$$u = (x^2 - y^2), \quad v = 2xy.$$

(i) first quardant of z plane is $x \ge 0, \; y \ge 0$



z plane $\rightarrow$

(1) $x \ge 0, y \ge 0$. (1) $v \ge 0$

(ii) $u \ge 0$, when $x > y$

$u \le 0$, when $x < y$

OR.

$w = z^2$

$z = re^{i\theta}$, $w = Re^{i\phi}$

$w = z^2$ ~~ie~~ $r^2 e^{2i\theta}$

$= Re^{i\phi}$

$\Rightarrow R = r^2$, $\phi = 2\theta$

ie

$x \ge 0, y \ge 0$ mapped to ~~first~~ upper half of w-plane.

(2) region $1 \le x \le 2$, $1 \le y \le 2$

we have $u = x^2 - y^2$, $v = 2xy$.

(i) $x = 1$, mapped to

$u = 1 - y^2$, $v = 2y$

ie $u = 1 - \cancel{4}\frac{v^2}{4}$

$\frac{v^2}{4} = 1 - u \Rightarrow \boxed{v^2 = -4(u-1)}$ parabola

(ii) $x = 2$, mapped to

$u = 4 - y^2$, $v = 4y$

$u = 4 - \frac{v^2}{16}$

$\frac{v^2}{16} = 4 - u \Rightarrow \boxed{v^2 = -16(u-4)}$ parabola

(iii) $y = 1$ mapped to

$u = x^2 - 1$, $v = 2x$

$\Rightarrow u = \frac{v^2}{4} - 1 \Rightarrow \frac{v^2}{4} = u + 1$

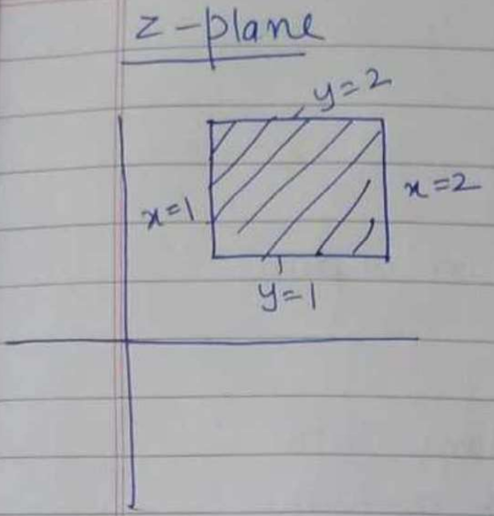
$\Rightarrow \boxed{v^2 = 4(u+1)}$ parabola

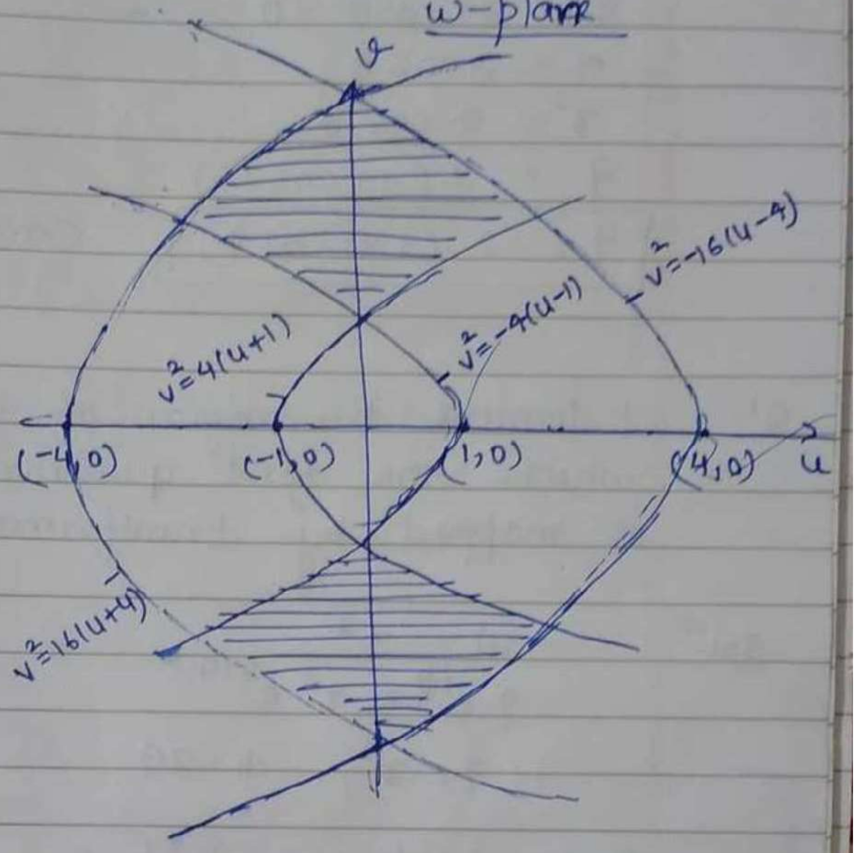
(iv) $y = 2$ mapped to

$u = x^2 - 4$, $v = 4x$

$u = \frac{v^2}{16} - 4$

$\frac{v^2}{16} = u + 4 \Rightarrow \boxed{v^2 = 16(u+4)}$ parabola





thus the region bounded by lines $1 \le x \le 2$, $1 \le y \le 2$ will mapped to the region bounded by prabolas $v^2 = -4(u-1)$, $v^2 = -16(u-4)$, $v^2 = 4(u+1)$ and $v^2 = 16(u+4)$.

Q:- Show that the transformation $w = z^2$ maps the circle $|z-1| = 1$ into the cardoid $q = 2(1+\cos\phi)$ where $w = q e^{i\phi}$ in w-plane.

Sol^n

$$w = z^2$$
$$q e^{i\phi} = r^2 e^{2i\theta}$$
$$\boxed{q = r^2}, \quad \boxed{\phi = 2\theta}$$

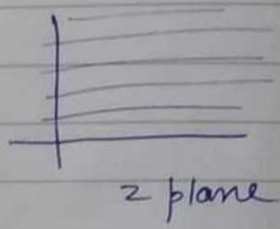
$$|z-1| = 1$$
$$|x+iy-1| = 1$$
$$|(x-1) + iy| = 1$$
$$(x-1)^2 + y^2 = 1$$
$$x^2 + y^2 - 2x = 0$$
$$r^2\cos^2\theta + r^2\sin\theta - 2r\cos\theta = 0$$

$r^2 - 2r\cos\theta = 0$
$r = 2\cos\theta$
$r^2 = 4\cos^2\theta$
$q = 2(1+\cos 2\theta)$
$\boxed{q = 2(1+\cos\phi)}$ Cardoid in w-plane.

Q:- Determine the region of w-plane into which the first quadrant of z-plane is mapped by transformation $w = z^2$.

Sol^n

$w = z^2$
$q e^{i\phi} = r^2 e^{2i\theta}$
$q = r^2, \quad \phi = 2\theta$



for first quardant in z-plane
$0 \le \theta \le \pi/2$

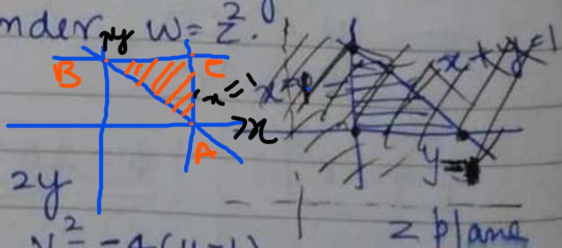
i.e $\quad 0 \le 2\theta \le \pi$
$0 \le \phi \le \pi$

in w-plane
$q = r^2, \quad 0 \le \phi \le \pi$ which is ~~first~~ upper half of w-plane.

—— x —— x —— x —— x ——

Imp Q- find image of region bounded by lines $x=1$, $y=1$ and $x+y=1$ under $w = z^2$.

Sol^n
$w = z^2$



$\Rightarrow \quad u = x^2 - y^2, \quad v = 2xy$

$x = 1 \quad \Rightarrow \quad u = 1 - y^2, \quad v = 2y$

$\Rightarrow \quad u = 1 - \frac{v^2}{4} \Rightarrow v^2 = -4(u-1)$

$x=1$ maps to $\boxed{v^2 = -4(u-1)}$

Parabola passes through $(0, \pm 2)$ and vertex $(1, 0)$

(ii) $y=1 \Rightarrow v^2=4(u+1)$ which is parabola having vertex at $(-1,0)$ and passes through $(0,\pm 2)$

(iii) $x+y=1 \Rightarrow y=1-x$

$u = x^2-y^2$

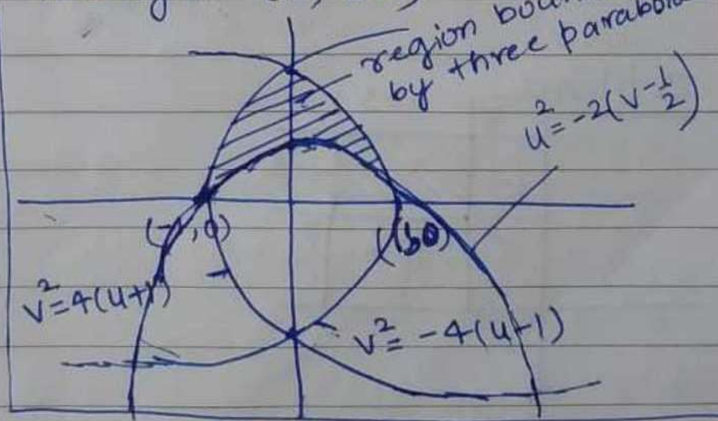
$u = x^2-(1-x)^2$

$u = 2x-1 \Rightarrow \boxed{x=\frac{u+1}{2}}$

& $v = 2xy = 2x(1-x)$

$v = \frac{\cancel{2}(u+1)}{\cancel{2}}\left(1-\frac{u+1}{2}\right)$

$v = \frac{1}{2}(1-u^2) \Rightarrow \boxed{u^2=-2\left(v-\frac{1}{2}\right)}$ which is Parabola Vertex $(0,\frac{1}{2})$ and Passes through $(\pm 1, 0)$



(6) Transformation $w = e^{z}$

$$u + iv = e^{x+iy}$$
$$u + iv = e^{x} \cdot e^{iy}$$
$$u + iv = e^{x}(\cos y + i \sin y)$$

$$u = e^{x}\cos y \qquad v = e^{x}\sin y \quad \text{—— (1)}, \quad \theta = y$$

~~from (1)~~ $u^2 + v^2 = e^{2x}$, $\frac{v}{u} = \tan y$ or $\theta = \tan^{-1}\left(\frac{v}{u}\right) = y$

or in polar:-

$$w = e^{z}$$
$$Re^{i\phi} = e^{x+iy}$$
$$Re^{i\phi} = e^{x} \cdot e^{iy}$$

$$\boxed{R = e^{x}} \qquad\qquad \boxed{\phi = y}$$

$R = e^{x}$ represent a circle of radius e^{x} in w-plane
$\phi = y$ " " line making an angle y (radians) from x axis

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Q1 find image ~~of~~ and draw a rough sketch of mapping of the ~~r~~egion $1 \le x \le 2$ & $2 \le y \le 3$ under the mapping $w = e^{z}$.

Sol^n:-

$$w = e^{z}$$

$$R e^{i\phi} = e^{x} \cdot e^{iy}$$

$$\boxed{R = e^{x}} \qquad \boxed{\phi = y}$$

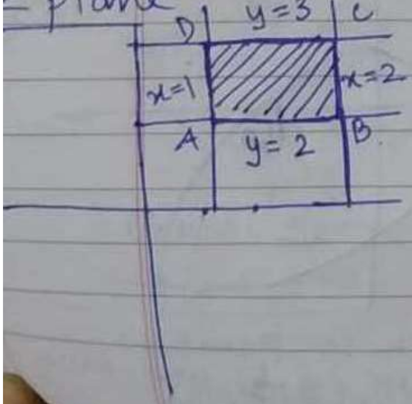
z-plane $\xrightarrow{\text{mapped to}}$ w plane

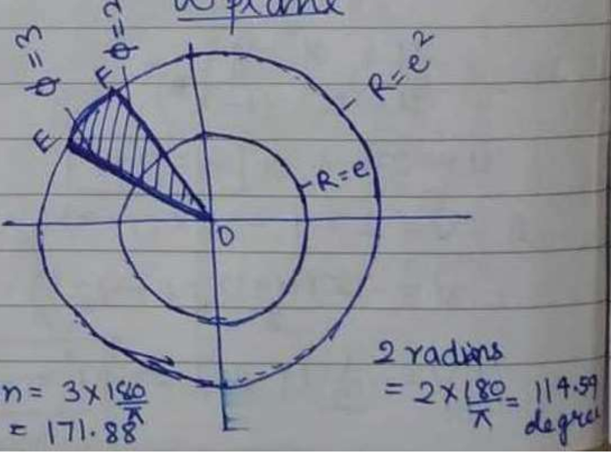
(1) $x = 1$ — $R = e^{1}$ is circle of radius $e = 2.7$.

(2) $x = 2$ — $R = e^{2} = 7.4$, $R = 7.4$ is a circle of radius 7.4.

(3) $y = 2$ — $\phi = 2$ is a line making an angle 2 radians with x axis.

(4) $y = 3$ — $\phi = 3$ is a line making an angle 3 radiam from x axis.





3 radian $= 3 \times \frac{180}{\pi} = 171.88$

2 radians $= 2 \times \frac{180}{\pi} = 114.59$ degree

shaded region in z-plane (rectangle) map to the shaded region in w-plane.

Q2 Find image of semi-infinite strip $x \geq 0$, $0 \leq y \leq \pi$ under the transformation $w = e^z$ and lable corresponding portion of boundaries.

Sol^n $\quad w = e^z$.

$$R e^{i\phi} = e^x \cdot e^{iy}$$

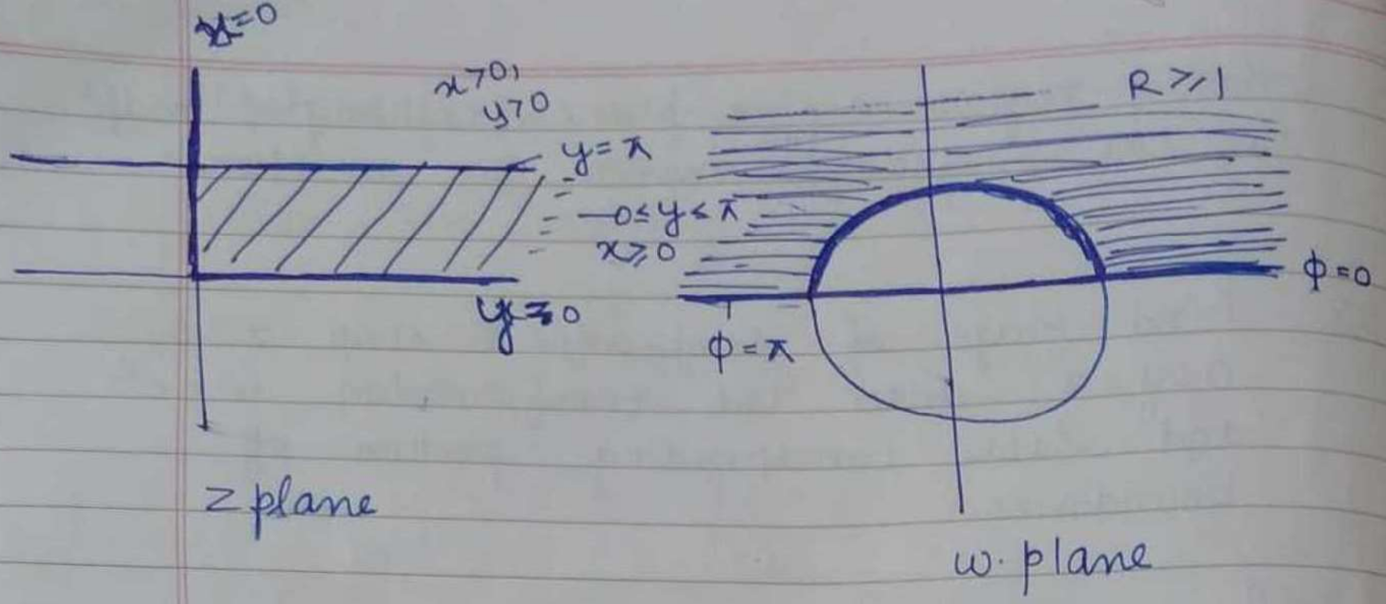
$$\boxed{R = e^x} \qquad \boxed{\phi = y}$$

z-plane | w-plane

① $x \geq 0$ — $R \geq 1$ which is a region outside circle of radius 1 incluing boundary of $R = 1$.

② $y = 0$ — $\phi = 0$ ie +ive u axis or a line making 0 angle.

③ $y = \pi$ — $\phi = \pi$ ie neg u axis or line making angle π radian from x axis



In w plane region is upper half plane excluding the unit circle.

Mapping of some simple curves under $w = e^z$:-

	z plane	w-plane
(1)	straight line. $x = c$	circle $R = e^c$
(2)	y axis $x = 0$	unit circle $R = 1$
(3)	Region b/w $y = 0$ & $y = \pi$ ie $0 \le y \le \pi$	upper half plane
(4)	Region b/w $y = 0$ and $y = -\pi$ $-\pi \le y \le 0$	Lower half plane
(5)	Region b/w $y = c$ and $y = c + 2\pi$ $c \le y \le c + 2\pi$.	Whole plane.

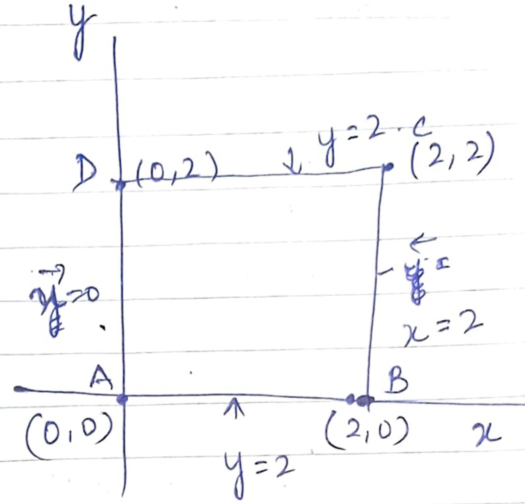
Q:- Find the image of square with vertices $(0,0)$, $(2,0)$, $(2,2)$, $(0,2)$ in z-plane under the transformation $w=z^2$.

Soln:

$w = z^2$

$u+iv = (x+iy)^2$

$u = x^2-y^2; \quad v = 2xy$



z-plane

ABCD is a square in z-plane with vertices $(0,0)$, $(2,0)$, $(0,2)$, $(2,2)$.
ie bounded by four lines. $x=0$, $y=0$, $x=2$, $y=2$.

(1) $x=0$ maps to:-

$u = \cancel{x^2}-y^2, \quad v=0.$

which represents the left part of u axis. ie $u<0$

(2) $x=2$ maps to:-

$u = 4-y^2, \quad v = 4y.$

ie $u = 4 - \frac{v^2}{16}$

ie $\boxed{v^2 = -16(u-4)}$

parabola with vertex $(4,0)$, opening to the left of the vertex.

(3) $y=0$ maps to

$$u = x^2, \quad v = 0$$

which represent right part of u axis ie $u > 0$.

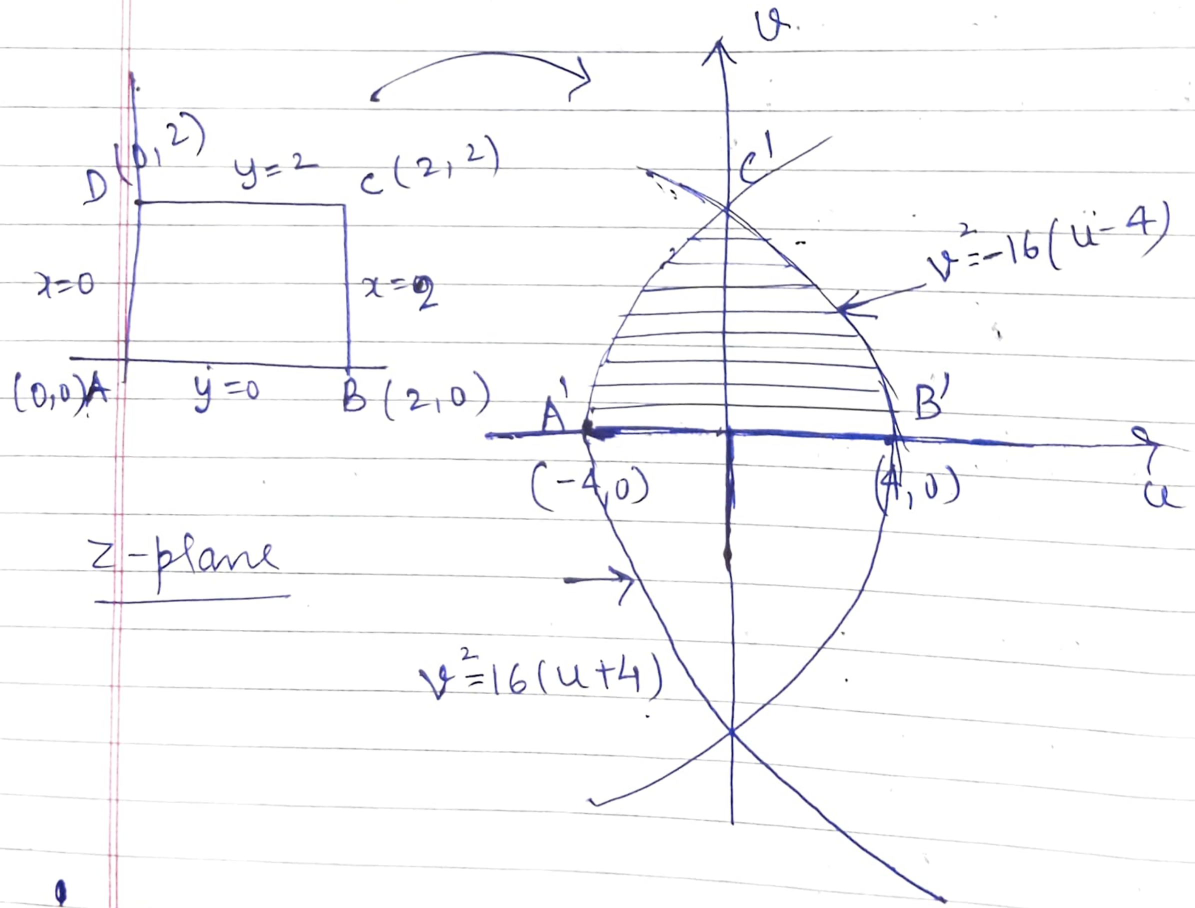
(4) $y=2$ maps to:

$$u = x^2 - 4, \quad v = 4x$$

$$u = \frac{v^2}{16} - 4$$

$$\boxed{v^2 = 16(u+4)}$$

which is parabola vertex $(-4,0)$, opening to right of the vertex.



$$\boxed{\begin{array}{l} x > 0,\ y > 0 \\ \Rightarrow v > 0 \end{array}}$$

w-plane

Q:- Find the image of the region bounded by $1 \le r \le 2$, $\frac{\pi}{6} \le \theta \le \frac{\pi}{3}$ in the z-plane under the transformation $w = z^2$;

Soln:-

$$w = z^2$$

$$Re^{i\phi} = r^2 e^{2i\theta}$$

$$\boxed{R = r^2} \quad \boxed{\phi = 2\theta}.$$

given $1 \le r \le 2$

$$1 \le r^2 \le 4$$

$$1 \le R \le 4$$

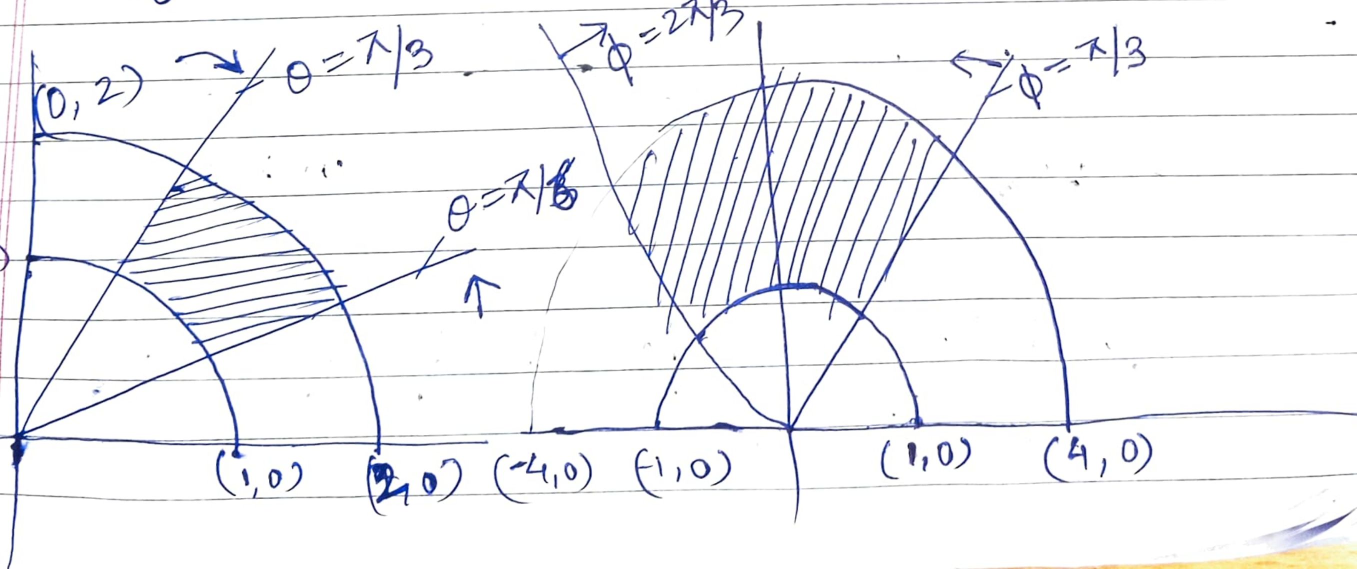
$1 \le r \le 2$ maps to $1 \le R \le 4$.

Now $\frac{\pi}{6} \le \theta \le \frac{\pi}{3}$

$$\frac{\pi}{3} \le 2\theta \le \frac{2\pi}{3}$$

ie $\frac{\pi}{3} \le \phi \le \frac{2\pi}{3}$

ie $\pi/6 \le \theta \le \pi/3$ maps to $\frac{\pi}{3} \le \phi \le \frac{2\pi}{3}$.



Q! Prove that the transformation $w = e^z$ transforms the region between the real axis and a line ||el to the real axis $y = \pi$ into the upper half of the w-plane.

Sol^n $w = e^z$

$$Re^{i\phi} = e^x \cdot e^{iy}$$

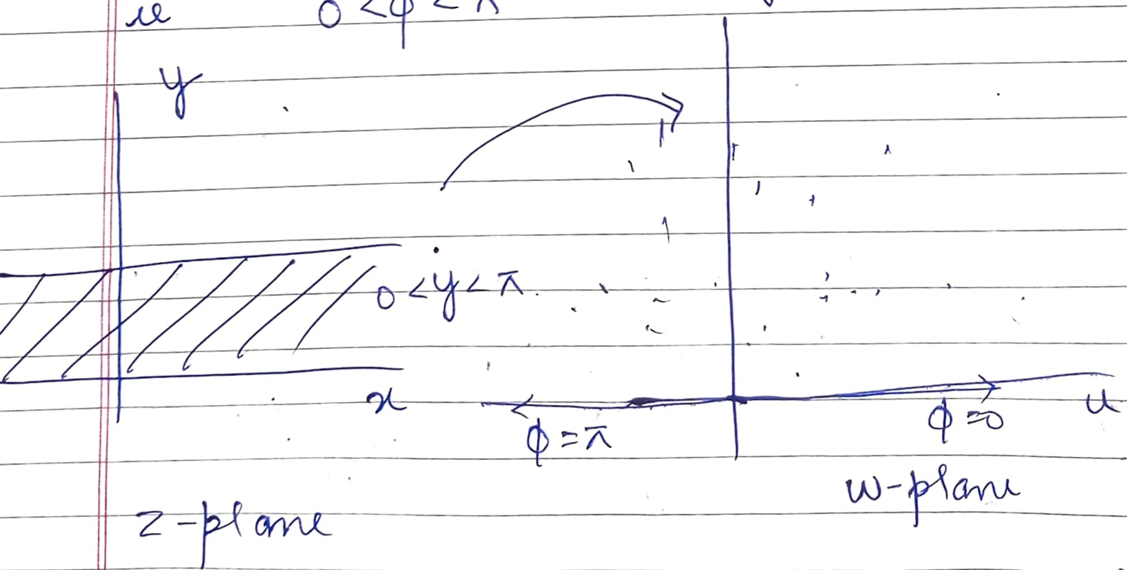
$$\boxed{R = e^x} \qquad \boxed{\phi = y}.$$

region in z-plane is $0 < y < \pi$
real axis $y \neq 0$

$$y > 0$$
$$\Rightarrow \quad \phi > 0$$

$$y < \pi \Rightarrow \phi < \pi$$

ie $\quad 0 < \phi < \pi$



$0 < \phi < \pi$ is upper half of w-plane.
ie $\boxed{v > 0}$