Volumes Using Cross-Sections

$$\text{Volume} = \text{area} \times \text{height} = A \cdot h.$$

DEFINITION The **volume** of a solid of integrable cross-sectional area $A(x)$ from $x = a$ to $x = b$ is the integral of A from a to b,

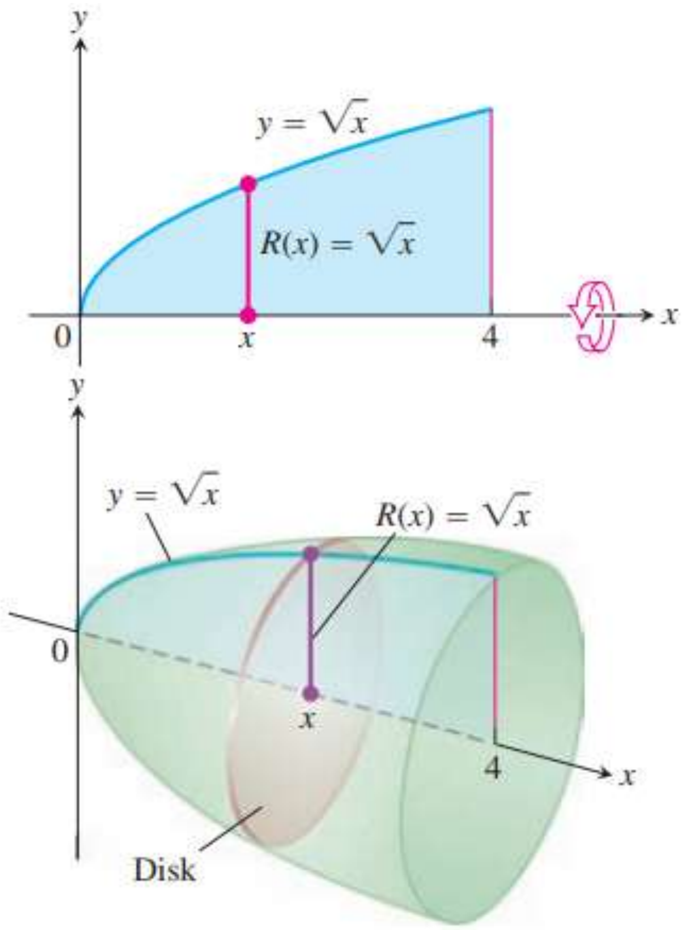
$$V = \int_a^b A(x)\, dx.$$

Calculating the Volume of a Solid

1. *Sketch the solid and a typical cross-section.*
2. *Find a formula for* $A(x)$, the area of a typical cross-section.
3. *Find the limits of integration.*
4. *Integrate* $A(x)$ to find the volume.

Solids of Revolution: The Disk Method

The solid generated by rotating (or revolving) a plane region about an axis in its plane is called a solid of revolution



Volume by Disks for Rotation About the x-axis

$$V = \int_a^b A(x)\,dx = \int_a^b \pi[R(x)]^2\,dx.$$

EXAMPLE The region between the curve $y = \sqrt{x}$, $0 \le x \le 4$, and the x-axis is revolved about the x-axis to generate a solid. Find its volume.

Solution We draw figures showing the region, a typical radius, and the generated solid The volume is

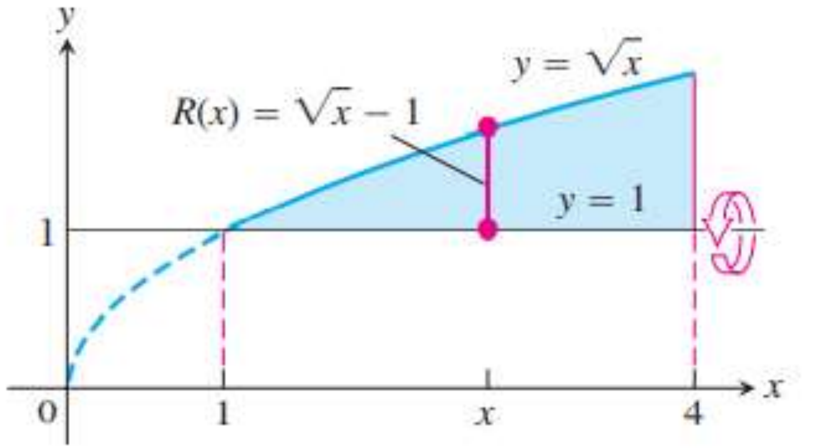
$$V = \int_a^b \pi[R(x)]^2\,dx$$

$$= \int_0^4 \pi\left[\sqrt{x}\right]^2 dx$$

Radius $R(x) = \sqrt{x}$ for rotation around x-axis

$$= \pi \int_0^4 x\,dx = \pi \frac{x^2}{2}\bigg]_0^4 = \pi \frac{(4)^2}{2} = 8\pi.$$

EXAMPLE Find the volume of the solid generated by revolving the region bounded by $y = \sqrt{x}$ and the lines $y = 1$, $x = 4$ about the line $y = 1$.



(a)

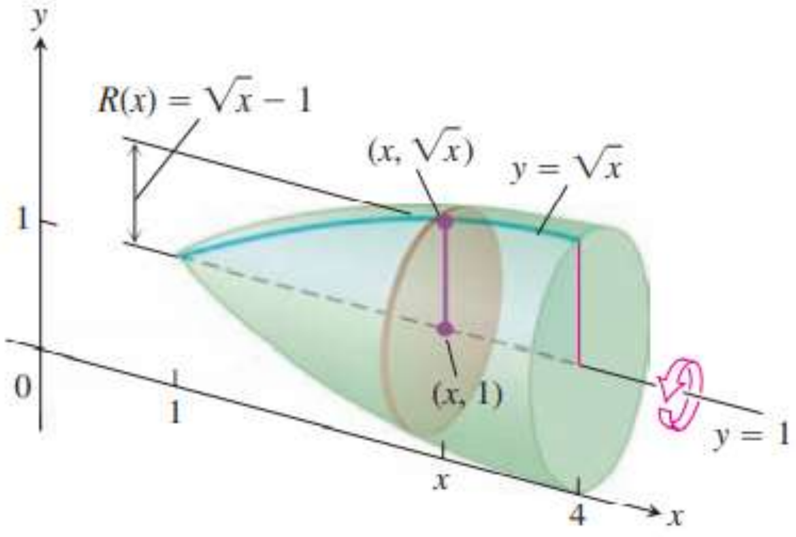
Solution The volume is

$$V = \int_1^4 \pi[R(x)]^2\,dx$$

$$= \int_1^4 \pi\left[\sqrt{x} - 1\right]^2 dx \qquad \text{Radius } R(x) = \sqrt{x} - 1 \text{ for rotation around } y = 1$$

$$= \pi \int_1^4 \left[x - 2\sqrt{x} + 1\right] dx \qquad \text{Expand integrand.}$$

$$= \pi\left[\frac{x^2}{2} - 2 \cdot \frac{2}{3}x^{3/2} + x\right]_1^4 = \frac{7\pi}{6}. \qquad \text{Integrate.}$$



(b)

EXAMPLE Find the volume of the solid generated by revolving the region between the parabola $x = y^2 + 1$ and the line $x = 3$ about the line $x = 3$.

Solution Note that the cross-sections are perpendicular to the line $x = 3$ and have y-coordinates from $y = -\sqrt{2}$ to $y = \sqrt{2}$. The volume is

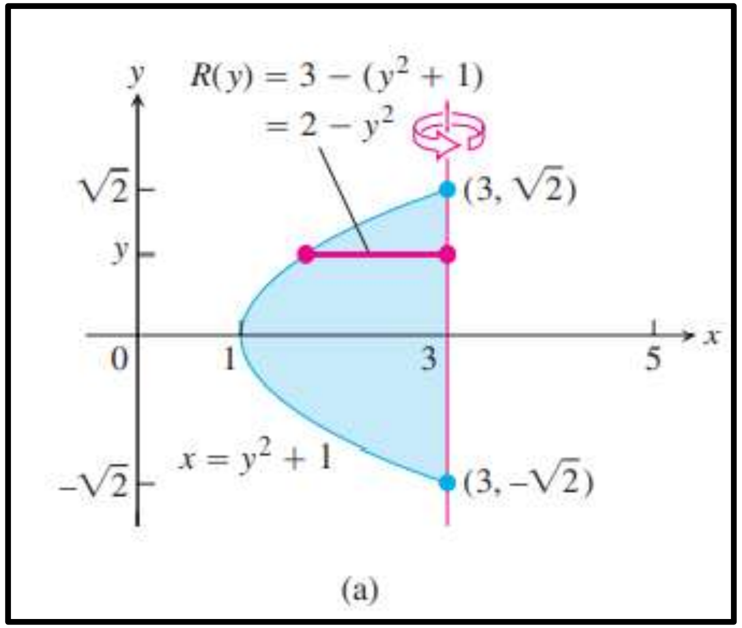
$$V = \int_{-\sqrt{2}}^{\sqrt{2}} \pi[R(y)]^2\, dy \qquad y = \pm\sqrt{2} \text{ when } x = 3$$

$$= \int_{-\sqrt{2}}^{\sqrt{2}} \pi[2 - y^2]^2\, dy \qquad \text{Radius } R(y) = 3 - (y^2 + 1) \text{ for rotation around axis } x = 3$$

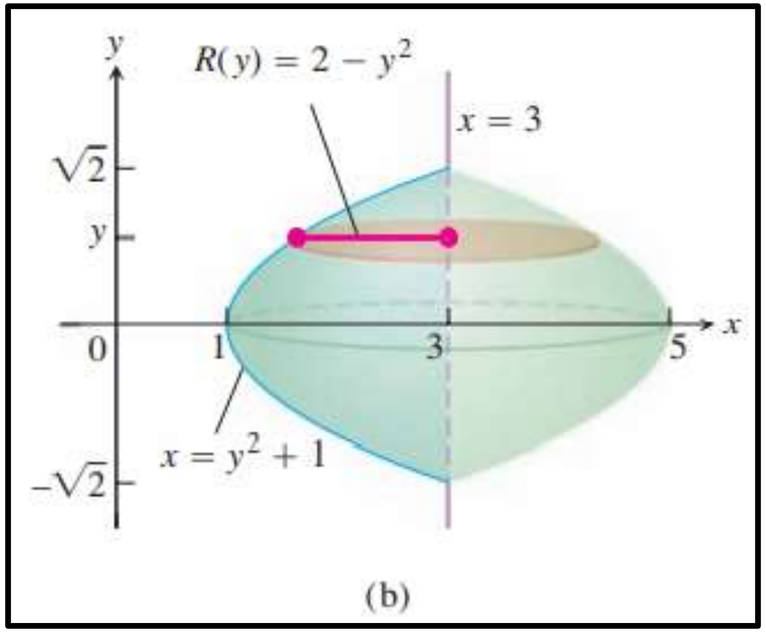
$$= \pi \int_{-\sqrt{2}}^{\sqrt{2}} [4 - 4y^2 + y^4]\, dy \qquad \text{Expand integrand.}$$

$$= \pi \left[4y - \frac{4}{3}y^3 + \frac{y^5}{5} \right]_{-\sqrt{2}}^{\sqrt{2}} \qquad \text{Integrate.}$$

$$= \frac{64\pi\sqrt{2}}{15}.$$



(a)



(b)

EXAMPLE The region bounded by the parabola $y = x^2$ and the line $y = 2x$ in the first quadrant is revolved about the y-axis to generate a solid. Find the volume of the solid.

Solution First we sketch the region and draw a line segment across it perpendicular to the axis of revolution (the y-axis).

The radii of the washer swept out by the line segment are $R(y) = \sqrt{y}, r(y) = y/2$

The line and parabola intersect at $y = 0$ and $y = 4$, so the limits of integration are $c = 0$ and $d = 4$. We integrate to find the volume:

$$V = \int_c^d \pi([R(y)]^2 - [r(y)]^2)\,dy$$

$$= \int_0^4 \pi\left(\left[\sqrt{y}\right]^2 - \left[\frac{y}{2}\right]^2\right) dy$$

$$= \pi \int_0^4 \left(y - \frac{y^2}{4}\right) dy = \pi\left[\frac{y^2}{2} - \frac{y^3}{12}\right]_0^4 = \frac{8}{3}\pi.$$