Fourier Transform

- The Fourier transform allows us to extend the concept of a frequency spectrum to nonperiodic functions.
- The transform assumes that a nonperiodic function is a periodic function with an infinite period.
- Thus, the Fourier transform is an integral representation of a nonperiodic function that is analogous to a Fourier series representation of a periodic function.
- The Fourier transform is an *integral transform* like the Laplace transform. It transforms a function in the time domain into the frequency domain.
- The Fourier transform is very useful in communications systems and digital signal processing, in situations where the Laplace transform does not apply.
- While the Laplace transform can only handle circuits with inputs for $t > 0$ with initial conditions, the Fourier transform can handle circuits with inputs for $t < 0$ as well as those for $t > 0$

$$F(\omega) = \mathcal{F}[f(t)] = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt \tag{17.8}$$

where $\mathcal{F}$ is the Fourier transform operator. It is evident from Eq. (17.8) that:

The Fourier transform is an integral transformation of $f(t)$ from the time domain to the frequency domain.

In general, $F(\omega)$ is a complex function; its magnitude is called the *amplitude spectrum*, while its phase is called the *phase spectrum*. Thus $F(\omega)$ is the *spectrum*.

Equation (17.7) can be written in terms of $F(\omega)$, and we obtain the *inverse Fourier transform* as

$$f(t) = \mathcal{F}^{-1}[F(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}\,d\omega \tag{17.9}$$

The function $f(t)$ and its transform $F(\omega)$ form the Fourier transform pairs:

$$f(t) \qquad \Longleftrightarrow \qquad F(\omega) \tag{17.10}$$

since one can be derived from the other.

The Fourier transform $F(\omega)$ exists when the Fourier integral in Eq. (17.8) converges. A sufficient but not necessary condition that $f(t)$ has a Fourier transform is that it be completely integrable in the sense that

$$\int_{-\infty}^{\infty} |f(t)|\, dt < \infty \tag{17.11}$$

TABLE 17.1 Properties of the Fourier transform.

Property	$f(t)$	$F(\omega)$
Linearity	$a_1 f_1(t) + a_2 f_2(t)$	$a_1 F_1(\omega) + a_2 F_2(\omega)$
Scaling	$f(at)$	$\frac{1}{\|a\|} F\left(\frac{\omega}{a}\right)$
Time shift	$f(t-a)u(t-a)$	$e^{-j\omega a} F(\omega)$
Frequency shift	$e^{j\omega_0 t} f(t)$	$F(\omega - \omega_0)$
Modulation	$\cos(\omega_0 t)\ f(t)$	$\frac{1}{2}[F(\omega + \omega_0) + F(\omega - \omega_0)]$

TABLE 17.1 *(continued)*

Property	$f(t)$	$F(\omega)$
Time differentiation	$\frac{df}{dt}$	$j\omega F(\omega)$
	$\frac{d^n f}{dt^n}$	$(j\omega)^n F(\omega)$
Time integration	$\int_{-\infty}^{t} f(t)\,dt$	$\frac{F(\omega)}{j\omega} + \pi F(0)\,\delta(\omega)$
Frequency differentiation	$t^n f(t)$	$(j)^n \frac{d^n}{d\omega^n} F(\omega)$
Reversal	$f(-t)$	$F(-\omega)$ or $F^*(\omega)$
Duality	$F(t)$	$2\pi f(-\omega)$
Convolution in t	$f_1(t) * f_1(t)$	$F_1(\omega) F_2(\omega)$
Convolution in ω	$f_1(t) f_1(t)$	$\frac{1}{2\pi} F_1(\omega) * F_2(\omega)$

TABLE 17.2 Fourier transform pairs.

$f(t)$	$F(\omega)$
$\delta(t)$	1
1	$2\pi\delta(\omega)$
$u(t)$	$\pi\delta(\omega) + \frac{1}{j\omega}$
$u(t+\tau) - u(t-\tau)$	$2\frac{\sin\omega\tau}{\omega}$
$\|t\|$	$\frac{-2}{\omega^2}$
$\text{sgn}(t)$	$\frac{2}{j\omega}$
$e^{-at}u(t)$	$\frac{1}{a+j\omega}$
$e^{at}u(-t)$	$\frac{1}{a-j\omega}$
$t^n e^{-at}u(t)$	$\frac{n!}{(a+j\omega)^{n+1}}$
$e^{-a\|t\|}$	$\frac{2a}{a^2+\omega^2}$
$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$
$\sin\omega_0 t$	$j\pi[\delta(\omega+\omega_0) - \delta(\omega-\omega_0)]$
$\cos\omega_0 t$	$\pi[\delta(\omega+\omega_0) + \delta(\omega-\omega_0)]$
$e^{-at}\sin\omega_0 t\, u(t)$	$\frac{\omega_0}{(a+j\omega)^2+\omega_0^2}$
$e^{-at}\cos\omega_0 t\, u(t)$	$\frac{a+j\omega}{(a+j\omega)^2+\omega_0^2}$

17.4 CIRCUIT APPLICATIONS

The Fourier transform generalizes the phasor technique to nonperiodic functions. Therefore, we apply Fourier transforms to circuits with nonsinusoidal excitations in exactly the same way we apply phasor techniques to circuits with sinusoidal excitations. Thus, Ohm's law is still valid:

$$V(\omega) = Z(\omega)I(\omega) \tag{17.52}$$

where $V(\omega)$ and $I(\omega)$ are the Fourier transforms of the voltage and current and $Z(\omega)$ is the impedance. We get the same expressions for the impedances of resistors, inductors, and capacitors as in phasor analysis, namely,

$$\begin{array}{lcl} R & \Longrightarrow & R \\ L & \Longrightarrow & j\omega L \\ C & \Longrightarrow & \dfrac{1}{j\omega C} \end{array} \tag{17.53}$$

Once we transform the functions for the circuit elements into the frequency domain and take the Fourier transforms of the excitations, we can use circuit techniques such as voltage division, source transformation, mesh analysis, node analysis, or Thevenin's theorem, to find the unknown response (current or voltage). Finally, we take the inverse Fourier transform to obtain the response in the time domain.

Although the Fourier transform method produces a response that exists for $-\infty < t < \infty$, Fourier analysis cannot handle circuits with initial conditions.

The transfer function is again defined as the ratio of the output response $Y(\omega)$ to the input excitation $X(\omega)$, that is,

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} \tag{17.54}$$

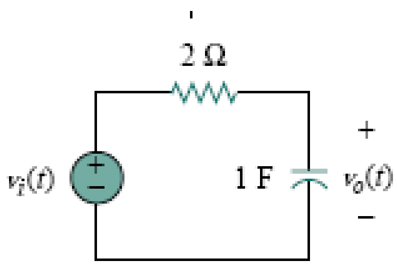

Figure 17.18 For Example 17.7.

Find $v_o(t)$ in the circuit of Fig. 17.18 for $v_i(t) = 2e^{-3t}u(t)$.

Solution:

The Fourier transform of the input voltage is

$$V_i(\omega) = \frac{2}{3 + j\omega}$$

and the transfer function obtained by voltage division is

$$H(\omega) = \frac{V_o(\omega)}{V_i(\omega)} = \frac{1/j\omega}{2 + 1/j\omega} = \frac{1}{1 + j2\omega}$$

Hence,

$$V_o(\omega) = V_i(\omega)H(\omega) = \frac{2}{(3 + j\omega)(1 + j2\omega)}$$

or

$$V_o(\omega) = \frac{1}{(3 + j\omega)(0.5 + j\omega)}$$

By partial fractions,

$$V_o(\omega) = \frac{-0.4}{3 + j\omega} + \frac{0.4}{0.5 + j\omega}$$

Taking the inverse Fourier transform yields

$$v_o(t) = 0.4(e^{-0.5t} - e^{-3t})u(t)$$

Using the Fourier transform method, find $i_o(t)$ in Fig. 17.20 when $i_s(t) = 10 \sin 2t$ A.

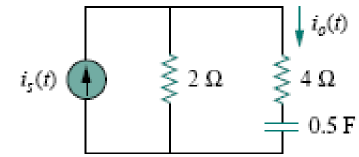

Figure 17.20 For Example 17.8.

Solution:

By current division,

$$H(\omega) = \frac{I_o(\omega)}{I_s(\omega)} = \frac{2}{2 + 4 + 2/j\omega} = \frac{j\omega}{1 + j\omega 3}$$

If $i_s(t) = 10 \sin 2t$, then

$$I_s(\omega) = j\pi 10[\delta(\omega + 2) - \delta(\omega - 2)]$$

Hence,

$$I_o(\omega) = H(\omega) I_s(\omega) = \frac{10\pi\omega[\delta(\omega - 2) - \delta(\omega + 2)]}{1 + j\omega 3}$$

The inverse Fourier transform of $I_o(\omega)$ cannot be found using Table 17.2. We resort to the inverse Fourier transform formula in Eq. (17.9) and write

$$i_o(t) = \mathcal{F}^{-1}[I_o(\omega)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{10\pi\omega[\delta(\omega - 2) - \delta(\omega + 2)]}{1 + j\omega 3} e^{j\omega t}\, d\omega$$

We apply the sifting property of the impulse function, namely,

$$\delta(\omega - \omega_0) f(\omega) = f(\omega_0)$$

or

$$\int_{-\infty}^{\infty} \delta(\omega - \omega_0) f(\omega)\, d\omega = f(\omega_0)$$

and obtain

$$\begin{aligned} i_o(t) &= \frac{10\pi}{2\pi} \left[\frac{2}{1 + j6} e^{j2t} - \frac{-2}{1 - j6} e^{-j2t} \right] \\ &= 10 \left[\frac{e^{j2t}}{6.082 e^{j80.54^\circ}} + \frac{e^{-j2t}}{6.082 e^{-j80.54^\circ}} \right] \\ &= 1.644[e^{j(2t - 80.54^\circ)} + e^{-j(2t - 80.54^\circ)}] \\ &= 3.288 \cos(2t - 80.54^\circ) \text{ A} \end{aligned}$$