Hence, the series is convergent inside the circle with its centre at $\left(\frac{4}{3}, 0\right)$ and a radius of $\frac{2}{3}$, i.e., in the region $\left|z - \frac{4}{3}\right| < \frac{2}{3}$ and is divergent in the region $\left|z - \frac{4}{3}\right| > \frac{2}{3}$.

4.5 TAYLOR'S SERIES

If $f(z)$ is analytic inside a circle C with centre at $z = a$ then for each z inside C, $f(z)$ can be expanded as a power series about $z = a$ as

$$f(z) = f(a) + (z-a) f'(a) + \frac{(z-a)^2}{2!} f''(a) + \cdots + \frac{(z-a)^n}{n!} f^{(n)}(a) + \cdots$$

Proof Let z be any point inside the circle C. Draw a circle C_1 inside C with the centre at $z = a$, enclosing the point z (Fig. 4.1).

Let w be any point on C_1.

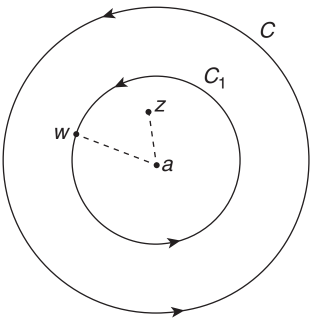

Fig. 4.1

$$|z-a| < |w-a|$$

$$\frac{|z-a|}{|w-a|} < 1$$

$$\left|\frac{z-a}{w-a}\right| < 1$$

By Cauchy's integral formula,

$$f(z) = \frac{1}{2\pi i} \int_{C_1} \frac{f(w)}{w-z} \mathrm{d}w \qquad \text{...(4.1)}$$

Consider
$$\frac{1}{w-z} = \frac{1}{(w-a)-(z-a)}$$

$$= \frac{1}{(w-a)\left(1 - \frac{z-a}{w-a}\right)}$$

$$= \frac{1}{w-a}\left(1 - \frac{z-a}{w-a}\right)^{-1}$$

$$= \frac{1}{w-a}\left[1 + \left(\frac{z-a}{w-a}\right) + \left(\frac{z-a}{w-a}\right)^2 + \cdots + \left(\frac{z-a}{w-a}\right)^n + \cdots\right]$$

$$\left[\because \left|\frac{z-a}{w-a}\right| < 1 \quad \text{Using binomial expansion}\right]$$

$$= \frac{1}{(w-a)} \sum_{n=0}^{\infty} \left(\frac{z-a}{w-a} \right)^n$$

$$= \sum_{n=0}^{\infty} \frac{(z-a)^n}{(w-a)^{n+1}}$$

Substituting $\frac{1}{w-z}$ in Eq. (4.1),

$$f(z) = \frac{1}{2\pi i} \int_{C_1} f(w) \left[\sum_{n=0}^{\infty} \frac{(z-a)^n}{(w-a)^{n+1}} \right] \mathrm{d}w$$

$$= \frac{1}{2\pi i} \sum_{n=0}^{\infty} (z-a)^n \left[\int_{C_1} \frac{f(w)}{(w-a)^{n+1}} \mathrm{d}w \right]$$

$$= \frac{1}{2\pi i} \sum_{n=0}^{\infty} (z-a)^n \left[2\pi i \frac{f^{(n)}(a)}{n!} \right] \quad \left[\begin{array}{l} \text{Using Cauchy's integral formula} \\ \text{for derivatives} \end{array} \right]$$

$$= \sum_{n=0}^{\infty} \frac{(z-a)^n}{n!} f^{(n)}(a)$$

$$= f(a) + (z-a) f'(a) + \frac{(z-a)^2}{2!} f''(a) + \cdots + \frac{(z-a)^n}{n!} f^{(n)}(a) + \cdots$$

Example 1

Expand $f(z) = e^z$ in a Taylor series about $z = 0$.

Solution

$$f(z) = e^z, \quad a = 0$$

By Taylor's series,

$$f(z) = f(a) + f'(a)(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \frac{f'''(a)}{3!}(z-a)^3 + \cdots$$

Putting $a = 0$,

$$f(z) = f(0) + f'(0)z + \frac{f''(0)}{2!} z^2 + \frac{f'''(0)}{3!} z^3 + \cdots \qquad \text{...(1)}$$

$$f(z) = e^z \qquad f(0) = e^0 = 1$$

$$f'(z) = e^z \qquad f'(0) = e^0 = 1$$

$$f''(z) = e^z \qquad f''(0) = e^0 = 1$$

$$f'''(z) = e^z \qquad f'''(0) = e^0 = 1$$

Substituting in Eq. (1),

$$f(z) = 1 + z + \frac{1}{2!}z^2 + \frac{1}{3!}z^3 + \ldots$$

$$e^z = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \ldots$$

Example 2

Expand $\frac{1}{z+2}$ at $z = 1$ *in Taylor's series.*

Solution

Let $f(z) = \frac{1}{z+2}, \quad a = 1$

By Taylor's series,

$$f(z) = f(a) + f'(a)(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \frac{f'''(a)}{3!}(z-a)^3 + \cdots$$

Putting $a = 1$,

$$f(z) = f(1) + f'(1)(z-1) + \frac{f''(1)}{2!}(z-1)^2 + \frac{f'''(1)}{3!}(z-1)^3 + \cdots \qquad \ldots(1)$$

$$f(z) = \frac{1}{z+2} \qquad f(1) = \frac{1}{1+2} = \frac{1}{3}$$

$$f'(z) = -\frac{1}{(z+2)^2} \qquad f'(1) = -\frac{1}{(1+2)^2} = -\frac{1}{9}$$

$$f''(z) = \frac{2}{(z+2)^3} \qquad f''(1) = \frac{2}{(1+2)^3} = \frac{2}{27}$$

$$f'''(z) = -\frac{6}{(z+2)^4} \qquad f'''(1) = -\frac{6}{(1+2)^4} = -\frac{6}{81} = -\frac{2}{27}$$

Substituting in Eq. (1),

$$f(z) = \frac{1}{3} - \frac{1}{9}(z-1) + \frac{1}{2!}\frac{2}{27}(z-1)^2 + \frac{1}{3!}\left(-\frac{2}{27}\right)(z-1)^3 + \cdots$$

$$\frac{1}{z+2} = \frac{1}{3} - \frac{1}{9}(z-1) + \frac{1}{27}(z-1)^2 - \frac{1}{81}(z-1)^3 + \cdots$$

Aliter:

$$f(z) = \frac{1}{z+2}$$

Let $z - 1 = t$ then $z = t + 1$.

$$f(z) = \frac{1}{t+1+2}$$

$$\frac{1}{z+2} = \frac{1}{3+t}$$

$$= \frac{1}{3\left(1+\frac{t}{3}\right)}$$

$$= \frac{1}{3}\left(1+\frac{t}{3}\right)^{-1}$$

$$= \frac{1}{3}\left(1-\frac{t}{3}+\frac{t^2}{9}-\frac{t^3}{27}+\cdots\right)$$

$$= \frac{1}{3}-\frac{1}{9}t+\frac{1}{27}t^2-\frac{1}{81}t^3+\cdots$$

$$= \frac{1}{3}-\frac{1}{9}(z-1)+\frac{1}{27}(z-1)^2-\frac{1}{81}(z-1)^3+\ldots$$

Example 3

Obtain the Taylor series of $f(z) = \dfrac{1-z}{z^2}$ *in powers of* $z - 1$.

Solution

$$f(z) = \frac{1-z}{z^2} = \frac{1}{z^2} - \frac{1}{z}, \quad a = 1$$

By Taylor's series,

$$f(z) = f(a) + f'(a)(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \frac{f'''(a)}{3!}(z-a)^3 + \ldots$$

Putting $a = 1$,

$$f(z) = f(1) + f'(1)(z-1) + \frac{f''(1)}{2!}(z-1)^2 + \frac{f'''(1)}{3!}(z-1)^3 + \cdots \qquad \ldots(1)$$

$$f(z) = \frac{1}{z^2} - \frac{1}{z} \qquad f(1) = \frac{1}{1} - \frac{1}{1} = 0$$

$$f'(z) = -\frac{2}{z^3} + \frac{1}{z^2} \qquad f'(1) = -2 + 1 = -1$$

$$f''(z) = \frac{6}{z^4} - \frac{2}{z^3} \qquad f''(1) = 6 - 2 = 4$$

$$f'''(z) = -\frac{24}{z^5} + \frac{6}{z^4} \qquad f'''(1) = -24 + 6 = -18$$

Substituting in Eq. (1),

$$f(z) = 0 - 1(z-1) + \frac{4}{2!}(z-1)^2 - \frac{18}{3!}(z-1)^3 + \cdots$$

$$\frac{1-z}{z^2} = -(z-1) + 2(z-1)^2 - 3(z-1)^3 + \cdots$$

Example 4

Expand $\frac{z-1}{z+1}$ *in a Taylor series about the point* $z = 1$.

Solution

Let $f(z) = \frac{z-1}{z+1}, \quad a = 1$

By Taylor's series,

$$f(z) = f(a) + f'(a)(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \frac{f'''(a)}{3!}(z-a)^3 + \ldots$$

Putting $a = 1$,

$$f(z) = f(1) + f'(1)(z-1) + \frac{f''(1)}{2!}(z-1)^2 + \cdots \qquad \text{...(1)}$$

$$f(z) = \frac{z-1}{z+1} \qquad f(1) = 0$$

$$f'(z) = \frac{(z+1)(1) - (z-1)(1)}{(z+1)^2} \qquad f'(1) = \frac{1}{2}$$

$$= \frac{2}{(z+1)^2}$$

$$f''(z) = \frac{0 - 2\{2(z+1)\}}{(z+1)^4} \qquad f''(1) = -\frac{4}{2^3} = -\frac{1}{2}$$

$$= -\frac{4}{(z+1)^3}$$

Substituting in Eq. (1),

$$f(z) = 0 + \frac{1}{2}(z-1) - \frac{1}{2!}\frac{1}{2}(z-1)^2 + \cdots$$

$$\frac{z-1}{z+1} = \frac{1}{2}(z-1) - \frac{1}{4}(z-1)^2 + \cdots$$

Example 5

Expand $\log(1+z)$ *as a Taylor series about* $z=0$ *when* $|z|<1$.

Solution

Let $f(z)=\log(1+z),\ a=0$

By Taylor's series,

$$f(z)=f(a)+f'(a)(z-a)+\frac{f''(a)}{2!}(z-a)^2+\frac{f'''(a)}{3!}(z-a)^3+\dots$$

Putting $a=0$,

$$f(z)=f(0)+f'(0)z+\frac{f''(0)}{2!}z^2+\frac{f'''(0)}{3!}z^3+\dots \qquad \text{...(1)}$$

$$f(z)=\log(1+z) \qquad f(0)=\log 1=0$$

$$f'(z)=\frac{1}{1+z} \qquad f'(0)=\frac{1}{1+0}=1$$

$$f''(z)=-\frac{1}{(1+z)^2} \qquad f''(0)=-\frac{1}{1}=-1$$

$$f'''(z)=\frac{2}{(1+z)^3} \qquad f'''(0)=2$$

Substituting in Eq. (1),

$$f(z)=0+z-\frac{z^2}{2!}+\frac{2}{3!}z^3+\dots$$

$$\log(1+z)=z-\frac{z^2}{2}+\frac{z^3}{3}+\dots$$

Example 6

Expand $f(z)=\sin z$ *about* $z=\frac{\pi}{4}$.

Solution

$$f(z)=\sin z, \qquad a=\frac{\pi}{4}$$

By Taylor's series,

$$f(z)=f(a)+f'(a)(z-a)+\frac{f''(a)}{2!}(z-a)^2+\frac{f'''(a)}{3!}(z-a)^3+\dots$$

Putting $a = \frac{\pi}{4}$,

$$f(z) = f\left(\frac{\pi}{4}\right) + f'\left(\frac{\pi}{4}\right)\left(z - \frac{\pi}{4}\right) + \frac{f''\left(\frac{\pi}{4}\right)}{2!}\left(z - \frac{\pi}{4}\right)^2 + \frac{f'''\left(\frac{\pi}{4}\right)}{3!}\left(z - \frac{\pi}{4}\right)^3 + \cdots \quad ...(1)$$

$$f(z) = \sin z \qquad f\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

$$f'(z) = \cos z \qquad f'\left(\frac{\pi}{4}\right) = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

$$f''(z) = -\sin z \qquad f''\left(\frac{\pi}{4}\right) = -\sin\frac{\pi}{4} = -\frac{1}{\sqrt{2}}$$

$$f'''(z) = -\cos z \qquad f'''\left(\frac{\pi}{4}\right) = -\cos\frac{\pi}{4} = -\frac{1}{\sqrt{2}}$$

Substituting in Eq. (1),

$$f(z) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\left(z - \frac{\pi}{4}\right) - \frac{1}{2!}\frac{1}{\sqrt{2}}\left(z - \frac{\pi}{4}\right)^2 - \frac{1}{3!}\frac{1}{\sqrt{2}}\left(z - \frac{\pi}{4}\right)^3 + \cdots$$

$$\sin z = \frac{1}{\sqrt{2}}\left[1 + \left(z - \frac{\pi}{4}\right) - \frac{1}{2}\left(z - \frac{\pi}{4}\right)^2 - \frac{1}{6}\left(z - \frac{\pi}{4}\right)^3 + \cdots\right]$$

Example 7

Expand $f(z) = \frac{z}{(z+1)(z-3)}$ *as a Taylor's series about* $z = 0$.

Solution

$$f(z) = \frac{z}{(z+1)(z-3)}$$

$$= \frac{A}{z+1} + \frac{B}{z-3}$$

$$z = A(z-3) + B(z+1)$$

Putting $z = -1$,

$$-1 = -4A$$

$$A = \frac{1}{4}$$

Putting $z = 3$,

$$3 = 4B$$

$$B = \frac{3}{4}$$

$$\therefore f(z) = \frac{1}{4}\frac{1}{z+1} + \frac{3}{4}\frac{1}{z-3}$$

$$= \frac{1}{4}\frac{1}{1+z} + \frac{3}{4}\frac{1}{-3\left(1-\frac{z}{3}\right)}$$

$$= \frac{1}{4}(1+z)^{-1} - \frac{1}{4}\left(1-\frac{z}{3}\right)^{-1}$$

$$= \frac{1}{4}(1-z+z^2-\ldots) - \frac{1}{4}\left[1+\frac{z}{3}+\left(\frac{z}{3}\right)^2+\ldots\right]$$

$$= -\frac{1}{3}z + \frac{2}{9}z^2 - \ldots$$

Example 8

Expand $f(z) = \dfrac{z+1}{(z-3)(z-4)}$ *as a Taylor's series about* $z = 2$.

Solution

$$f(z) = \frac{z+1}{(z-3)(z-4)}$$

$$= \frac{A}{z-3} + \frac{B}{z-4}$$

$$z+1 = A(z-4) + B(z-3)$$

Putting $z = 3$,

$$4 = -A$$

$$A = -4$$

Putting $z = 4$,

$$5 = B(1)$$

$$B = 5$$

$$\therefore f(z) = -\frac{4}{z-3} + \frac{5}{z-4}$$

Let $z-2=t$ then $t=z+2$.

$$f(z)=-\frac{4}{z-2-1}+\frac{5}{z-2-2}$$

$$=-\frac{4}{t-1}+\frac{5}{t-2}$$

$$=\frac{4}{1-t}-\frac{5}{2\left(1-\frac{t}{2}\right)}$$

$$=4(1-t)^{-1}-\frac{5}{2}\left(1-\frac{t}{2}\right)^{-1}$$

$$=4(1-t+t^2-\cdots)-\frac{5}{2}\left[1-\frac{t}{2}+\left(\frac{t}{2}\right)^2-\cdots\right]$$

$$=\frac{3}{2}-\frac{11}{4}t+\frac{27}{8}t^2+\cdots$$

$$=\frac{3}{2}-\frac{11}{4}(z-2)+\frac{27}{8}(z-2)^2+\cdots$$

Example 9

Find the Taylor series to represent the function $\frac{z^2-1}{(z+2)(z+3)}$ in $|z|<2$.

Solution

Let $$f(z)=\frac{z^2-1}{(z+2)(z+3)}=\frac{z^2-1}{z^2+5z+6}$$

Since the degrees of the numerator and denominator are same, partial fraction cannot be applied. Dividing to reduce the degree of numerator,

$$f(z)=1+\frac{-5z-7}{(z+2)(z+3)}$$

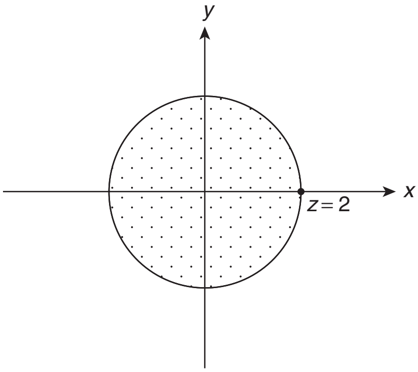

Fig. 4.2

Let $$\frac{-5z-7}{(z+2)(z+3)}=\frac{A}{z+2}+\frac{B}{z+3}$$

$$-5z-7=A(z+3)+B(z+2)$$

Putting $z=-2$,

$$-5(-2)-7=A(-2+3)$$

$$A=3$$

Putting $z = -3$,

$$-5(-3) - 7 = B(-3+2)$$

$$B = -8$$

$$\therefore f(z) = 1 + \frac{3}{z+2} - \frac{8}{z+3}$$

$f(z)$ is not analytic at $z = -2$ and $z = -3$. But $f(z)$ is analytic inside the region $|z| < 2$ (Fig. 4.2).

$$|z| < 2, \quad \left|\frac{z}{2}\right| < 1, \quad \left|\frac{z}{3}\right| < 1$$

$$f(z) = 1 + \frac{3}{2\left(1+\frac{z}{2}\right)} - \frac{8}{3\left(1+\frac{z}{3}\right)}$$

$$= 1 + \frac{3}{2}\left(1+\frac{z}{2}\right)^{-1} - \frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$= 1 + \frac{3}{2}\left[1 - \frac{z}{2} + \left(\frac{z}{2}\right)^2 - \cdots\right] - \frac{8}{3}\left[1 - \frac{z}{3} + \left(\frac{z}{3}\right)^2 - \cdots\right]$$

$$= -\frac{1}{6} + \frac{5}{36}z + \frac{17}{216}z^2 + \cdots$$

EXERCISE 4.1

Find the Taylor's series for the following functions about the indicated points:

1. $f(z) = \cos z$ about $z = \frac{\pi}{4}$

$$\left[\textbf{Ans.: } \frac{1}{\sqrt{2}}\left[1 - \left(z - \frac{\pi}{4}\right) + \frac{1}{2}\left(z - \frac{\pi}{4}\right)^2 - \cdots\right]\right]$$

2. $f(z) = \cos z$ about $z = 0$

$$\left[\textbf{Ans.: } 1 - \frac{z^2}{2} + \frac{z^4}{24} - \cdots\right]$$

3. $f(z) = e^{2z}$ about $z = 2i$

$$\left[\textbf{Ans.: } e^{4i}\left\{1 + 2(z-2i) - (z-2i)^2 + \cdots\right\}\right]$$

4. $f(z) = \tanh z$ about $z = 0$

$$\left[\textbf{Ans.: } z - \frac{z^3}{3} + \cdots\right]$$

5. $f(z) = \dfrac{z-1}{z+1}$ in the region $|z| < 1$

$$[\textbf{Ans.: } 1 - 2(1 + z + z^2 + z^3 + \cdots)]$$

6. $f(z) = \dfrac{1}{z^2+4}$ about $z = -i$

$$\left[\textbf{Ans.: } \frac{1}{3} + \frac{1}{18}(z+i).i + \frac{7}{27}(z+1)^2 - \cdots\right]$$

7. $f(z) = \dfrac{z-1}{z^2}$ about $z = 1$

$$[\textbf{Ans.: } (z-1) - 2(z-1)^2 + 3(z-1)^3 - \cdots]$$

8. $f(z) = \dfrac{1}{z^2 - 4z + 3}$ about $z = 4$

$$\left[\textbf{Ans.: } \frac{1}{3} - \frac{4}{9}(z-4) + \frac{13}{27}(z-4)^2 - \cdots\right]$$

4.6 LAURENT'S SERIES

If $f(z)$ is analytic on two concentric circles C_1 and C_2 with the centre at $z = a$ and radii $r_1, r_2 (r_2 < r_1)$ and in the annular region R between C_1 and C_2 then for all z in R,

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots + \frac{b_1}{z-a} + \frac{b_2}{(z-a)^2} + \cdots$$

$$= \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n}$$

where
$$a_n = \frac{1}{2\pi i} \int_{C_1} \frac{f(w)}{(w-a)^{n+1}} \mathrm{d}w$$

$$b_n = \frac{1}{2\pi i} \int_{C_2} \frac{f(w)}{(w-a)^{-n+1}} \mathrm{d}w$$

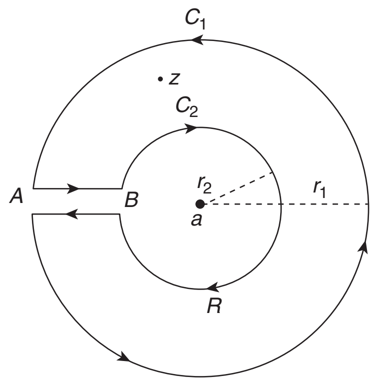

Fig. 4.3

Proof

Let C_1, C_2 be two concentric circles with centre at $z = a$ and radii r_1 and $r_2 (r_2 < r_1)$ respectively (Fig. 4.3).

The annular region R between C_1 and C_2 is a multiply connected region. This multiply connected region is converted to a simply connected region by introducing a cross-cut AB.

$f(z)$ is analytic in this simply connected region.

By Cauchy's integral formula,

$$f(z) = \frac{1}{2\pi i}\int_{C_1}\frac{f(w)}{w-z}\mathrm{d}w + \frac{1}{2\pi i}\int_{AB}\frac{f(w)}{w-z}\mathrm{d}w - \frac{1}{2\pi i}\int_{C_2}\frac{f(w)}{w-z}\mathrm{d}w + \frac{1}{2\pi i}\int_{BA}\frac{f(w)}{w-z}\mathrm{d}w$$

$$= \frac{1}{2\pi i}\int_{C_1}\frac{f(w)}{w-z}\mathrm{d}w - \frac{1}{2\pi i}\int_{C_2}\frac{f(w)}{w-z}\mathrm{d}w$$

$$\left[\because \text{ integrals around } AB \text{ and } BA \text{ cancel each other}\right]$$

$$= f_1(z) - f_2(z)$$

(i) $$f_1(z) = \frac{1}{2\pi i}\int_{C_1}\frac{f(w)}{w-z}\mathrm{d}w \qquad \text{...(4.2)}$$

z is any point in R and for this integral, w lies on C_1.

$$\therefore \quad |z-a| < |w-a|$$

$$\left|\frac{z-a}{w-a}\right| < 1$$

Consider $$\frac{1}{w-z} = \frac{1}{(w-a)-(z-a)}$$

$$= \frac{1}{(w-a)\left(1-\dfrac{z-a}{w-a}\right)}$$

$$= \frac{1}{w-a}\left(1-\frac{z-a}{w-a}\right)^{-1}$$

$$= \frac{1}{(w-a)}\left[1+\left(\frac{z-a}{w-a}\right)+\left(\frac{z-a}{w-a}\right)^2+\cdots+\left(\frac{z-a}{w-a}\right)^n+\cdots\right]$$

$$= \frac{1}{(w-a)}\sum_{n=0}^{\infty}\left(\frac{z-a}{w-a}\right)^n$$

$$= \sum_{n=0}^{\infty}\frac{(z-a)^n}{(w-a)^{n+1}}$$

Substituting $\dfrac{1}{w-z}$ in Eq. (4.2),

$$f_1(z) = \frac{1}{2\pi i}\int_{C_1} f(w)\left[\sum_{n=0}^{\infty}\frac{(z-a)^n}{(w-a)^{n+1}}\right]\mathrm{d}w$$

$$= \sum_{n=0}^{\infty}\left[(z-a)^n\left\{\frac{1}{2\pi i}\int_{C_1}\frac{f(w)}{(w-a)^{n+1}}\,\mathrm{d}w\right\}\right]$$

$$= \sum_{n=0}^{\infty}(z-a)^n a_n \qquad \cdots(4.3)$$

where $\quad a_n = \dfrac{1}{2\pi i}\displaystyle\int_{C_1}\frac{f(w)}{(w-a)^{n+1}}\,\mathrm{d}w$

(ii) $$f_2(z) = \frac{1}{2\pi i}\int_{C_2}\frac{f(w)}{w-z}\,\mathrm{d}w \qquad \cdots(4.4)$$

z is any point in R and for this integral, w lies on C_2.

$$\therefore \quad |z-a| > |w-a|$$

$$1 > \left|\frac{w-a}{z-a}\right|$$

$$\left|\frac{w-a}{z-a}\right| < 1$$

Consider $\quad \dfrac{1}{w-z} = \dfrac{1}{(w-a)-(z-a)}$

$$= \frac{1}{(z-a)\left(\dfrac{w-a}{z-a}-1\right)}$$

$$= -\frac{1}{(z-a)}\cdot\frac{1}{\left(1-\dfrac{w-a}{z-a}\right)}$$

$$= -\frac{1}{(z-a)}\left(1-\frac{w-a}{z-a}\right)^{-1}$$

$$= -\frac{1}{(z-a)}\left[1+\left(\frac{w-a}{z-a}\right)+\left(\frac{w-a}{z-a}\right)^2+\cdots+\left(\frac{w-a}{z-a}\right)^{n-1}+\cdots\right]$$

$$= -\frac{1}{(z-a)}\sum_{n=1}^{\infty}\left(\frac{w-a}{z-a}\right)^{n-1}$$

$$= \sum_{n=1}^{\infty} -\frac{(w-a)^{n-1}}{(z-a)^n}$$

Substituting $\frac{1}{w-z}$ in Eq. (4.4),

$$f_2(z) = \frac{1}{2\pi i}\int_{C_2} f(w)\left[\sum_{n=1}^{\infty} -\frac{(w-a)^{n-1}}{(z-a)^n}\right]dw$$

$$= \sum_{n=1}^{\infty}\left[\frac{-1}{(z-a)^n}\left\{\frac{1}{2\pi i}\int_{C_2}\frac{f(w)}{(w-a)^{-n+1}}dw\right\}\right]$$

$$= \sum_{n=1}^{\infty} -\frac{1}{(z-a)^n} b_n$$

where $$b_n = \frac{1}{2\pi i}\int_{C_2}\frac{f(w)}{(w-a)^{-n+1}}dw$$

$$\therefore f(z) = f_1(z) - f_2(z)$$

$$= \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty}\frac{b_n}{(z-a)^n}$$

where $$a_n = \frac{1}{2\pi i}\int_{C_1}\frac{f(w)}{(w-a)^{n+1}}dw$$

$$b_n = \frac{1}{2\pi i}\int_{C_2}\frac{f(w)}{(w-a)^{-n+1}}dw$$

The second term $\sum_{n=1}^{\infty}\frac{b_n}{(z-a)^n}$ is known as the principal part of Laurent's series.

Example 1

Find the Laurent's series of $f(z) = z^2 e^{\frac{1}{z}}$ *about* $z = 0$.

Solution

$$f(z) = z^2 e^{\frac{1}{z}}$$

$f(z)$ is analytic for $|z| > 0$.

$$f(z) = z^2\left(1 + \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{6z^3} + \cdots\right)$$

$$= z^2 + z + \frac{1}{2} + \frac{1}{6z} + \cdots$$

Example 2

Find the Laurent's series of $f(z)=\dfrac{e^{2z}}{(z-1)^3}$ *about* $z = 1$.

Solution

$$f(z)=\frac{e^{2z}}{(z-1)^3}$$

$f(z)$ is not analytic at $z = 1$. But it is analytic in the region $|z - 1| > 0$ about $z = 1$.
Let $z - 1 = t$ then $z = t + 1$.

$$f(z)=\frac{e^{2(t+1)}}{t^3}$$

$$=\frac{e^2}{t^3}e^{2t}$$

$$=\frac{e^2}{t^3}\left[1+2t+\frac{(2t)^2}{2!}+\frac{(2t)^3}{3!}+\cdots\right]$$

$$=e^2\left[\frac{1}{t^3}+\frac{2}{t^2}+\frac{2}{t}+\frac{4}{3}+\cdots\right]$$

$$=e^2\left[\frac{1}{(z-1)^3}+\frac{2}{(z-1)^2}+\frac{2}{(z-1)}+\frac{4}{3}+\cdots\right]$$

Example 3

Find Laurent's series expansion in powers of z that represent $f(z)=\dfrac{1}{z^2(1-z)}$ *for the following domains: (i)* $|z| < 1$ *(ii)* $|z| > 1$.

[Summer 2013]

Solution

$$f(z)=\frac{1}{z^2(1-z)}$$

$$=\frac{A}{z}+\frac{B}{z^2}+\frac{C}{1-z}$$

$$1=Az(1-z)+B(1-z)+Cz^2$$

Putting $z = 0$,

$$1=B$$

Putting $z = 1$,

$$1 = C$$

Putting $z = -1$,

$$\begin{aligned} 1 &= -2A + 2B + C \\ &= -2A + 2 + 1 \\ A &= 1 \end{aligned}$$

$$\therefore \quad f(z) = \frac{1}{z} + \frac{1}{z^2} + \frac{1}{1-z}$$

(i) $|z| < 1$

$f(z)$ is analytic in the region $|z| < 1$ about $z = 0$ (Fig. 4.4).

$$\begin{aligned} f(z) &= \frac{1}{z} + \frac{1}{z^2} + (1-z)^{-1} \\ &= \frac{1}{z} + \frac{1}{z^2} + (1 + z + z^2 + \cdots) \end{aligned}$$

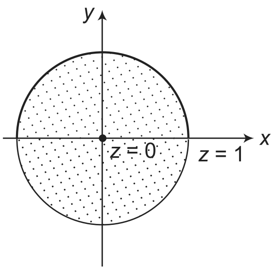

Fig. 4.4

(ii) $|z| > 1$

$f(z)$ is analytic in the region $|z| > 1$ about $z = 0$ (Fig. 4.5).

$$\because \quad |z| > 1, \qquad \left|\frac{1}{z}\right| < 1$$

$$\begin{aligned} f(z) &= \frac{1}{z} + \frac{1}{z^2} + \frac{1}{-z\left(-\frac{1}{z} + 1\right)} \\ &= \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z}\left(1 - \frac{1}{z}\right)^{-1} \\ &= \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4} + \cdots\right) \\ &= \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z} - \frac{1}{z^2} - \frac{1}{z^3} - \frac{1}{z^4} - \frac{1}{z^5} - \cdots \\ &= -\frac{1}{z^3} - \frac{1}{z^4} - \frac{1}{z^5} - \cdots \end{aligned}$$

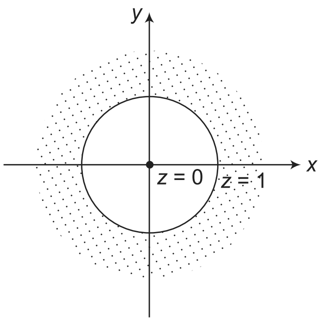

Fig. 4.5

Example 4

Expand $f(z) = \dfrac{1}{(z+1)(z-2)}$ *in Laurent's series in the following regions:*

(i) $|z| < 1$ *(ii)* $1 < |z| < 2$ *(iii)* $|z| > 2$. **[Summer 2015]**

Solution

$$f(z)=\frac{1}{(z+1)(z+2)}$$
$$=\frac{1}{3}\frac{(z+1)-(z-2)}{(z+1)(z-2)}$$
$$=\frac{1}{3}\left[\frac{1}{z-2}-\frac{1}{z+1}\right]$$

$f(z)$ is not analytic at $z = 2$ and $z = -1$.

(i) $|z| < 1$

$f(z)$ is analytic in the region $|z| < 1$ about $z = 0$ (Fig. 4.6).

$$\because \quad |z|<1<2, \qquad \left|\frac{z}{2}\right|<1$$

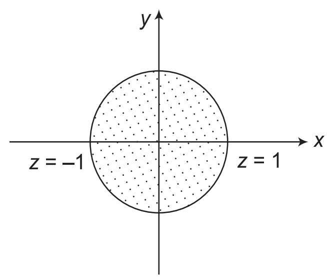

Fig. 4.6

$$f(z)=\frac{1}{3}\left[\frac{1}{-2\left(1-\frac{z}{2}\right)}-\frac{1}{z+1}\right]$$
$$=-\frac{1}{3}\left[\frac{1}{2}\left(1-\frac{z}{2}\right)^{-1}+(1+z)^{-1}\right]$$
$$=-\frac{1}{3}\left[\frac{1}{2}\left\{1+\frac{z}{2}+\left(\frac{z}{2}\right)^{2}+\cdots\right\}+(1-z+z^{2}-\cdots)\right]$$
$$=-\frac{1}{3}\left(\frac{3}{2}-\frac{3}{4}z+\frac{9}{8}z^{2}-\cdots\right)$$
$$=-\frac{1}{2}+\frac{1}{4}z-\frac{3}{8}z^{2}+\cdots$$

(ii) $1 < |z| < 2$

$f(z)$ is analytic in the annular region $1 < |z| < 2$ about $z = 0$ (Fig. 4.7).

$$1<|z|,\ \left|\frac{1}{z}\right|<1$$
$$|z|<2,\ \left|\frac{z}{2}\right|<1$$

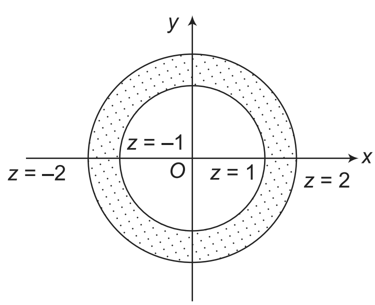

Fig. 4.7

$$f(z)=\frac{1}{3}\left[\frac{1}{-2\left(1-\frac{z}{2}\right)}-\frac{1}{z\left(1+\frac{1}{z}\right)}\right]$$
$$=\frac{1}{3}\left[-\frac{1}{2}\left(1-\frac{z}{2}\right)^{-1}-\frac{1}{z}\left(1+\frac{1}{z}\right)^{-1}\right]$$
$$=\frac{1}{3}\left[-\frac{1}{2}\left\{1+\left(\frac{z}{2}\right)+\left(\frac{z}{2}\right)^{2}+\cdots\right\}-\frac{1}{z}\left(1-\frac{1}{z}+\frac{1}{z^{2}}-\cdots\right)\right]$$

$$= -\frac{1}{3}\left[\left(\frac{1}{2}+\frac{z}{4}+\frac{z^2}{8}+\cdots\right)+\left(\frac{1}{z}-\frac{1}{z^2}+\frac{1}{z^3}-\cdots\right)\right]$$

(iii) $|z| > 2$

$f(z)$ is analytic in the region $|z| > 2$ about $z = 0$ (Fig. 4.8).

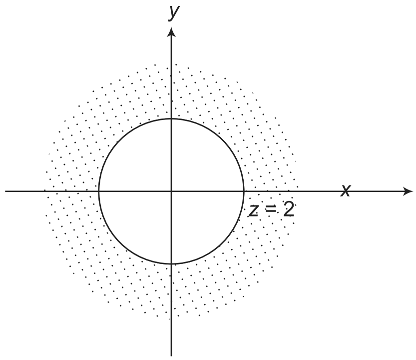

Fig. 4.8

$$\because \quad |z| > 2 > 1, \quad \left|\frac{2}{z}\right| < 1, \quad \left|\frac{1}{z}\right| < 1$$

$$f(z) = \frac{1}{3}\left[\frac{1}{z\left(1-\frac{2}{z}\right)} - \frac{1}{z\left(1+\frac{1}{z}\right)}\right]$$

$$= \frac{1}{3}\left[\frac{1}{z}\left(1-\frac{2}{z}\right)^{-1} - \frac{1}{z}\left(1+\frac{1}{z}\right)^{-1}\right]$$

$$= \frac{1}{3}\left[\frac{1}{z}\left\{1+\left(\frac{2}{z}\right)+\left(\frac{2}{z}\right)^2+\cdots\right\} - \frac{1}{z}\left(1-\frac{1}{z}+\frac{1}{z^2}-\cdots\right)\right]$$

$$= \frac{1}{3}\left[\left(\frac{1}{z}+\frac{2}{z^2}+\frac{4}{z^3}+\cdots\right)-\left(\frac{1}{z}-\frac{1}{z^2}+\frac{1}{z^3}-\cdots\right)\right]$$

$$= \frac{1}{3}\left(\frac{3}{z^2}+\frac{3}{z^3}+\cdots\right)$$

$$= \frac{1}{z^2}+\frac{1}{z^3}+\cdots$$

Example 5

Find all possible Laurent's expansion of $f(z) = \dfrac{4-3z}{z(1-z)(2-z)}$ *about* $z = 0$ *Indicate the region of convergence in each case.*

Solution

$$f(z) = \frac{4-3z}{z(1-z)(2-z)}$$

$$= \frac{A}{z} + \frac{B}{1-z} + \frac{C}{2-z}$$

$$4-3z = A(1-z)(2-z) + Bz(2-z) + Cz(1-z)$$

Putting $z = 0$,

$$4 = 2A$$

$$A = 2$$

Putting $z = 1$,

$$B = 1$$

Putting $z = 2$,

$$4 - 3(2) = -2C$$
$$C = 1$$

$$\therefore \quad f(z) = \frac{2}{z} + \frac{1}{1-z} + \frac{1}{2-z}$$

$f(z)$ is not analytic at $z = 0$, $z = 1$ and $z = 2$.

(i) $|z| < 1$

$f(z)$ is analytic in the region $|z| < 1$ about $z = 0$ (Fig. 4.9).

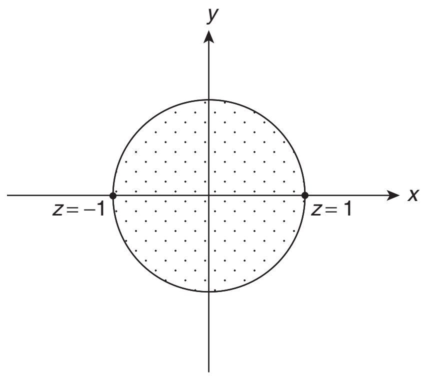

Fig. 4.9

$$\because \quad |z| < 1 < 2, \quad \left|\frac{z}{2}\right| < 1$$

$$f(z) = \frac{2}{z} + \frac{1}{1-z} + \frac{1}{2}\frac{1}{\left(1-\frac{z}{2}\right)}$$

$$= \frac{2}{z} + (1-z)^{-1} + \frac{1}{2}\left(1-\frac{z}{2}\right)^{-1}$$

$$= \frac{2}{z} + \left(1 + z + z^2 + \cdots\right) + \frac{1}{2}\left[1 + \frac{z}{2} + \left(\frac{z}{2}\right)^2 + \cdots\right]$$

$$= \left(\frac{3}{2} + \frac{5}{4}z + \frac{9}{8}z^2 + \cdots\right) + \frac{2}{z}$$

(ii) $1 < |z| < 2$

$f(z)$ is analytic in the annular region $1 < |z| < 2$ about $z = 0$ (Fig. 4.10).

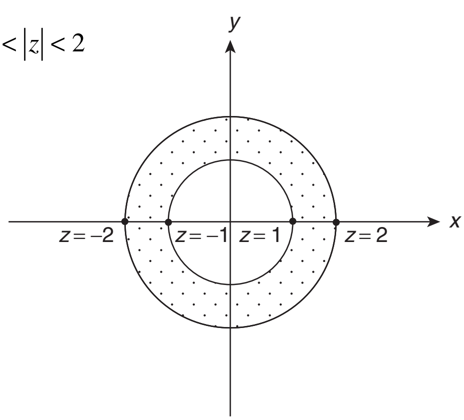

Fig. 4.10

$$1 < |z|, \quad \left|\frac{1}{z}\right| < 1$$

$$|z| < 2, \quad \left|\frac{z}{2}\right| < 1$$

$$f(z) = \frac{2}{z} - \frac{1}{z}\frac{1}{1-\frac{1}{z}} + \frac{1}{2}\frac{1}{1-\frac{z}{2}}$$

$$= \frac{2}{z} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1} + \frac{1}{2}\left(1-\frac{z}{2}\right)^{-1}$$

$$= \frac{2}{z} - \frac{1}{z}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \cdots\right) + \frac{1}{2}\left[1 + \left(\frac{z}{2}\right) + \left(\frac{z}{2}\right)^2 + \cdots\right]$$

$$= \frac{1}{2}\left(1 + \frac{z}{2} + \frac{z^2}{4} + \cdots\right) + \left(\frac{1}{z} - \frac{1}{z^2} - \frac{1}{z^3} - \cdots\right)$$

(iii) $|z| > 2$

$f(z)$ is analytic in the region $|z| > 2$ about $z = 0$ (Fig. 4.11).

$$\because \quad |z| > 2 > 1, \quad \left|\frac{2}{z}\right| < 1, \quad \left|\frac{1}{z}\right| < 1$$

$$f(z) = \frac{2}{z} - \frac{1}{z}\frac{1}{\left(1-\frac{1}{z}\right)} - \frac{1}{z}\frac{1}{\left(1-\frac{2}{z}\right)}$$

$$= \frac{2}{z} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1} - \frac{1}{z}\left(1-\frac{2}{z}\right)^{-1}$$

$$= \frac{2}{z} - \frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\cdots\right)$$

$$-\frac{1}{z}\left[1+\frac{2}{z}+\left(\frac{2}{z}\right)^2+\cdots\right]$$

$$= -\frac{3}{z^2} - \frac{5}{z^3} - \cdots$$

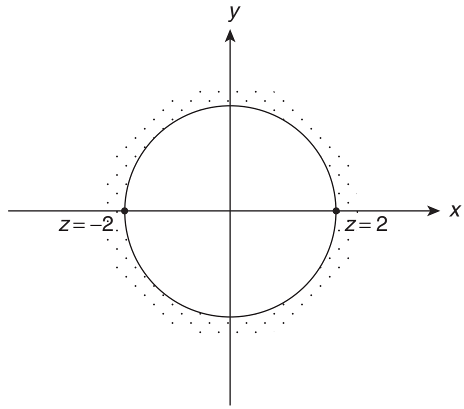

Fig. 4.11

Example 6

Find the Laurent's series expansion of

$$f(z) = \frac{4z+1}{z(z^2+z-2)} \text{ in } |z| > 2.$$

Solution

$$f(z) = \frac{4z+1}{z(z^2+z-2)}$$

$$= \frac{4z+1}{z(z-1)(z+2)}$$

$$= \frac{A}{z} + \frac{B}{z-1} + \frac{C}{z+2}$$

$$4z+1 = A(z+2)(z-1) + Bz(z+2) + Cz(z-1)$$

Putting $z = 0$,

$$1 = (2)(-1)A$$

$$A = -\frac{1}{2}$$

Putting $z = 1$,

$$5 = 3B$$

$$B = \frac{5}{3}$$

Putting $z = -2$,

$$-7 = (-2)(-3)C$$

$$C = -\frac{7}{6}$$

$$\therefore \quad f(z) = -\frac{1}{2}\frac{1}{z} + \frac{5}{3}\frac{1}{z-1} - \frac{7}{6}\frac{1}{z+2}$$

$f(z)$ is analytic in the region $|z| > 2$ about $z = 0$ (Fig. 4.12).

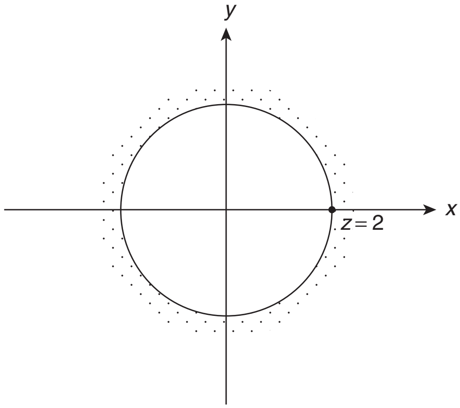

Fig. 4.12

$$\because \quad |z| > 2 > 1, \quad \left|\frac{2}{z}\right| < 1, \quad \left|\frac{1}{z}\right| < 1$$

$$f(z) = -\frac{1}{2z} + \frac{5}{3z\left(1-\frac{1}{z}\right)} - \frac{7}{6z\left(1+\frac{2}{z}\right)}$$

$$= -\frac{1}{2z} + \frac{5}{3z}\left(1-\frac{1}{z}\right)^{-1} - \frac{7}{6z}\left(1+\frac{2}{z}\right)^{-1}$$

$$= -\frac{1}{2z} + \frac{5}{3z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\cdots\right) - \frac{7}{6z}\left[1-\frac{2}{z}+\left(\frac{2}{z}\right)^2 - \cdots\right]$$

$$= -\frac{1}{2z} + \left(\frac{5}{3z}+\frac{5}{3z^2}+\frac{5}{3z^3}+\cdots\right) - \left(\frac{7}{6z}-\frac{7}{3z^2}+\frac{14}{3z^3}-\cdots\right)$$

$$= \frac{4}{z^2} - \frac{3}{z^3} + \cdots$$

Example 7

Evaluate $f(z) = \dfrac{1}{(z+1)(z+3)}$ *in the Laurent series valid in the regions* $|z| > 3$ and $1 < |z| < 3$.

Solution

$$f(z) = \frac{1}{(z+1)(z+3)}$$

$$= \frac{A}{z+1} + \frac{B}{z+3}$$

$$1 = A(z+3) + B(z+1)$$

Putting $z = -1$,

$$1 = A(-1+3)$$

$$A = \frac{1}{2}$$

Putting $z = -3$,

$$1 = B(-3+1)$$

$$B = -\frac{1}{2}$$

$$\therefore f(z) = \frac{1}{2}\frac{1}{z+1} - \frac{1}{2}\frac{1}{z+3}$$

$f(z)$ is not analytic at $z = -1$ and $z = -3$.

(i) $|z| > 3$

$f(z)$ is analytic in the region $|z| > 3$ about $z = 0$ (Fig. 4.13).

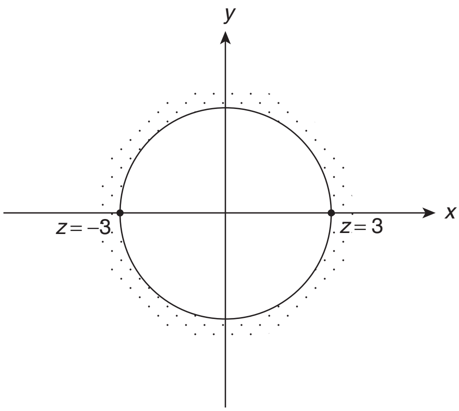

Fig. 4.13

$$\because \quad |z| > 3 > 1, \left|\frac{3}{z}\right| < 1, \left|\frac{1}{z}\right| < 1$$

$$f(z) = \frac{1}{2}\frac{1}{z\left(1+\frac{1}{z}\right)} - \frac{1}{2}\frac{1}{z\left(1+\frac{3}{z}\right)}$$

$$= \frac{1}{2z}\left(1+\frac{1}{z}\right)^{-1} - \frac{1}{2z}\left(1+\frac{3}{z}\right)^{-1}$$

$$= \frac{1}{2z}\left[1-\frac{1}{z}+\left(\frac{1}{z}\right)^2 - \cdots\right] - \frac{1}{2z}\left[1-\frac{3}{z}+\left(\frac{3}{z}\right)^2 - \cdots\right]$$

$$= \left(\frac{1}{2z}-\frac{1}{2z^2}+\frac{1}{2z^3}-\cdots\right) - \left(\frac{1}{2z}-\frac{3}{2z^2}+\frac{9}{2z^3}-\cdots\right)$$

$$= \frac{1}{z^2}-\frac{4}{z^3}-\cdots$$

(ii) $1 < |z| < 3$

$f(z)$ is analytic in the annular region $1 < |z| < 3$ about $z = 0$ (Fig. 4.14).

$$1 < |z|, \ \left|\frac{1}{z}\right| < 1$$

$$|z| < 3, \ \left|\frac{z}{3}\right| < 1$$

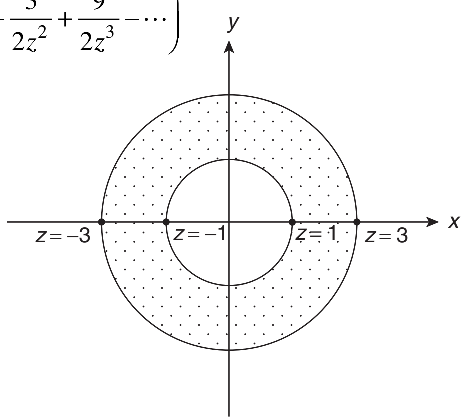

Fig. 4.14

$$f(z)=\frac{1}{2}\frac{1}{z\left(1+\frac{1}{z}\right)}-\frac{1}{2(3)}\frac{1}{\left(1+\frac{z}{3}\right)}$$

$$=\frac{1}{2z}\left(1+\frac{1}{z}\right)^{-1}-\frac{1}{6}\left(1+\frac{z}{3}\right)^{-1}$$

$$=\frac{1}{2z}\left[1-\frac{1}{z}+\left(\frac{1}{z}\right)^2-\cdots\right]-\frac{1}{6}\left[1-\frac{z}{3}+\left(\frac{z}{3}\right)^2-\cdots\right]$$

$$=\left(\frac{1}{2z}-\frac{1}{2z^2}+\frac{1}{2z^3}-\cdots\right)-\left(\frac{1}{6}-\frac{z}{18}+\frac{z^2}{54}-\cdots\right)$$

$$=\left(-\frac{1}{6}+\frac{z}{18}-\frac{z^2}{54}+\cdots\right)+\left(\frac{1}{2z}-\frac{1}{2z^2}+\frac{1}{2z^3}-\cdots\right)$$

Example 8

Expand $f(z)=\frac{1}{(z+2)(z+4)}$ *valid for the following regions:*

(i) $|z|<2$ *(ii)* $2<|z|<4$ *(iii)* $|z|>4$.

Solution

$$f(z)=\frac{1}{(z+2)(z+4)}$$

$$=\frac{(z+4)-(z+2)}{2(z+2)(z+4)}$$

$$=\frac{1}{2}\left[\frac{1}{z+2}-\frac{1}{z+4}\right]$$

(i) $|z|<2$

$f(z)$ is analytic in the region $|z|<2$ about $z=0$ (Fig. 4.15).

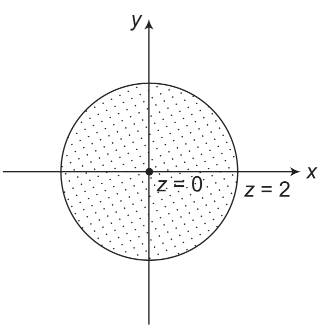

Fig. 4.15

$$\because\quad |z|<2<4,\quad \left|\frac{z}{2}\right|<1,\quad \left|\frac{z}{4}\right|<1$$

$$f(z)=\frac{1}{2}\frac{1}{2\left(\frac{z}{2}+1\right)}-\frac{1}{2}\frac{1}{4\left(\frac{z}{4}+1\right)}$$

$$=\frac{1}{4}\left(1+\frac{z}{2}\right)^{-1}-\frac{1}{8}\left(1+\frac{z}{4}\right)^{-1}$$

$$= \frac{1}{4}\left[1 - \frac{z}{2} + \left(\frac{z}{2}\right)^2 - \cdots\right] - \frac{1}{8}\left[1 - \frac{z}{4} + \left(\frac{z}{4}\right)^2 - \cdots\right]$$

$$= \left(\frac{1}{4} - \frac{z}{8} + \frac{z^2}{16} - \cdots\right) - \left(\frac{1}{8} - \frac{z}{32} + \frac{z^2}{128} - \cdots\right)$$

$$= \frac{1}{8} - \frac{3z}{32} + \frac{7z^2}{128} - \cdots$$

(ii) $2 < |z| < 4$

$f(z)$ is analytic in the annular region $2 < |z| < 4$ about $z = 0$ (Fig. 4.16).

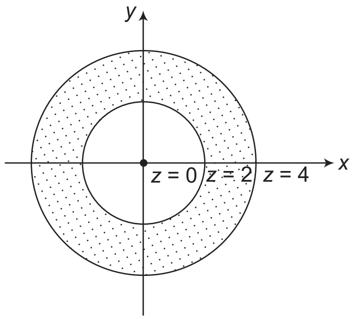

Fig. 4.16

$$2 < |z|, \qquad \left|\frac{2}{z}\right| < 1$$

$$|z| < 4, \qquad \left|\frac{z}{4}\right| < 1$$

$$f(z) = \frac{1}{2}\frac{1}{z\left(1 + \frac{2}{z}\right)} - \frac{1}{2}\frac{1}{4\left(\frac{z}{4} + 1\right)}$$

$$= \frac{1}{2z}\left(1 + \frac{2}{z}\right)^{-1} - \frac{1}{8}\left(1 + \frac{z}{4}\right)^{-1}$$

$$= \frac{1}{2z}\left[1 - \frac{2}{z} + \left(\frac{2}{z}\right)^2 - \cdots\right] - \frac{1}{8}\left[1 - \frac{z}{4} + \left(\frac{z}{4}\right)^2 - \cdots\right]$$

$$= \left(\frac{1}{2z} - \frac{1}{z^2} + \frac{2}{z^3} - \cdots\right) - \frac{1}{8}\left(1 - \frac{z}{4} + \frac{z^2}{16} - \cdots\right)$$

(iii) $|z| > 4$

$f(z)$ is analytic in the region $|z| > 4$ about $z = 0$ (Fig. 4.17).

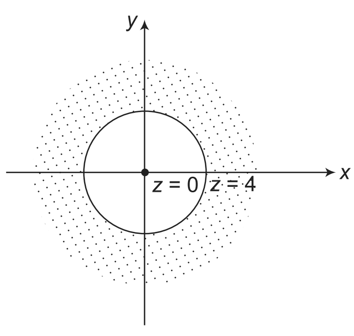

Fig. 4.17

$$\because \quad |z| > 4 > 2, \quad \left|\frac{4}{z}\right| < 1, \quad \left|\frac{2}{z}\right| < 1$$

$$f(z) = \frac{1}{2}\frac{1}{z\left(1 + \frac{2}{z}\right)} - \frac{1}{2}\frac{1}{z\left(1 + \frac{4}{z}\right)}$$

$$= \frac{1}{2z}\left(1 + \frac{2}{z}\right)^{-1} - \frac{1}{2z}\left(1 + \frac{4}{z}\right)^{-1}$$

$$= \frac{1}{2z}\left[1 - \frac{2}{z} + \left(\frac{2}{z}\right)^2 - \cdots\right] - \frac{1}{2z}\left[1 - \frac{4}{z} + \left(\frac{4}{z}\right)^2 - \cdots\right]$$

$$= \left(\frac{1}{2z} - \frac{1}{z^2} + \frac{2}{z^3} - \cdots\right) - \left(\frac{1}{2z} - \frac{2}{z^2} + \frac{8}{z^3} - \cdots\right)$$

$$= \frac{1}{z^2} - \frac{6}{z^3} + \cdots$$

Example 9

Expand $f(z) = \dfrac{z^2 - 1}{(z+2)(z+3)}$ *as a Laurent's series if* $(i)\ 2 < |z| < 3$, *and* $(ii)\ |z| > 3$.

Solution

Let $$f(z) = \frac{z^2 - 1}{(z+2)(z+3)} = \frac{z^2 - 1}{z^2 + 5z + 6}$$

Since the degrees of the numerator and denominator are same, partial fraction cannot be applied. Dividing to reduce the degree of the numerator,

$$f(z) = 1 + \frac{-5z - 7}{(z+2)(z+3)}$$

Let $$\frac{-5z - 7}{(z+2)(z+3)} = \frac{A}{z+2} + \frac{B}{z+3}$$

$$-5z - 7 = A(z+3) + B(z+2)$$

Putting $z = -2$,

$$-5(-2) - 7 = A(-2+3)$$

$$A = 3$$

Putting $z = -3$,

$$-5(-3) - 7 = B(-3+2)$$

$$B = -8$$

$$\therefore f(z) = 1 + \frac{3}{z+2} - \frac{8}{z+3}$$

$f(z)$ is not analytic at $z = -2$ and $z = -3$.

(i) $2 < |z| < 3$

$f(z)$ is analytic in the annular region $2 < |z| < 3$ about $z = 0$ (Fig. 4.18).

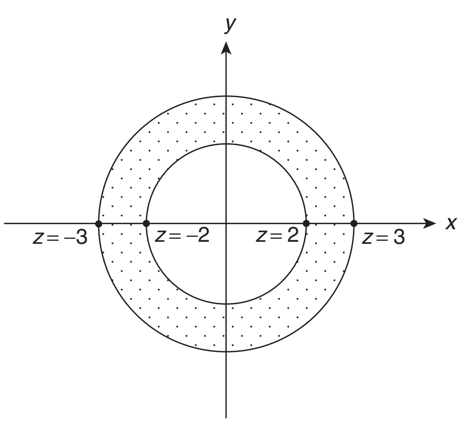

Fig. 4.18

$$2 < |z|,\ \left|\frac{2}{z}\right| < 1$$

$$|z| < 3,\ \left|\frac{z}{3}\right| < 1$$

$$f(z)=1+\frac{3}{z\left(1+\frac{2}{z}\right)}-\frac{8}{3\left(1+\frac{z}{3}\right)}$$

$$=1+\frac{3}{z}\left(1+\frac{2}{z}\right)^{-1}-\frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$=1+\frac{3}{z}\left[1-\frac{2}{z}+\left(\frac{2}{z}\right)^2-\left(\frac{2}{z}\right)^3+\cdots\right]-\frac{8}{3}\left[1-\frac{z}{3}+\left(\frac{z}{3}\right)^2-\left(\frac{z}{3}\right)^3+\cdots\right]$$

$$=1+3\left(\frac{1}{z}-\frac{2}{z^2}+\frac{4}{z^3}-\frac{8}{z^4}+\cdots\right)-\frac{8}{3}\left(1-\frac{z}{3}+\frac{z^2}{9}-\frac{z^3}{27}+\cdots\right)$$

(ii) $|z|>3$

$f(z)$ is analytic in the region $|z|>3$ about $z=0$ (Fig. 4.19).

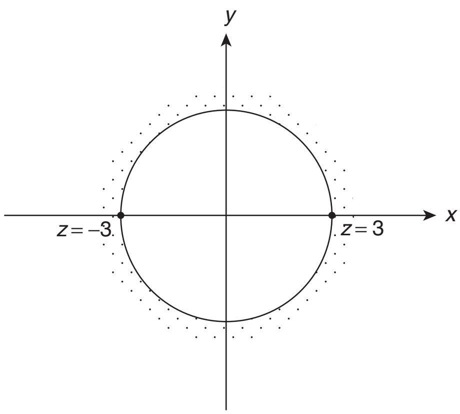

Fig. 4.19

$$\because\quad |z|>3>1,\quad \left|\frac{3}{z}\right|<1,\quad \left|\frac{1}{z}\right|<1$$

$$f(z)=1+\frac{3}{z\left(1+\frac{2}{z}\right)}-\frac{8}{z\left(1+\frac{3}{z}\right)}$$

$$=1+\frac{3}{z}\left(1+\frac{2}{z}\right)^{-1}-\frac{8}{z}\left(1+\frac{3}{z}\right)^{-1}$$

$$=1+\frac{3}{z}\left[1-\frac{2}{z}+\left(\frac{2}{z}\right)^2-\left(\frac{2}{z}\right)^3+\cdots\right]-\frac{8}{z}\left[1-\frac{3}{z}+\left(\frac{3}{z}\right)^2-\left(\frac{3}{z}\right)^3+\cdots\right]$$

$$=1+3\left(\frac{1}{z}-\frac{2}{z^2}+\frac{4}{z^3}-\frac{8}{z^4}+\cdots\right)-8\left(\frac{1}{z}-\frac{3}{z^2}+\frac{9}{z^3}-\frac{27}{z^4}+\cdots\right)$$

$$=1-\frac{5}{z}+\frac{18}{z^2}-\frac{60}{z^3}+\frac{192}{z^4}-\cdots$$

Example 10

Find the Laurent's series of $f(z)=\dfrac{z}{(z^2+1)(z^2+4)}$ in $1<|z|<2$.

Solution

$$f(z) = \frac{z}{(z^2+1)(z^2+4)}$$

$$= \frac{Az+B}{z^2+1} + \frac{Cz+D}{z^2+4}$$

$$z = (Az+B)(z^2+4) + (Cz+D)(z^2+1)$$

$$= Az^3 + Bz^2 + 4Az + 4B + Cz^3 + Dz^2 + Cz + D$$

$$= (A+C)z^3 + (B+D)z^2 + (4A+C)z + 4B + D$$

Equating coefficients of z^3,

$$0 = A + C$$

$$A = -C$$

Equating coefficients of z^2,

$$0 = B + D$$

$$B = -D$$

Equating coefficients of z,

$$1 = 4A + C$$

$$1 = 4A - A$$

$$3A = 1$$

$$A = \frac{1}{3}$$

$$C = -\frac{1}{3}$$

Equating constant term,

$$0 = 4B + D$$

$$0 = -4D + D$$

$$D = 0$$

$$B = 0$$

$$\therefore\ f(z) = \frac{1}{3}\frac{z}{z^2+1} - \frac{1}{3}\frac{z}{z^2+4}$$

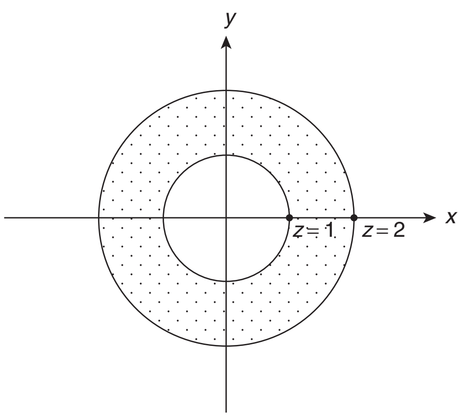

Fig. 4.20

$f(z)$ is analytic in the annular region $1 < |z| < 2$ about $z = 0$ (Fig. 4.20).

$$1 < |z|,\quad \left|\frac{1}{z}\right| < 1,\quad \left|\frac{1}{z^2}\right| < 1$$

$$|z| < 2,\quad |z^2| < 4,\quad \left|\frac{z^2}{4}\right| < 1 \qquad \left[\because\ |z|^2 = |z^2|\right]$$

$$f(z) = \frac{z}{3}\frac{1}{z^2\left(1+\frac{1}{z^2}\right)} - \frac{z}{3}\frac{1}{4\left(1+\frac{z^2}{4}\right)}$$

$$= \frac{1}{3z}\left(1+\frac{1}{z^2}\right)^{-1} - \frac{z}{12}\left(1+\frac{z^2}{4}\right)^{-1}$$

$$= \frac{1}{3z}\left(1-\frac{1}{z^2}+\frac{1}{z^4}-\cdots\right) - \frac{z}{12}\left[1-\frac{z^2}{4}+\left(\frac{z^2}{4}\right)^2-\cdots\right]$$

$$= \left(\frac{1}{3z}-\frac{1}{3z^3}+\frac{1}{3z^5}-\cdots\right) - \left(\frac{z}{12}-\frac{z^3}{48}+\frac{z^5}{192}-\cdots\right)$$

$$= -\left(\frac{z}{12}-\frac{z^3}{48}+\frac{z^5}{192}-\cdots\right) + \left(\frac{1}{3z}-\frac{1}{3z^3}+\frac{1}{3z^5}-\cdots\right)$$

Example 11

Find the series of $f(z) = \frac{z}{(z-1)(z-4)}$ *in terms of* $(z+3)$ *valid for* $|z+3| < 4$.

Solution

$$f(z) = \frac{z}{(z-1)(z-4)}$$

$$= \frac{A}{z-1} + \frac{B}{z-4}$$

$$z = A(z-4) + B(z-1)$$

Putting $z = 1$,

$$1 = -3A$$

$$A = -\frac{1}{3}$$

Putting $z = 4$,

$$4 = 3B$$

$$B = \frac{4}{3}$$

$$\therefore \quad f(z) = -\frac{1}{3}\frac{1}{z-1} + \frac{4}{3}\frac{1}{z-4}$$

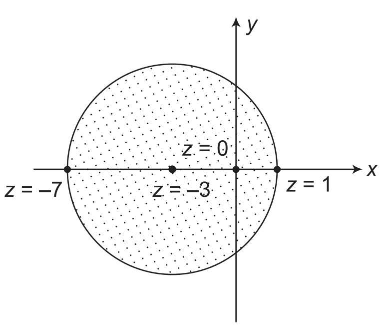

Fig. 4.21

$f(z)$ is analytic in the region $|z+3| < 4$ about $z = -3$ (Fig. 4.21).

Let $z + 3 = t$ then $z = t - 3$

$$f(z) = -\frac{1}{3}\frac{1}{t-4} + \frac{4}{3}\frac{1}{t-7}$$

$$\because \quad |z+3| < 4, \text{ i.e., } |t| < 4 < 7, \quad \left|\frac{t}{4}\right| < 1, \quad \left|\frac{t}{7}\right| < 1$$

$$f(z) = -\frac{1}{3}\left[\frac{1}{-4\left(1-\frac{t}{4}\right)}\right] + \frac{4}{3}\left[\frac{1}{-7\left(1-\frac{t}{7}\right)}\right]$$

$$= \frac{1}{12}\left(1-\frac{t}{4}\right)^{-1} - \frac{4}{21}\left(1-\frac{t}{7}\right)^{-1}$$

$$= \frac{1}{12}\left[1+\frac{t}{4}+\left(\frac{t}{4}\right)^2+\cdots\right] - \frac{4}{21}\left[1+\frac{t}{7}+\left(\frac{t}{7}\right)^2+\cdots\right]$$

$$= \left(\frac{1}{12}+\frac{t}{48}+\frac{t^2}{192}+\cdots\right) - \left(\frac{4}{21}+\frac{4t}{147}+\frac{4t^2}{1029}+\cdots\right)$$

$$= -\frac{3}{28} - \frac{5t}{784} + \frac{29t^2}{21952} + \cdots$$

$$= -\frac{3}{28} - \frac{5}{784}(z+3) + \frac{29}{21952}(z+3)^2 + \cdots$$

Example 12

Find Laurent's series expansion in powers of z that represent $f(z) = \dfrac{1}{z(z-1)}$ *for the following domains:* (*i*) $0 < |z| < 1$ (*ii*) $0 < |z-1| < 1$.

[Winter 2013]

Solution

$$f(z) = \frac{1}{z(z-1)}$$

$$= \frac{z-(z-1)}{z(z-1)}$$

$$= \frac{1}{z-1} - \frac{1}{z}$$

$f(z)$ is not analytic at $z = 1$ and $z = 0$.

(i) $0 < |z| < 1$

$f(z)$ is analytic in the region $0 < |z| < 1$ (Fig. 4.22).

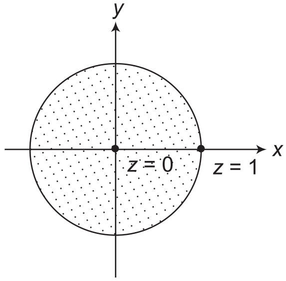

Fig. 4.22

$$f(z) = -\frac{1}{(1-z)} - \frac{1}{z}$$

$$= -(1-z)^{-1} - \frac{1}{z}$$

$$= -(1+z+z^2+z^3+\cdots) - \frac{1}{z}$$

(ii) $0 < |z - 1| < 1$

$f(z)$ is analytic in the region $0 < |z - 1| < 1$ (Fig. 4.23).

Let $z - 1 = t$ then $z = t + 1$

$0 < |z - 1| < 1$, $0 < |t| < 1$

$$f(z) = \frac{1}{z-1} - \frac{1}{z}$$

$$= \frac{1}{t} - \frac{1}{1+t}$$

$$= \frac{1}{t} - (1+t)^{-1}$$

$$= \frac{1}{t} - (1 - t + t^2 - t^3 + \cdots)$$

$$= \frac{1}{z-1} - \left[1 - (z-1) + (z-1)^2 - (z-1)^3 + \cdots\right]$$

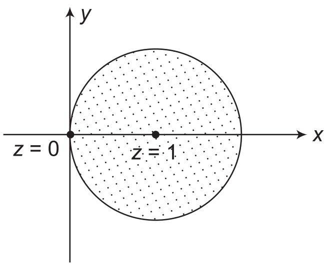

Fig. 4.23

Example 13

Find the Laurent's series expansion of $f(z) = \dfrac{7z-2}{z(z-2)(z+1)}$ *in* $1 < |z+1| < 3$.

Solution

$$f(z) = \frac{7z-2}{z(z-2)(z+1)}$$

By partial fraction expansion,

$$f(z) = \frac{A}{z} + \frac{B}{z-2} + \frac{C}{z+1}$$

$$7z - 2 = A(z-2)(z+1) + Bz(z+1) + Cz(z-2)$$

Putting $z = 0$,

$$-2 = A(-2)(1)$$

$$A = 1$$

Putting $z = 2$,

$$14 - 2 = B(2)(2+1)$$

$$B = 2$$

Purring $z = -1$,

$$-7 - 2 = C(-1)(-1-2)$$

$$C = -3$$

$$\therefore f(z) = \frac{1}{z} + \frac{2}{z-2} - \frac{3}{z+1}$$

$f(z)$ is not analytic at $z = 0$, $z = 2$ and $z = -1$.
$f(z)$ is analytic in the annular region $1 < |z + 1| < 3$ about $z = -1$ (Fig. 4.24).
Let $z + 1 = t$ then $z = t - 1$.

$$1 < |z+1| < 3, \text{ i.e., } 1 < |t| < 3$$

$$1 < |t|, \quad \left|\frac{1}{t}\right| < 1$$

$$|t| < 3, \quad \left|\frac{t}{3}\right| < 1$$

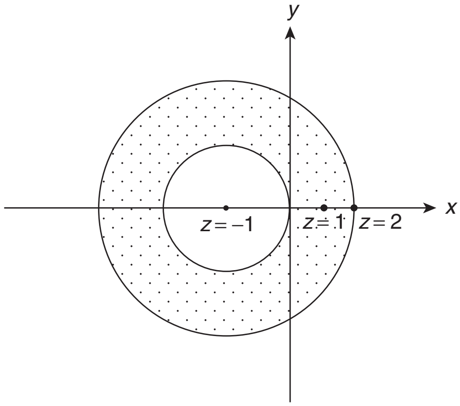

Fig. 4.24

$$\begin{aligned}
f(z) &= \frac{1}{t-1} + \frac{2}{t-3} - \frac{3}{t} \\
&= \frac{1}{t\left(1-\frac{1}{t}\right)} + \frac{2}{-3\left(1-\frac{t}{3}\right)} - \frac{3}{t} \\
&= \frac{1}{t}\left(1-\frac{1}{t}\right)^{-1} - \frac{2}{3}\left(1-\frac{t}{3}\right)^{-1} - \frac{3}{t} \\
&= \frac{1}{t}\left(1+\frac{1}{t}+\frac{1}{t^2}+\cdots\right) - \frac{2}{3}\left[1+\frac{t}{3}+\left(\frac{t}{3}\right)^2+\cdots\right] - \frac{3}{t} \\
&= \left(-\frac{2}{t}+\frac{1}{t^2}+\frac{1}{t^3}+\cdots\right) - \frac{2}{3}\left[1+\frac{t}{3}+\frac{t^2}{9}+\cdots\right] \\
&= \left(-\frac{2}{3}-\frac{2}{9}t-\frac{2}{27}t^2-\cdots\right) + \left[-\frac{2}{t}+\frac{1}{t^2}+\frac{1}{t^3}+\cdots\right] \\
&= \left[-\frac{2}{3}-\frac{2}{9}(z+1)-\frac{2}{27}(z+1)^2-\cdots\right] + \left[-\frac{2}{z+1}+\frac{1}{(z+1)^2}+\frac{1}{(z+1)^3}+\cdots\right]
\end{aligned}$$

Example 14

Find the Laurent's expansion of $f(z) = \dfrac{1}{z - z^3}$ *in the region* $1 < |z - 1| < 2$.

Solution

$$\begin{aligned}
f(z) &= \frac{1}{z - z^3} \\
&= \frac{1}{z(1 - z^2)}
\end{aligned}$$

$$= \frac{1}{z(1-z)(1+z)}$$
$$= \frac{A}{z} + \frac{B}{1-z} + \frac{C}{1+z}$$
$$1 = A(1-z)(1+z) + Bz(1+z) + Cz(1-z)$$

Putting $z = 0$,
$$A = 1$$
Putting $z = 1$,
$$1 = 2B$$
$$B = \frac{1}{2}$$
Putting $z = -1$,
$$1 = -2C$$
$$C = -\frac{1}{2}$$
$$\therefore \quad f(z) = \frac{1}{z} + \frac{1}{2}\frac{1}{1-z} - \frac{1}{2}\frac{1}{1+z}$$
$$= \frac{1}{z} - \frac{1}{2}\frac{1}{z-1} - \frac{1}{2}\frac{1}{z+1}$$

$f(z)$ is not analytic at $z = 0$, $z = 1$ and $z = -1$.

(i) $1 < |z-1| < 2$

$f(z)$ is analytic in the annular region $1 < |z-1| < 2$ about $z = 1$ (Fig. 4.25).

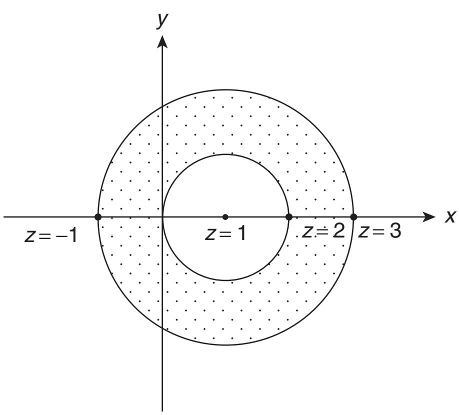

Fig. 4.25

Let $z - 1 = t$ then $z = t + 1$.
$$1 < |t| < 2$$
$$1 < |t|, \quad \left|\frac{1}{t}\right| < 1$$
$$|t| < 2, \quad \left|\frac{t}{2}\right| < 1$$
$$f(z) = \frac{1}{t+1} - \frac{1}{2}\frac{1}{t} - \frac{1}{2}\frac{1}{t+2}$$
$$= \frac{1}{t}\frac{1}{\left(1+\frac{1}{t}\right)} - \frac{1}{2}\frac{1}{t} - \frac{1}{4}\frac{1}{\left(1+\frac{t}{2}\right)}$$
$$= \frac{1}{t}\left(1+\frac{1}{t}\right)^{-1} - \frac{1}{2t} - \frac{1}{4}\left(1+\frac{t}{2}\right)^{-1}$$

$$= \frac{1}{t}\left(1 - \frac{1}{t} + \frac{1}{t^2} - \cdots\right) - \frac{1}{2t} - \frac{1}{4}\left[1 - \frac{t}{2} + \left(\frac{t}{2}\right)^2 - \cdots\right]$$

$$= \left(\frac{1}{t} - \frac{1}{t^2} + \frac{1}{t^3} - \cdots\right) - \frac{1}{2t} + \left(-\frac{1}{4} + \frac{t}{8} - \frac{t^2}{16} + \cdots\right)$$

$$= \left(-\frac{1}{4} + \frac{t}{8} - \frac{t^2}{16} + \cdots\right) + \left(\frac{1}{2t} - \frac{1}{t^2} + \frac{1}{t^3} - \cdots\right)$$

$$= \left[-\frac{1}{4} + \frac{z-1}{8} - \frac{(z-1)^2}{16} + \cdots\right] + \left[\frac{1}{2(z-1)} - \frac{1}{(z-1)^2} + \frac{1}{(z-1)^3} - \cdots\right]$$

Example 15

Find the Laurent's series expansion of $f(z) = \dfrac{z+4}{(z+3)(z-1)^2}$ *in the region* $|z-1| > 4$.

Solution

$$f(z) = \frac{z+4}{(z+3)(z-1)^2}$$

$$= \frac{A}{z+3} + \frac{B}{z-1} + \frac{C}{(z-1)^2}$$

$$z + 4 = A(z-1)^2 + B(z+3)(z-1) + C(z+3)$$

Putting $z = -3$,

$$1 = 16A$$

$$A = \frac{1}{16}$$

Putting $z = 1$,

$$5 = 4C$$

$$C = \frac{5}{4}$$

Putting $z = 0$,

$$4 = A - 3B + 3C$$

$$4 = \frac{1}{16} - 3B + \frac{15}{4}$$

$$B = -\frac{1}{16}$$

$$\therefore f(z) = \frac{1}{16}\frac{1}{z+3} - \frac{1}{16}\frac{1}{z-1} + \frac{5}{4}\frac{1}{(z-1)^2}$$

$f(z)$ is not analytic at $z = 1$ and $z = -3$.

$f(z)$ is analytic in the region $|z-1|>4$ about $z=1$ (Fig. 4.26).
Let $z-1=t$ then $z=t+1$.

$$|z-1|>4, \text{ i.e., } |t|>4, \qquad \left|\frac{1}{t}\right|<\frac{1}{4}<1$$

$$f(z)=\frac{1}{16}\frac{1}{t+4}-\frac{1}{16}\frac{1}{t}+\frac{5}{4}\frac{1}{t^2}$$

$$=\frac{1}{16}\frac{1}{t\left(1+\frac{4}{t}\right)}-\frac{1}{16}\frac{1}{t}+\frac{5}{4}\frac{1}{t^2}$$

$$=\frac{1}{16t}\left(1+\frac{4}{t}\right)^{-1}-\frac{1}{16t}+\frac{5}{4t^2}$$

$$=\frac{1}{16t}\left[1-\frac{4}{t}+\left(\frac{4}{t}\right)^2-\cdots\right]-\frac{1}{16t}+\frac{5}{4t^2}$$

$$=\frac{1}{16t}\left(-\frac{4}{t}+\frac{16}{t^2}-\cdots\right)+\frac{5}{4t^2}$$

$$=\frac{1}{16(z-1)}\left[-\frac{4}{(z-1)}+\frac{16}{(z-1)^2}-\cdots\right]+\frac{5}{4(z-1)^2}$$

$$=\frac{1}{(z-1)^2}+\frac{1}{(z-1)^3}-\cdots$$

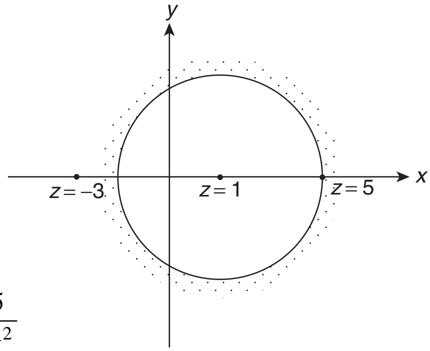

Fig. 4.26

Example 16

Find the Laurent's series of $f(z)=\dfrac{1}{z(1-z)}$ *valid in the following region*

(i) $|z+1|<1$ *(ii)* $|z+1|>2$ *(iii)* $1<|z+1|<2$.

Solution

$$f(z)=\frac{1}{z(1-z)}$$

$$=\frac{1-z+z}{z(1-z)}$$

$$=\frac{1}{z}+\frac{1}{1-z}$$

$f(z)$ is not analytic at $z=0$ and $z=1$.

(i) $|z+1|<1$

$f(z)$ is analytic in the region $|z+1|<1$ about $z=-1$ (Fig. 4.27).

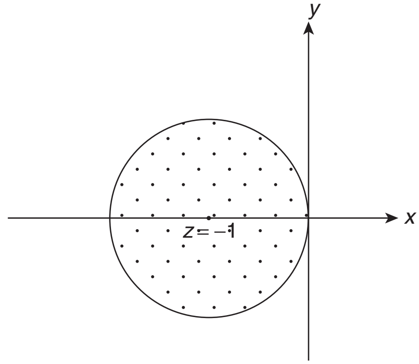

Fig. 4.27

Let $z + 1 = t$ then $z = t - 1$.

$$|z+1| < 1, \text{ i.e., } |t| < 1 < 2,$$

$$\left|\frac{t}{2}\right| < 1$$

$$f(z) = \frac{1}{t-1} + \frac{1}{1-(t-1)}$$

$$= \frac{1}{t-1} + \frac{1}{2-t}$$

$$= -\frac{1}{1-t} + \frac{1}{2\left(1-\frac{t}{2}\right)}$$

$$= -(1-t)^{-1} + \frac{1}{2}\left(1-\frac{t}{2}\right)^{-1}$$

$$= -(1+t+t^2+\cdots) + \frac{1}{2}\left[1+\frac{t}{2}+\left(\frac{t}{2}\right)^2+\cdots\right]$$

$$= -\frac{1}{2}\left(1+\frac{3}{2}t+\frac{7}{4}t^2+\cdots\right)$$

$$= -\frac{1}{2}\left[1+\frac{3}{2}(z+1)+\frac{7}{4}(z+1)^2+\cdots\right]$$

(ii) $|z + 1| > 2$

$f(z)$ is analytic in the region $|z + 1| > 2$ about $z = -1$ (Fig. 4.28).

Let $z+1=t$ then $z=t-1$.

$$|z+1| > 2, \text{ i.e., } |t| > 2, \quad \left|\frac{1}{t}\right| < \frac{1}{2} < 1$$

$$f(z) = \frac{1}{t-1} + \frac{1}{2-t}$$

$$= \frac{1}{t\left(1-\frac{1}{t}\right)} - \frac{1}{t\left(1-\frac{2}{t}\right)}$$

$$= \frac{1}{t}\left(1-\frac{1}{t}\right)^{-1} - \frac{1}{t}\left(1-\frac{2}{t}\right)^{-1}$$

$$= \frac{1}{t}\left(1+\frac{1}{t}+\frac{1}{t^2}+\cdots\right)$$

$$-\frac{1}{t}\left[1+\frac{2}{t}+\left(\frac{2}{t}\right)^2+\cdots\right]$$

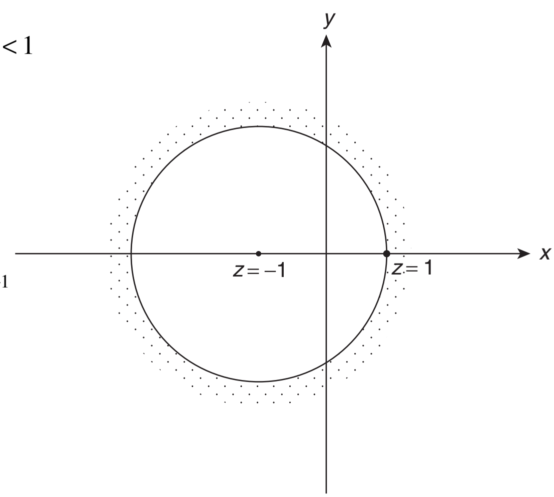

Fig. 4.28

$$= \frac{1}{t} + \frac{1}{t^2} + \frac{1}{t^3} + \cdots - \left(\frac{1}{t} + \frac{2}{t^2} + \frac{4}{t^3} + \cdots \right)$$

$$= -\frac{1}{t^2} - \frac{3}{t^3} - \cdots$$

$$= -\frac{1}{(z+1)^2} - \frac{3}{(z+1)^3} - \cdots$$

(iii) $1 < |z+1| < 2$

$f(z)$ is analytic in the annular region $1 < |z+1| < 2$ about $z = -1$ (Fig. 4.29).

Let $z+1 = t$ then $z = t-1$.

$$1 < |z+1| < 2 \text{ i.e.,} 1 < |t| < 2$$

$$1 < |t|, \quad \left|\frac{1}{t}\right| < 1$$

$$|t| < 2, \quad \left|\frac{t}{2}\right| < 1$$

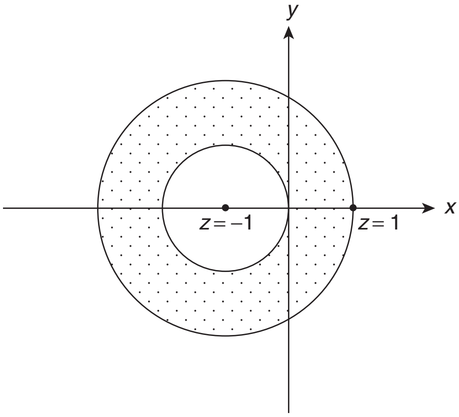

Fig. 4.29

$$f(z) = \frac{1}{t-1} + \frac{1}{2-t}$$

$$= \frac{1}{t\left(1 - \frac{1}{t}\right)} + \frac{1}{2\left(1 - \frac{t}{2}\right)}$$

$$= \frac{1}{t}\left(1 - \frac{1}{t}\right)^{-1} + \frac{1}{2}\left(1 - \frac{t}{2}\right)^{-1}$$

$$= \frac{1}{t}\left(1 + \frac{1}{t} + \frac{1}{t^2} + \cdots\right) + \frac{1}{2}\left[1 + \frac{t}{2} + \left(\frac{t}{2}\right)^2 + \cdots\right]$$

$$= \left(\frac{1}{t} + \frac{1}{t^2} + \frac{1}{t^3} + \cdots\right) + \left(\frac{1}{2} + \frac{t}{4} + \frac{t^2}{8} + \cdots\right)$$

$$= \left[\frac{1}{z+1} + \frac{1}{(z+1)^2} + \frac{1}{(z+1)^3} + \cdots\right] + \left[\frac{1}{2} + \frac{z+1}{4} + \frac{(z+1)^2}{8} + \cdots\right]$$

$$= \left[\frac{1}{2} + \frac{z+1}{4} + \frac{(z+1)^2}{8} + \cdots\right] + \left[\frac{1}{z+1} + \frac{1}{(z+1)^2} + \frac{1}{(z+1)^3} + \cdots\right]$$

Example 17

Expand $-\dfrac{1}{(z-1)(z-2)}$ *in Laurent's series valid for*

(i) $|z-1|<1$ *(ii)* $1<|z|<2$ *(iii)* $|z|>2$

Solution

Let
$$f(z) = -\frac{1}{(z-1)(z-2)}$$
$$= -\frac{(z-1)-(z-2)}{(z-1)(z-2)}$$
$$= -\frac{1}{z-2} + \frac{1}{z-1}$$

$f(z)$ is not analytic at $z=2$ and $z=1$.

(i) $|z-1|<1$

$f(z)$ is analytic in the region $|z-1|<1$ about $z=1$ (Fig. 4.30).

Let $z-1=t$ then $z=t+1$.

$|z-1|<$ i.e., $|t|<1$

$$f(z) = -\frac{1}{t+1-2} + \frac{1}{t}$$
$$= -\frac{1}{t-1} + \frac{1}{t}$$
$$= \frac{1}{1-t} + \frac{1}{t}$$
$$= (1-t)^{-1} + \frac{1}{t}$$
$$= (1+t+t^2+\cdots) + \frac{1}{t}$$
$$= \left[1+(z-1)+(z-1)^2+\cdots\right] + \frac{1}{z-1}$$

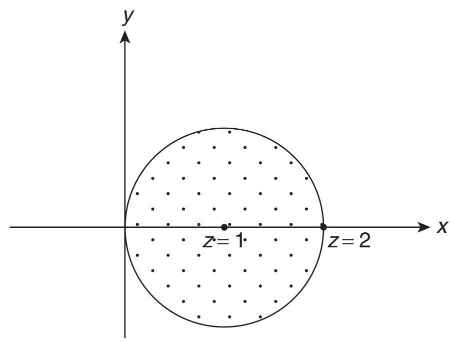

Fig. 4.30

(ii) $1<|z|<2$

$f(z)$ is analytic in the annular region $1<|z|<2$ about $z=0$ (Fig. 4.31).

$$1<|z|, \quad \left|\frac{1}{z}\right|<1$$
$$|z|<2, \quad \left|\frac{z}{2}\right|<1$$

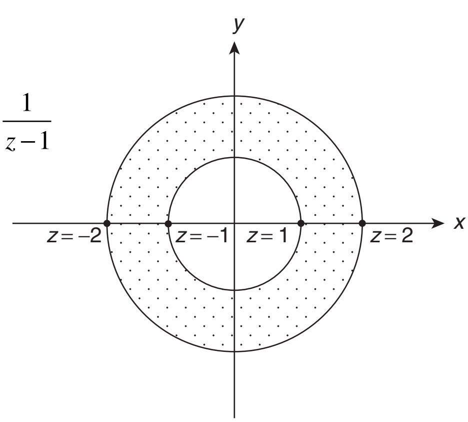

Fig. 4.31

$$f(z) = -\frac{1}{-2\left(1-\frac{z}{2}\right)} + \frac{1}{z\left(1-\frac{1}{z}\right)}$$

$$= \frac{1}{2}\left(1-\frac{z}{2}\right)^{-1} + \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}$$

$$= \frac{1}{2}\left[1+\left(\frac{z}{2}\right)+\left(\frac{z}{2}\right)^2+\cdots\right] + \frac{1}{z}\left[1+\frac{1}{z}+\left(\frac{1}{z}\right)^2+\cdots\right]$$

$$= \frac{1}{2}\left(1+\frac{z}{2}+\frac{z^2}{4}+\cdots\right) + \left(\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right)$$

(iii) $|z| > 2$

$f(z)$ is analytic in the region $|z| > 2$ about $z = 0$ (Fig. 4.32).

$\because \quad |z| > 2 > 1, \quad \left|\frac{2}{z}\right| < 1, \quad \left|\frac{1}{z}\right| < 1$

$$f(z) = -\frac{1}{z\left(1-\frac{2}{z}\right)} + \frac{1}{z\left(1-\frac{1}{z}\right)}$$

$$= -\frac{1}{z}\left(1-\frac{2}{z}\right)^{-1} + \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}$$

$$= -\frac{1}{z}\left[1+\frac{2}{z}+\left(\frac{2}{z}\right)^2+\cdots\right] + \frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\cdots\right)$$

$$= -\left(\frac{1}{z}+\frac{2}{z^2}+\frac{4}{z^3}+\cdots\right) + \left(\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right)$$

$$= -\frac{1}{z^2}+\frac{3}{z^3}+\cdots$$

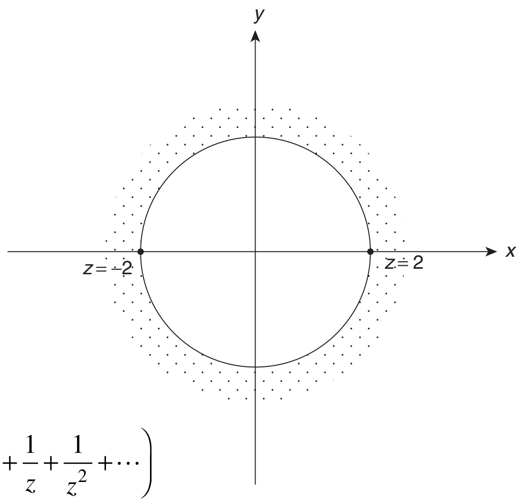

Fig. 4.32

Example 18

Expand $f(z) = \dfrac{z}{(z-1)(z-3)}$ *as Laurent's series valid in the following regions:*

(i) $1 < |z| < 3$ *(ii)* $0 < |z-1| < 2$ *(iii)* $|z| > 3.$ **[Winter 2012]**

Solution

$$f(z) = \frac{z}{(z-1)(z-3)}$$

$$= \frac{A}{z-1} + \frac{B}{z-3}$$

$$z = A(z-3) + B(z-1)$$

Putting $z = 1$,

$$1 = A(1-3)$$

$$A = -\frac{1}{2}$$

Putting $z = 3$,

$$3 = B(3-1)$$

$$B = \frac{3}{2}$$

$$\therefore \quad f(z) = -\frac{1}{2}\frac{1}{z-1} + \frac{3}{2}\frac{1}{z-3}$$

$f(z)$ is not analytic at $z = 1$ and $z = 3$.

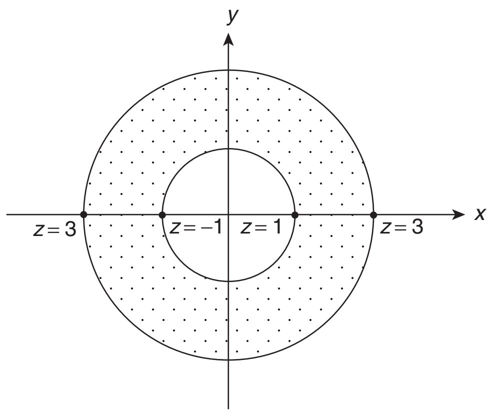

Fig. 4.33

(i) $1 < |z| < 3$

$f(z)$ is analytic in the annular region $1 < |z| < 3$ about $z = 0$ (Fig. 4.33).

$$1 < |z|, \qquad \left|\frac{1}{z}\right| < 1$$

$$|z| < 3, \qquad \left|\frac{z}{3}\right| < 1$$

$$f(z) = -\frac{1}{2}\frac{1}{z\left(1-\frac{1}{z}\right)} + \frac{3}{2}\frac{1}{(-3)\left(1-\frac{z}{3}\right)}$$

$$= -\frac{1}{2z}\left(1-\frac{1}{z}\right)^{-1} - \frac{1}{2}\left(1-\frac{z}{3}\right)^{-1}$$

$$= -\frac{1}{2z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\cdots\right) - \frac{1}{2}\left[1+\frac{z}{3}+\left(\frac{z}{3}\right)^2+\cdots\right]$$

$$= -\frac{1}{2}\left(\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right) - \frac{1}{2}\left(1+\frac{z}{3}+\frac{z^2}{9}+\cdots\right)$$

$$= -\frac{1}{2}\left(1+\frac{z}{3}+\frac{z^2}{9}+\cdots\right) - \frac{1}{2}\left(\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right)$$

(ii) $0 < |z-1| < 2$

$f(z)$ is analytic in the annular region $0 < |z-1| < 2$ about $z = 1$ (Fig. 4.34).

Let $z-1 = t$ then $z = t+1$.

$0 < |z-1| < 2$, i.e., $0 < |t| < 2$

$|t| < 2, \quad \left|\frac{t}{2}\right| < 1$

$$f(z) = -\frac{1}{2}\frac{1}{t} + \frac{3}{2}\frac{1}{t+1-3}$$

$$= -\frac{1}{2t} + \frac{3}{2}\frac{1}{t-2}$$

$$= -\frac{1}{2t} + \frac{3}{2}\frac{1}{(-2)\left(1-\frac{t}{2}\right)}$$

$$= -\frac{1}{2t} - \frac{3}{4}\left(1-\frac{t}{2}\right)^{-1}$$

$$= -\frac{1}{2t} - \frac{3}{4}\left[1+\frac{t}{2}+\left(\frac{t}{2}\right)^2+\cdots\right]$$

$$= -\frac{1}{2(z-1)} - \frac{3}{4}\left[1+\frac{(z-1)}{2}+\frac{(z-1)^2}{4}+\cdots\right]$$

$$= -\frac{3}{4}\left[1+\frac{(z-1)}{2}+\frac{(z-1)^2}{4}+\cdots\right] - \frac{1}{2(z-1)}$$

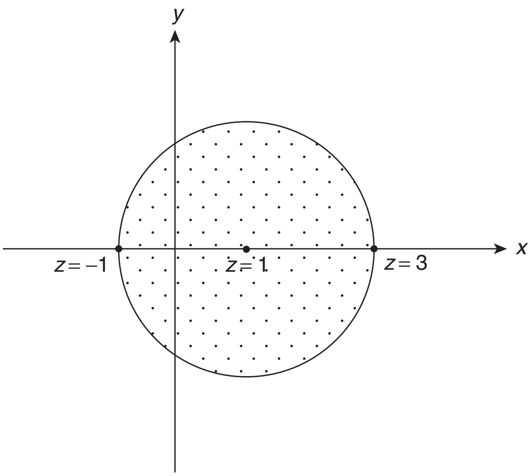

Fig. 4.34

(iii) $|z| > 3$

$f(z)$ is analytic in the region $|z| > 3$ about $z = 0$ (Fig. 4.35).

$|z| > 3, \left|\frac{1}{z}\right| < \frac{1}{3} < 1$

$$f(z) = -\frac{1}{2z\left(1-\frac{1}{z}\right)} + \frac{3}{2}\frac{1}{z\left(1-\frac{3}{z}\right)}$$

$$= -\frac{1}{2z}\left(1-\frac{1}{z}\right)^{-1} + \frac{3}{2z}\left(1-\frac{3}{z}\right)^{-1}$$

$$= -\frac{1}{2z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\cdots\right) + \frac{3}{2z}\left[1+\frac{3}{z}+\left(\frac{3}{z}\right)^2+\cdots\right]$$

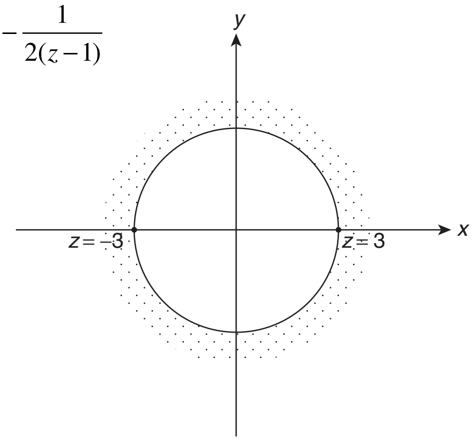

Fig. 4.35

$$= -\frac{1}{2}\left(\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right)+\frac{3}{2}\left(\frac{1}{z}+\frac{3}{z^2}+\frac{9}{z^3}+\cdots\right)$$

$$= \frac{1}{z}+\frac{4}{z^2}+\frac{13}{z^3}+\cdots$$

EXERCISE 4.2

Expand the following functions in Laurent's series:

1. $f(z) = \dfrac{z-1}{(z+2)(z+3)}, \qquad 2 < |z| < 3$

$$\left[\textbf{Ans.: } -\frac{3}{z}\left[1-\frac{2}{z}+\frac{4}{z^2}\cdots\right]+\frac{4}{3}\left[1-\frac{z}{3}+\frac{z^2}{9}\cdots\right]\right]$$

2. $f(z) = \dfrac{1}{z(z-1)}$ for $0 < |z| < 1$ and $0 < |z-1| < 1$ **[Winter 2013]**

$$\left[\textbf{Ans.: } -\frac{1}{z}(1+z+z^2+\cdots) \text{ and } \frac{1}{z-1}\left[1-(z-1)+(z-1)^2-\cdots\right]\right]$$

3. $f(z) = \dfrac{z-1}{z^2}$ in $|z-1| > 1$

$$\left[\textbf{Ans.: } \frac{1}{z-1}-\frac{2}{(z-1)^2}+\frac{3}{(z-1)^3}-\cdots\right]$$

4. $f(z) = \dfrac{z+3}{z(z^2-z-2)}$ in $1 < |z| < 2$

$$\left[\textbf{Ans.: } -\frac{5}{12}\left(1+\frac{z}{2}+\frac{z^2}{4}+\cdots\right)-\frac{3}{2z}+\frac{2}{3z}\left(1-\frac{1}{z}+\frac{1}{z^2}+\cdots\right)\right]$$

5. $f(z) = \dfrac{1}{z(1-z)^2}$ in the region $0 < |z| < 1$ and $0 < |z-1| < 1$

$$\left[\textbf{Ans.: } \text{(i) } \frac{1}{z}+2+3z+4z^2+\cdots \text{ (ii) } \frac{1}{(z-1)^2}-\frac{1}{z-1}+1-(z-1)+(z-1)^2-\cdots\right]$$

6. $f(z) = \dfrac{1}{z(1-z)^2}$ in the region $|z| > 1$

$$\left[\textbf{Ans.: } \frac{1}{z^3}+\frac{2}{z^4}+\frac{3}{z^5}+\cdots\right]$$

7. $f(z) = \dfrac{1}{z^3 - 3z^2 + 2z}$ about $z = 0$ for (i) $1 < |z| < 2$ (ii) $|z| > 2$

$$\left[\textbf{Ans.: } \text{(i) } -\frac{1}{4} - \frac{1}{8}z - \frac{1}{16}z^2 - \cdots - \frac{1}{z^2} - \frac{1}{z^3} - \frac{1}{z^4} - \cdots \text{ (ii) } \frac{1}{z^3} + \frac{3}{z^4} + \frac{7}{z^5} + \cdots\right]$$

8. $f(z) = \dfrac{z-1}{z^2 - 2z - 3} f$ or (i) $1 < |z| < 3$ (ii) $|z| > 3$

$$\left[\textbf{Ans.: } \text{(i) } \frac{1}{2}\left(-\frac{1}{3} - \frac{z}{9} - \frac{z^2}{27} - \cdots - \frac{1}{z^2} + \frac{1}{z^3} - \cdots\right) \text{ (ii) } \frac{1}{z} + \frac{1}{z^2} + \frac{5}{z^3} + \cdots\right]$$

9. $f(z) = \dfrac{1}{z^2(z-2)}; 0 < |z| < 2$

$$\left[\textbf{Ans.: } -\frac{1}{2z^2}\left(1 + \frac{z}{2} + \frac{z^2}{4} + \cdots\right)\right]$$

10. $f(z) = \dfrac{1+2z}{z+z^2}$; $0 < |z| < 1$

$$\left[\textbf{Ans.: } \frac{1}{z}(1 + z + z^2 + \cdots)\right]$$

11. $f(z) = \dfrac{z}{(z+1)(z+2)}$; $0 < |z+2| < 1$

$$\left[\textbf{Ans.: } \frac{2}{z+2} + 1 + (z+2) + (z+2)^2 + \cdots\right]$$

12. $f(z) = \dfrac{(z-2)(z+2)}{(z+1)(z+4)}$ in (i) $1 < |z| < 4$ (ii) $|z| > 4$

$$\left[\textbf{Ans.: } \text{(i) } 1 - \left(\frac{1}{z} - \frac{1}{z^2} + \cdots\right) - \left(1 - \frac{z}{4} + \frac{z^2}{16} - \cdots\right)\right.$$

$$\left.\text{(ii) } 1 - \left(\frac{1}{z} - \frac{1}{z^2} + \cdots\right) - 4\left(\frac{1}{z} - \frac{4}{z^2} + \cdots\right)\right]$$