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Scalar Function: This is a function from $\mathbb{R}^n$ to $\mathbb{R}$ such that it takes values from one or more variables and produces a single value.

$f(x,y,z) = x^2 + 2yz^5$ is an example of a scalar-valued function.

An n-variable scalar-valued function acts as a map from $\mathbb{R}^n$ to $\mathbb{R}$ so that

$$f : \mathbb{R}^n \to \mathbb{R}$$

Vector-valued functions: The function whose values are vectors depending on the point P in the space is a vector-valued function.

$v = v(P) = v_1(P)\hat{i} + v_2(P)\hat{j} + v_3(P)\hat{k}$, where P is any arbitrary point in the space.

$$v = x^2 y\hat{i} + xy^2 z\hat{j} - z^3\hat{k}$$

Gradient

The gradient vector of $f(x,y)$ at a point $P(x_0,y_0)$ is the vector

$$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}$$

where $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are partial derivatives of f at P. The symbol '∇' is called nabla, and '∇f' is read as 'grad f'.

Properties:

1. Sum rule: $\nabla(f \pm g) = \nabla f \pm \nabla g$
2. Product Rule: $\nabla(fg) = f\nabla g + g\nabla f$
3. Quotient Rule: $\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$

g^2

4. Constant multiple rule: $\nabla(Kf) = K\nabla f$

Prob: Given $u = x^3 + 3yz^2$. Find ∇u.

Sol^n:

$$\nabla u = \frac{\partial u}{\partial x}\hat{i} + \frac{\partial u}{\partial y}\hat{j} + \frac{\partial u}{\partial z}\hat{k}$$

$$= 3x^2\hat{i} + 3z^2\hat{j} + 6yz\hat{k}$$

Divergence of a vector function

The divergence of a continuously differentiable vector point function F is denoted by $\mathrm{div}\,F$ and defined by the equation:

$$\mathrm{div}\,F = \nabla\cdot F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z},$$ where F is given by.

$$F = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$$

Note:
1. The divergence of a vector field is a scalar function.
2. The vector field with zero divergence everywhere is known as solenoidal.

Prob: compute $\mathrm{div}\,F$ for $F = x^2y\,\hat{i} + xyz\,\hat{j} - x^2y^2\,\hat{k}$

Sol^n:

$$\mathrm{div}\,F = \nabla\cdot F = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\left(x^2y\,\hat{i} + xyz\,\hat{j} - x^2y^2\,\hat{k}\right)$$

$$= \frac{\partial}{\partial x}(x^2y) + \frac{\partial}{\partial y}(xyz) - \frac{\partial}{\partial z}(x^2y^2)$$

$$= 2xy + xz.$$

Curl of a vector

The curl of a continuously differentiable vector point function F is defined by

$$\text{curl } F = \nabla \times F$$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \times (F_1\hat{i} + F_2\hat{j} + F_3\hat{k}) \quad \text{where}$$

$F = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$.

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

Notes:

1. The curl of a vector field is a vector field itself.
2. If the curl is zero, then the object is not rotating or known as 'irrotational'.

Prob: Verify $\text{div}(\text{curl } F) = 0$ for

$$F = yz^2\hat{i} + xy\hat{j} + yz\hat{k}$$

Soln:

$$\text{curl } F = \nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz^2 & xy & yz \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(yz) - \frac{\partial}{\partial z}(xy)\right] - \hat{j}\left[\frac{\partial}{\partial x}(yz) - \frac{\partial}{\partial z}(yz^2)\right] + \hat{k}\left[\frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial y}(yz^2)\right]$$

$$\text{Curl } F = z\hat{i} + 2yz\hat{j} + (y - z^2)\hat{k}$$

$$\text{div}(\text{curl } F) = \nabla \cdot (\text{curl } F)$$

$$= \frac{\partial}{\partial x}(z) + \frac{\partial}{\partial y}(2yz) + \frac{\partial}{\partial z}(y - z^2)$$

$$= 2z - 2z = 0$$