

## Directional derivative

Suppose that the function  $f(x, y)$  is defined throughout a region  $R$  in the  $xy$ -plane, that  $P_0(x_0, y_0)$  is a point in  $R$ , and that  $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$  is a unit vector. Then the equations

$$x = x_0 + su_1, \quad y = y_0 + su_2$$

parametrize the line through  $P_0$  parallel to  $\mathbf{u}$ . If the parameter  $s$  measures arc length from  $P_0$  in the direction of  $\mathbf{u}$ , we find the rate of change of  $f$  at  $P_0$  in the direction of  $\mathbf{u}$  by calculating  $df/ds$  at  $P_0$ .

**DEFINITION** The derivative of  $f$  at  $P_0(x_0, y_0)$  in the direction of the unit vector  $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$  is the number

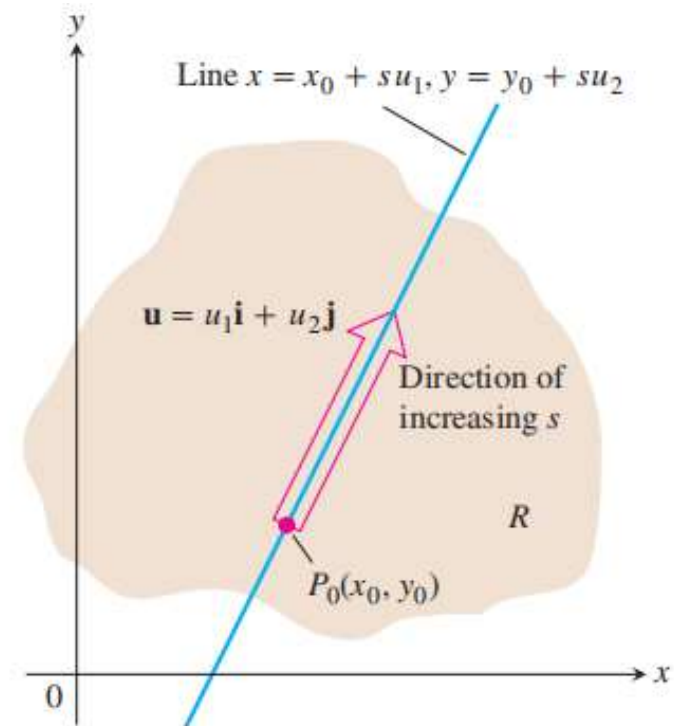
$$\left(\frac{df}{ds}\right)_{\mathbf{u}, P_0} = \lim_{s \rightarrow 0} \frac{f(x_0 + su_1, y_0 + su_2) - f(x_0, y_0)}{s}, \quad (1)$$

provided the limit exists.

The **directional derivative** defined by Equation (1) is also denoted by

$$(D_{\mathbf{u}}f)_{P_0} \quad \text{“The derivative of } f \text{ at } P_0 \text{ in the direction of } \mathbf{u}\text{”}$$

The partial derivatives  $f_x(x_0, y_0)$  and  $f_y(x_0, y_0)$  are the directional derivatives of  $f$  at  $P_0$  in the  $\mathbf{i}$  and  $\mathbf{j}$  directions.



**FIGURE** The rate of change of  $f$  in the direction of  $\mathbf{u}$  at a point  $P_0$  is the rate at which  $f$  changes along this line at  $P_0$ .

**EXAMPLE 1** Using the definition, find the derivative of

$$f(x, y) = x^2 + xy$$

at  $P_0(1, 2)$  in the direction of the unit vector  $\mathbf{u} = (1/\sqrt{2})\mathbf{i} + (1/\sqrt{2})\mathbf{j}$ .

**Solution** Applying the definition in Equation (1), we obtain

$$\begin{aligned}\left(\frac{df}{ds}\right)_{\mathbf{u}, P_0} &= \lim_{s \rightarrow 0} \frac{f(x_0 + su_1, y_0 + su_2) - f(x_0, y_0)}{s} \\ &= \lim_{s \rightarrow 0} \frac{f\left(1 + s \cdot \frac{1}{\sqrt{2}}, 2 + s \cdot \frac{1}{\sqrt{2}}\right) - f(1, 2)}{s} \\ &= \lim_{s \rightarrow 0} \frac{\left(1 + \frac{s}{\sqrt{2}}\right)^2 + \left(1 + \frac{s}{\sqrt{2}}\right)\left(2 + \frac{s}{\sqrt{2}}\right) - (1^2 + 1 \cdot 2)}{s} \\ &= \lim_{s \rightarrow 0} \frac{\left(1 + \frac{2s}{\sqrt{2}} + \frac{s^2}{2}\right) + \left(2 + \frac{3s}{\sqrt{2}} + \frac{s^2}{2}\right) - 3}{s} \\ &= \lim_{s \rightarrow 0} \frac{\frac{5s}{\sqrt{2}} + s^2}{s} = \lim_{s \rightarrow 0} \left(\frac{5}{\sqrt{2}} + s\right) = \frac{5}{\sqrt{2}}.\end{aligned}$$

The rate of change of  $f(x, y) = x^2 + xy$  at  $P_0(1, 2)$  in the direction  $\mathbf{u}$  is  $5/\sqrt{2}$ .

**THEOREM**    **The Directional Derivative Is a Dot Product**    If  $f(x, y)$  is differentiable in an open region containing  $P_0(x_0, y_0)$ , then

$$\left(\frac{df}{ds}\right)_{\mathbf{u}, P_0} = (\nabla f)_{P_0} \cdot \mathbf{u},$$

the dot product of the gradient  $\nabla f$  at  $P_0$  and  $\mathbf{u}$ .

**EXAMPLE**    Find the derivative of  $f(x, y) = xe^y + \cos(xy)$  at the point  $(2, 0)$  in the direction of  $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$ .

**Solution**    The direction of  $\mathbf{v}$  is the unit vector obtained by dividing  $\mathbf{v}$  by its length:

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\mathbf{v}}{5} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}.$$

The partial derivatives of  $f$  are everywhere continuous and at  $(2, 0)$  are given by

$$f_x(2, 0) = (e^y - y \sin(xy))_{(2,0)} = e^0 - 0 = 1$$

$$f_y(2, 0) = (xe^y - x \sin(xy))_{(2,0)} = 2e^0 - 2 \cdot 0 = 2.$$

The gradient of  $f$  at  $(2, 0)$  is

$$\nabla f|_{(2,0)} = f_x(2, 0)\mathbf{i} + f_y(2, 0)\mathbf{j} = \mathbf{i} + 2\mathbf{j}$$

(Figure 14.28). The derivative of  $f$  at  $(2, 0)$  in the direction of  $\mathbf{v}$  is therefore

$$\begin{aligned}(D_{\mathbf{u}}f)|_{(2,0)} &= \nabla f|_{(2,0)} \cdot \mathbf{u} \\ &= (\mathbf{i} + 2\mathbf{j}) \cdot \left(\frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}\right) = \frac{3}{5} - \frac{8}{5} = -1.\end{aligned}$$

**Properties of the Directional Derivative  $D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = |\nabla f| \cos \theta$**

1. The function  $f$  increases most rapidly when  $\cos \theta = 1$  or when  $\theta = 0$  and  $\mathbf{u}$  is the direction of  $\nabla f$ . That is, at each point  $P$  in its domain,  $f$  increases most rapidly in the direction of the gradient vector  $\nabla f$  at  $P$ . The derivative in this direction is

$$D_{\mathbf{u}}f = |\nabla f| \cos(0) = |\nabla f|.$$

2. Similarly,  $f$  decreases most rapidly in the direction of  $-\nabla f$ . The derivative in this direction is  $D_{\mathbf{u}}f = |\nabla f| \cos(\pi) = -|\nabla f|$ .
3. Any direction  $\mathbf{u}$  orthogonal to a gradient  $\nabla f \neq 0$  is a direction of zero change in  $f$  because  $\theta$  then equals  $\pi/2$  and

$$D_{\mathbf{u}}f = |\nabla f| \cos(\pi/2) = |\nabla f| \cdot 0 = 0.$$

**EXAMPLE** Find the directions in which  $f(x, y) = (x^2/2) + (y^2/2)$

- (a) increases most rapidly at the point  $(1, 1)$ .
- (b) decreases most rapidly at  $(1, 1)$ .
- (c) What are the directions of zero change in  $f$  at  $(1, 1)$ ?

**Solution**

- (a) The function increases most rapidly in the direction of  $\nabla f$  at  $(1, 1)$ . The gradient there is

$$(\nabla f)_{(1,1)} = (x\mathbf{i} + y\mathbf{j})_{(1,1)} = \mathbf{i} + \mathbf{j}.$$

Its direction is

$$\mathbf{u} = \frac{\mathbf{i} + \mathbf{j}}{|\mathbf{i} + \mathbf{j}|} = \frac{\mathbf{i} + \mathbf{j}}{\sqrt{(1)^2 + (1)^2}} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}.$$

- (b) The function decreases most rapidly in the direction of  $-\nabla f$  at  $(1, 1)$ , which is

$$-\mathbf{u} = -\frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}.$$

- (c) The directions of zero change at  $(1, 1)$  are the directions orthogonal to  $\nabla f$ :

$$\mathbf{n} = -\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j} \quad \text{and} \quad -\mathbf{n} = \frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}.$$

### EXAMPLE

- (a) Find the derivative of  $f(x, y, z) = x^3 - xy^2 - z$  at  $P_0(1, 1, 0)$  in the direction of  $\mathbf{v} = 2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$ .
- (b) In what directions does  $f$  change most rapidly at  $P_0$ , and what are the rates of change in these directions?

### Solution

- (a) The direction of  $\mathbf{v}$  is obtained by dividing  $\mathbf{v}$  by its length:

$$|\mathbf{v}| = \sqrt{(2)^2 + (-3)^2 + (6)^2} = \sqrt{49} = 7$$

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{2}{7}\mathbf{i} - \frac{3}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}.$$

The partial derivatives of  $f$  at  $P_0$  are

$$f_x = (3x^2 - y^2)|_{(1,1,0)} = 2, \quad f_y = -2xy|_{(1,1,0)} = -2, \quad f_z = -1|_{(1,1,0)} = -1.$$

The gradient of  $f$  at  $P_0$  is

$$\nabla f|_{(1,1,0)} = 2\mathbf{i} - 2\mathbf{j} - \mathbf{k}.$$

The derivative of  $f$  at  $P_0$  in the direction of  $\mathbf{v}$  is therefore

$$\begin{aligned}(D_{\mathbf{u}}f)_{(1,1,0)} &= \nabla f|_{(1,1,0)} \cdot \mathbf{u} = (2\mathbf{i} - 2\mathbf{j} - \mathbf{k}) \cdot \left(\frac{2}{7}\mathbf{i} - \frac{3}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}\right) \\ &= \frac{4}{7} + \frac{6}{7} - \frac{6}{7} = \frac{4}{7}.\end{aligned}$$

- (b) The function increases most rapidly in the direction of  $\nabla f = 2\mathbf{i} - 2\mathbf{j} - \mathbf{k}$  and decreases most rapidly in the direction of  $-\nabla f$ . The rates of change in the directions are, respectively,

$$|\nabla f| = \sqrt{(2)^2 + (-2)^2 + (-1)^2} = \sqrt{9} = 3 \quad \text{and} \quad -|\nabla f| = -3.$$