

Singular points or Singularity of $f(z)$:

A point at which $f(z)$ is not analytic is known as singular point or singularity of the function.

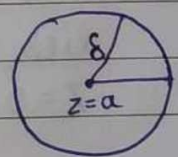
Ex:- $f(z) = \frac{1}{z-2}$

$z=2$ is singular point of $f(z)$.

Singular points are of two type:-

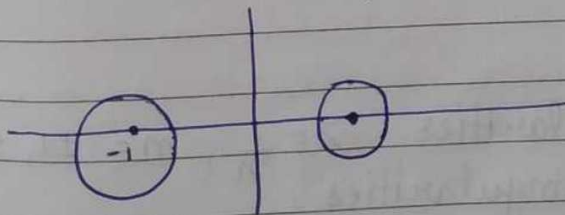
- (1) Non isolated singular points
- (2) isolated singular points.

If $z=a$ is a singular point of function $f(z)$ and if there is no other singularity in the neighbourhood of $z=a$, then $z=a$ is said to be isolated singular point otherwise it is non-isolated singular point of $f(z)$.



Ex:- $f(z) = \frac{z}{z^2-1}$

$z=1, -1$ are singularities



Since in the small neighbourhood of $z=1$ and $z=-1$ there are no other singularities lies therefore $z=1, -1$ are isolated singularities.

Example of function having nonisolated singularities

Ex:- $f(z) = \operatorname{cosec}\left(\frac{\pi}{z}\right)$

i.e. $f(z) = \frac{1}{\sin\left(\frac{\pi}{z}\right)}$

Singularities are points where

$$\sin\left(\frac{\pi}{z}\right) = 0$$

$$\frac{\pi}{z} = n\pi,$$

$$z = \frac{1}{n}, n = 0, \pm 1, \pm 2, \dots$$

singularities are

$$z = 0, \pm 1, \pm \frac{1}{2}, \pm \frac{1}{3}, \pm \frac{1}{4}, \dots, \infty$$

there are many ^{other} singularities
lies in the neighbourhood of
 $z = 0$.



therefore $z = 0$ is non-isolated
singularity.

and other singularities $z = \frac{1}{n}, n = \pm 1, \pm 2, \dots$
are isolated singularities.

Isolated singularities:-

Let $f(z)$ be analytic function in a domain D , except $z = z_0$ (which is isolated singularity). Then with $z = z_0$ as center we can draw two concentric circles C_1 and C_2 such that $f(z)$ is analytic in annulus region b/w C_1 and C_2 . In annulus region $f(z)$ can be expanded as Laurent series as

$$f(z) = \underbrace{\sum_{n=0}^{\infty} a_n (z-z_0)^n}_{\text{analytic part}} + \underbrace{\sum_{n=1}^{\infty} b_n (z-z_0)^{-n}}_{\substack{\text{principal part} \\ \text{or} \\ \text{singular part}}}$$

Depending on the nature of singular part, there are three type of singularities:-

(1) Removable singularity:-

A singularity $z = z_0$ is removable singularity if in Laurent series expansion of $f(z)$ principal part is absent or in other word if Laurent series expansion has no negative powers of $(z-z_0)$. i.e. $b_1, b_2, \dots, b_n$ all are zero.

i.e. $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$

$$f(z) = \frac{\sin(z-a)}{z-a}$$

$f(z)$ has singularity at $z=a$.

$$f(z) = \frac{1}{z-a} \left((z-a) - \frac{(z-a)^3}{3!} + \frac{(z-a)^5}{5!} - \dots \right)$$

$$f(z) = 1 - \frac{(z-a)^2}{3!} + \frac{(z-a)^4}{5!} - \dots$$

Series of $f(z)$ do not contain, negative powers of $(z-a)$, i.e. $z=a$ is removable singularity.

e:- Removable singularity can be made to disappear by redefining the function at $z=z_0$, such that $f(z)$ becomes analytic at $z=z_0$.

in above example, $f(z)$ can be redefined as

$$f(z) = \begin{cases} \frac{\sin(z-a)}{z-a}, & z \neq a \\ 1, & z = a \end{cases}$$

Note:- If $\lim_{z \rightarrow a} f(z)$ exist and finite

then $z=a$ is removable singular point.

Q: Is $z=0$ is removable singular point of the following function. If yes, then redefine the function at $z=0$, to make it analytic at $z=0$.

(1) $f(z) = \frac{1 - \cos z}{z}$; (2) $f(z) = \frac{z - \sin z}{z}$

Ans yes

$$f(z) = \begin{cases} \frac{1 - \cos z}{z}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$

Ans yes

$$f(z) = \begin{cases} \frac{z - \sin z}{z}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$

Def: Residue:- The coefficient of $\frac{1}{(z-z_0)}$ i.e. b_1 in Laurent's series expansion is called Residue of $f(z)$ at $z=z_0$.
Residue of $f(z)$ at $z=z_0$ is denoted as $\text{Res}_{z=z_0} f(z)$ or $\text{Res}[f(z); z=z_0]$

* If $z=z_0$ is removable singularity of $f(z)$ then $\boxed{\text{Res}_{z=z_0} f(z) = 0}$, as there are no negative powers of $(z-z_0)$.

2) Essential singularity:-

when principal part in Laurent's series expansion of function $f(z)$ contains infinite terms then $z=z_0$ is called Essential singularity.

(1) Ex:- $\sin\left(\frac{1}{z}\right)$

$z=0$ is singularity of $f(z)$

$$f(z) = \frac{1}{z} - \frac{1}{3!} \frac{1}{z^3} + \frac{1}{5!} \frac{1}{z^5} - \frac{1}{7!} \frac{1}{z^7} + \dots \infty$$

$f(z)$ has infinitely many negative powers of z , thus $z=0$, is essential singularity.

(2) Ex:- $f(z) = z \cos\left(\frac{1}{z}\right)$

$z=0$ is singularity

$$f(z) = z \left(1 - \frac{1}{2!} \frac{1}{z^2} + \frac{1}{4!} \frac{1}{z^4} - \dots \infty \right)$$

$$f(z) = z - \frac{1}{2!} \frac{1}{z} + \frac{1}{4!} \frac{1}{z^3} - \dots \infty$$

at $z=0$, $f(z)$ has essential singularity.

Ex

$$f(z) = z \sin\left(\frac{1}{z}\right)$$

(4) $\frac{e^{\frac{1}{z}}}{z^2}$

Note:- If $\lim_{z \rightarrow a} f(z)$ does not exist then

$f(z)$ has essential singularity.

(1) In example (1) i.e. $f(z) = \sin\left(\frac{1}{z}\right)$

$$\boxed{\text{Res } f(z) = 1 \atop z=0}$$

(2) In example (2) $f(z) = z \cos\left(\frac{1}{z}\right)$

$$\boxed{\text{Res } f(z) = -\frac{1}{2} \atop z=0}$$

Ex 3 $f(z) = \frac{e^{1/z}}{z^2}$

$z=0$ is singular point

$$f(z) = \frac{1}{z^2} \left(1 + \frac{1}{z} + \frac{1}{z^2} \cdot \frac{1}{2!} + \frac{1}{z^3} \cdot \frac{1}{3!} + \dots \infty \right)$$

$$f(z) = \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{2!} \frac{1}{z^4} + \frac{1}{3!} \frac{1}{z^5} + \dots \infty$$

$f(z)$ has infinitely many terms of negative powers of z , i.e. $z=0$ is essential singularity.

∴ Residue at $z=0$ is

$$\boxed{\text{Res } f(z) = 0 \atop z=0}$$

3. Pole :- If Principal part of Laurent's series expansion of $f(z)$ has finite no. of term ~~then~~ i.e.

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \dots + \frac{b_m}{(z-z_0)^m}$$

then, $z=z_0$ is called pole of order m.

when $m=1$ i.e.

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \frac{b_1}{z-z_0}$$

then $z=z_0$ is called pole of order 1 or simple pole.

Q:- Find pole of the function $\frac{e^{z-a}}{(z-a)^2}$

Solⁿ $z=a$ is singular point

$$f(z) = \frac{1}{(z-a)^2} \left[1 + (z-a) + \frac{(z-a)^2}{2!} + \frac{(z-a)^3}{3!} + \dots \infty \right]$$

$$f(z) = \left(\frac{1}{2!} + \frac{(z-a)}{3!} + \frac{(z-a)^2}{4!} + \dots \infty \right) + \left(\frac{1}{(z-a)} + \frac{1}{(z-a)^2} \right)$$

$f(z)$ has pole of order 2 at $z=a$.

Q:- Discuss singularity of $\sin\left(\frac{1}{z-a}\right)$.

Solⁿ $z=a$ is essential singularity.

Q:- Discuss singularity of $\frac{1 - \cosh z}{z^3}$.

soln:- $f(z) = \frac{1 - \cosh z}{z^3}$, $z=0$ is singularity.

$$= \frac{1 - \left(1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \dots - \infty\right)}{z^3}$$

$$= \frac{1}{z^3} \left(-\frac{z^2}{2!} - \frac{z^4}{4!} - \frac{z^6}{6!} - \dots - \infty \right)$$

$$= \left(-\frac{z}{4!} - \frac{z^3}{6!} - \dots - \infty \right) - \frac{1}{z \cdot 2!}$$

$z=0$ is pole of order 1 or simple pole.

Residue at $z=0$ is

$$\boxed{\text{Res } f(z)_{z=0} = -\frac{1}{2}}$$

Q:- Discuss the nature of singularity for following

(1) $\frac{z - \sin z}{z^3}$

Ans:- singular point $z=0$

$z=0$ is removable singularity

$$f(z) = \begin{cases} \frac{z - \sin z}{z^3}, & z \neq 0 \\ \frac{1}{6}, & z = 0. \end{cases}$$

$$(2) f(z) = \frac{e^{1/z}}{z^2}$$

Ans $z=0$ is singular point
 $f(z)$ has essential singularity at $z=0$.

$$(3) f(z) = e^{\left(\frac{1}{z-a}\right)}$$

Ans singular point is $z=a$
 $z=a$ is essential sing.

$$(4) f(z) = \frac{1}{\sin z - \cos z} \quad \text{at } z = \pi/4$$

$$f(z) = \frac{1}{\sin(t + \pi/4) - \cos(t + \pi/4)} \quad z = \pi/4 = t$$

$$f(z) = \frac{1}{2 \cdot \frac{1}{\sqrt{2}} \sin t} = \frac{1}{\sqrt{2} \sin t}$$

$$f(z) = \frac{1}{\sqrt{2} \left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \dots \right)}$$

$$= \frac{1}{\sqrt{2}} \left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \dots \right)^{-1}$$

$$= \frac{1}{\sqrt{2} t} \left(1 - \frac{t^2}{3!} + \frac{t^4}{5!} - \dots \right)^{-1}$$

$$= \frac{1}{\sqrt{2} t} \left(1 - \left(\frac{t^2}{3!} - \frac{t^4}{5!} - \dots \right) \right)^{-1}$$

$$= \frac{1}{\sqrt{2} t} \left[1 + \left(\frac{t^2}{3!} - \frac{t^4}{5!} - \dots \right) + \left(\frac{t^2}{3!} - \frac{t^4}{5!} - \dots \right)^2 + \dots \right]$$

$$= \frac{1}{12} \left[\frac{1}{t} + \left(\frac{t}{3!} - \frac{t^3}{5!} + \dots \right) + \frac{1}{t} \left(\frac{t^2}{3!} - \frac{t^4}{4!} + \dots \right) \right]$$

where $t = z - \pi/4$.

$z = \pi/4$ is simple pole.

(5) $f(z) = \frac{e^{2z}}{(z-1)^4}$

$f(z)$ is not analytic at $z=1$.

$$z-1=t \Rightarrow z=t+1$$

$$f(z) = \frac{e^{2(t+1)}}{t^4} = e^2 \cdot \frac{e^{2t}}{t^4}$$

$$= e^2 \frac{1}{t^4} \left[1 + 2t + \frac{(2t)^2}{2!} + \frac{(2t)^3}{3!} + \frac{(2t)^4}{4!} + \frac{(2t)^5}{5!} + \dots \right]$$

$$= e^2 \left[\left(\frac{2}{3} + \frac{4}{15}t + \dots \right) + \frac{4}{3t} + \frac{2}{t^2} + \frac{2}{t^3} + \frac{1}{t^4} \right]$$

$$= e^2 \left[\left(\frac{2}{3} + \frac{4}{15}(z-1) + \dots \right) + \frac{4}{3(z-1)} + \frac{2}{(z-1)^2} + \frac{2}{(z-1)^3} + \frac{1}{(z-1)^4} \right]$$

at $z=1$, $f(z)$ has pole of order 4.

$$\text{Res } f(z) = \frac{4}{3}$$

$z=1$

Q:- $z e^{\frac{1}{z^2}}$

Ans $f(z)$ is not analytic at $z=0$
 $z=0$ is essential singularity.

Q:- $f(z) = \frac{1}{1-e^z}$

$f(z)$ is not analytic at

$$1 - e^z = 0$$

$$e^z = 1 \Rightarrow e^z = e^{2n\pi i}, \quad n \in \mathbb{Z}$$

$$\Rightarrow \boxed{z = 2n\pi i} \quad n = 0, \pm 1, \pm 2, \dots$$

$z = 2n\pi i$, are poles of $f(z)$ and these are simple poles.
ie $z = 0, \pm 2\pi i, \pm 4\pi i, \dots$ all are simple poles.

Q:- Find Order of the pole and residue at each pole.

(1) $\frac{\sin^2 z}{z^3}$

Hint:- $\sin^2 z = \frac{1 - \cos 2z}{2}$

$$f(z) = \frac{1 - \cos 2z}{2z^3}$$

$z=0$ is simple pole

$$\boxed{\text{Res } f(z) = 1}$$

$z=0$

(2) $f(z) = \frac{1 - e^{2z}}{z^4}$

Hint:-

$z=0$ is pole of order 3.

$$\boxed{\text{Res } f(z) = -\frac{4}{3}}$$

$z=0$

Calculation of residue at pole:-

Residue:-

If $f(z)$ has a pole at point $z=z_0$ then coefficient of $\frac{1}{z-z_0}$, i.e. b_1 , in

Laurent's series expansion of $f(z)$ about $z=z_0$ is called residue of $f(z)$ at point $z=z_0$.

We know if $z=z_0$ is isolated singularity of $f(z)$ then

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^m b_n (z-z_0)^{-n}$$

where $b_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{-n+1}} dz$

$$b_1 = \frac{1}{2\pi i} \oint_C f(z) dz$$

b_1 is called residue of $f(z)$ at pole $z=z_0$ and denoted by

$$\text{Res}_{z=z_0} f(z) = b_1$$

$$\text{or } \text{Res}(f; z_0) = b_1$$

Methods of finding residue:-

classmate

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(1) Calculation of Residue at simple pole:-

(a) when $f(z)$ has simple pole at $z=z_0$ then from Laurent's series

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \frac{b_1}{z-z_0}$$

multiply by $(z-z_0)$

$$(z-z_0) f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^{n+1} + b_1$$

taking limit
as $z \rightarrow z_0$

$$b_1 = \lim_{z \rightarrow z_0} (z-z_0) f(z)$$

(b) when $f(z) = \frac{P(z)}{Q(z)}$ has simple pole
at $z=z_0$ and $P(z_0) \neq 0$

then

$$\operatorname{Res}_{z=z_0} f(z) = \frac{P(z_0)}{Q'(z_0)}$$

2) Residue at pole of order m:-

Let

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \dots + \frac{b_m}{(z-z_0)^m}$$

multiply by $(z-z_0)^m$

$$(z-z_0)^m \cdot f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^{m+n} + \frac{b_1}{(z-z_0)^{1-m}} + \frac{b_2}{(z-z_0)^{2-m}} + \dots + b_m$$

$$(z-z_0)^m f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^{m+n} + b_1 (z-z_0)^{m-1} + b_2 (z-z_0)^{m-2} + b_3 (z-z_0)^{m-3} + \dots + b_m$$

differentiating $(m-1)$ times and taking $\lim_{z \rightarrow z_0}$

$$\lim_{z \rightarrow z_0} \frac{d^{m-1}}{dz^{m-1}} [(z-z_0)^m f(z)] = 0 + (m-1)! b_1$$

$$b_1 = \frac{1}{(m-1)!} \left[\frac{d^{m-1}}{dz^{m-1}} \right]$$

$$b_1 = \frac{1}{(m-1)!} \left[\lim_{z \rightarrow z_0} \left\{ \frac{d^{m-1}}{dz^{m-1}} (z-z_0)^m f(z) \right\} \right]$$

For $m=1$

$$b_1 = \lim_{z \rightarrow z_0} [(z-z_0) f(z)]$$

$m=2$

$$b_1 = \lim_{z \rightarrow z_0} \frac{d}{dz} [(z-z_0)^2 f(z)]$$

Note:- when $z=z_0$ is an essential singularity of $f(z)$ then residue is obtained by expanding $f(z)$ about z_0 in Laurent's series & identifying b_1 (coeff. of $\frac{1}{z-z_0}$)

Examples

Q:- Find Residue at $z=0$ of $\frac{\sin z}{z^2}$

Soln

$$f(z) = \frac{\sin z}{z^2}$$

$$f(z) = \frac{1}{z^2} \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \dots \right)$$

$$f(z) = \frac{1}{z} - \frac{z}{3!} + \frac{z^3}{5!} - \frac{z^5}{7!} + \dots$$

$z=0$ is pole of order 1

$$\boxed{\text{Res } f(z) = 1 \atop z=0}$$

! - Determine the poles of the function $f(z) = \frac{z^2}{(z-1)^2(z+2)}$ and residue at each pole.

Ans:- $f(z) = \frac{z^2}{(z-1)^2(z+2)}$

$f(z)$ is not analytic at $z=1, z=-2$

$z=1$ is a pole of order 2.

$z=-2$ is a pole of order 1

Residue at $z = -2$ $z = -2$ is a simple pole

$$\text{Residue } f(z) = \lim_{z \rightarrow z_0} (z - z_0) f(z)$$

$$= \lim_{z \rightarrow -2} (z+2) \cdot \frac{z^2}{(z-1)^2 (z+2)}$$

$$= \lim_{z \rightarrow -2} \frac{z^2}{(z-1)^2} = \frac{(-2)^2}{(-2-1)^2} = \frac{4}{9}$$

$$\text{Res. } f(z) = \frac{4}{9}$$

$$z = -2$$

Residue at $z = 1$ $z = 1$ is a pole of order 2

$$\text{Res. } f(z) = \lim_{z \rightarrow z_0} \frac{d}{dz} ((z - z_0)^2 f(z))$$

$$= \lim_{z \rightarrow 1} \frac{d}{dz} ((z-1)^2 f(z))$$

$$= \lim_{z \rightarrow 1} \frac{d}{dz} \left[\frac{z^2}{(z-1)^2 (z-2)} \right]$$

$$= \lim_{z \rightarrow 1} \left[\frac{d}{dz} \left(\frac{z^2}{z-2} \right) \right]$$

$$= \lim_{z \rightarrow 1} \frac{z^2 + 4z}{(z+2)^2} = \frac{5}{9}$$

$$\boxed{\text{Res. } f(z) = \frac{5}{9} \text{ at } z=1}$$

Q:- Find residue of $\frac{1}{(z^2+1)^3}$ at pole $z=i$.

Solⁿ:-

$$f(z) = \frac{1}{(z^2+1)^3}$$

$f(z)$ is not analy. at $(z^2+1)^3 = 0$

$$(z^2 - i^2)^3 = 0$$

$$(z+i)^3 (z-i)^3 = 0$$

$z=i$ is pole of order 3

$z=-i$ is pole of order 3.

Residue at $z=i$:

$$\text{Res}(z=i) = \frac{1}{2!} \frac{d^2}{dz^2} \left((z-i)^3 \cdot \frac{1}{(z-i)^3 (z+i)^3} \right)$$

$$= \frac{1}{2!} \frac{d^2}{dz^2} \left[\frac{1}{(z+i)^3} \right]$$

$$= \frac{1}{2!} \frac{d}{dz} \left[\frac{-3}{(z+i)^4} \right]$$

$$= \frac{1}{2!} \lim_{z \rightarrow i} \left(\frac{12}{(z+i)^5} \right)$$

$$= \frac{1}{2!} \cdot \frac{12}{(2i)^5} = \frac{6}{32 i^5} = -\frac{6}{32 i} \quad (i^4 = 1)$$

$$\boxed{\text{Res. } f(z)_{z=i} = -\frac{3i}{16}}$$

Q: Determine residue at pole $z=i$ for

$$f(z) = \frac{9z+i}{z(z^2+1)}$$

Solⁿ:- $f(z)$ is not analytic at $z=0, z=i, -i$

~~$z=i$~~ $z=0, i, -i$ are simple poles

$$\text{Res at } z=i = \lim_{z \rightarrow i} \frac{(z/i) (9z+i)}{z(z+i)(z/i)}$$

$$= \lim_{z \rightarrow i} \frac{9z+i}{z(z+i)}$$

$$= \frac{10i}{2i^2} = -5i$$

Q: or using (b)

$$p(z) = 9z+i \quad q(z) = z(z^2+1)$$

$$z_0 = i \quad p(z_0) \neq 0, \quad q(z_0) = 0$$

$$\text{Res } f(z)_{z=i} = \frac{p(z_0)}{q'(z_0)} = \frac{(9z+i)}{(3z^2+1)}_{z=i}$$

$$= -5i$$

Q:- Determine residue of $f(z) = \frac{50z}{(z+4)(z-1)^2}$

Hint:- $z=1$ is pole of order 2

$$\boxed{\text{Res } f(z) = 8 \text{ at } z=1}$$

Q:- Find residue of $f(z) = \frac{(z+1)^3}{(z-1)^3}$ at its pole.

Ans:- $z=1$ is a pole of order 3.

$$\begin{aligned} * f(z) &= \left(\frac{z+1}{z-1} \right)^3 = \left(1 + \frac{2}{z-1} \right)^3 \\ &= 1 + \frac{8}{(z-1)^3} + \frac{6}{z-1} + \frac{24}{(z-1)^2} \end{aligned}$$

$z=1$ is pole of order 3

residue at $z=1$ is 6

Q:- Find residue of $f(z) = \frac{z^2 - 2z}{(z+1)^2(z^2+4)}$ at each of its pole.

soln

$f(z)$ is not analytic at $(z+1)^2(z^2+4)=0$

$$z = -1, \quad z = +2i, \quad z = -2i$$

$f(z)$ has simple pole at $z = 2i, -2i$
 $f(z)$ has pole of order 2 at $z = -1$

(a) Residue at $z = 2i$ (simple pole)

$$\text{Res } f(z) = \lim_{z \rightarrow z_0} (z - z_0) f(z)$$

$$= \lim_{z \rightarrow 2i} \cancel{(z - 2i)} \frac{z^2 - 2z}{(z+1)^2 \cancel{(z-2i)}(z+2i)}$$

$$= \lim_{z \rightarrow 2i} \frac{z^2 - 2z}{(z+1)^2 (z+2i)}$$

$$= \frac{-4 - 4i}{(-2i+1)^2 (4i)} = \frac{-4 - 4i}{(-4+1+4i)(4i)}$$

$$= \frac{-4(1+i)}{(-3+4i)4i}$$

$$= \frac{i(1+i)}{-3+4i}$$

$$= \frac{i(1+i)(-3-4i)}{(-3+4i)(-3-4i)}$$

$$= \frac{(i-1)(-3-4i)}{9+16} = \frac{-3i+4+3+4i}{25}$$

$$\text{Res } f(z)_{z=2i} = \frac{7+i}{25}$$

(b) Residue at $z = -2i$
Ans $\Rightarrow \frac{7-i}{25}$

(c) Residue at $z = -1$

$$\begin{aligned} &= \frac{1}{2-i} \lim_{z \rightarrow -1} \frac{d}{dz} \left((z+1)^2 f(z) \right) \\ &= 1 \cdot \lim_{z \rightarrow -1} \frac{d}{dz} \left(\frac{z^2 - 2z}{z^2 + 4} \right) \end{aligned}$$

$$\boxed{\text{Res } f(z) = \frac{-14}{25} \text{ at } z = -1}$$

Q! Find residue at poles for $\frac{e^z}{z^2 + \pi^2}$

Soln $f(z)$ is not analytic

$$z^2 + \pi^2 = 0$$

$$z = \pi i, -\pi i$$

a) Residue at $z = \pi i$:-

$$\text{Res. } f(z) = \lim_{z \rightarrow \pi i} (z - \pi i) f(z)$$

$$= \lim_{z \rightarrow \pi i} \cancel{(z - \pi i)} \frac{e^z}{(z + \pi i) \cancel{(z - \pi i)}}$$

$$\text{Res}_{z = \pi i} f(z) = \frac{e^{\pi i}}{2\pi i} = \frac{-1}{2\pi i} = \frac{1}{2\pi}$$

(b) Residue at $z = -\pi i$

$$\text{Res } f(z) = \lim_{z \rightarrow -\pi i} (z + \pi i) \frac{e^z}{(z + \pi i)(z - \pi i)}$$

$$= \frac{e^{-\pi i}}{-2\pi i} = \frac{\cos(-\pi) + i \sin(-\pi)}{-2\pi i}$$

$$= \frac{-1}{-2\pi i} = \frac{-i}{2\pi}$$

$$\boxed{\text{Res } f(z) = \frac{-i}{2\pi} \text{ at } z = -\pi i}$$

Other Problems :-

Residue using

$$\left\{ \text{Res } f(z) = \frac{P(z_0)}{Q'(z_0)} \right\} \text{ when}$$

$$f(z) = \frac{P(z)}{Q(z)} \text{ and } z_0 \text{ is}$$

simple pole of $f(z)$, with $P(z_0) \neq 0$, $Q(z_0) = 0$, and $Q'(z_0) \neq 0$. [Note if $P(z_0) \neq 0$, $Q(z_0) = 0$ & $Q'(z_0) \neq 0$ then z_0 is simple po

Q1 Determine poles and residue of $f(z) = \cot z$.

Soln

$$f(z) = \frac{\cos z}{\sin z}$$

singularities are $\sin z = 0$
 $z = n\pi, n = 0, \pm 1, \dots$

Here $z_0 = n\pi$ are simple poles as

$$P(z) = \cos z \quad Q(z) = \sin z$$

$$P(z_0) \neq 0, \quad Q(z_0) = 0, \quad Q'(z_0) \neq 0$$

$$\operatorname{Res}_{z=n\pi} f(z) = \frac{P(z_0)}{Q'(z_0)} = \frac{\cos(n\pi)}{\cos(n\pi)} = 1$$

$$\boxed{\operatorname{Res}_{z=n\pi} f(z) = 1}$$

$$(2) \quad f(z) = \frac{z}{\cos z}$$

$f(z)$ is not defined at
 $\cos z = 0$

$$z_0 = (2n+1)\frac{\pi}{2}, \quad n=0, \pm 1, \dots$$

$$P(z) = z, \quad P(z_0) \neq 0$$

$$Q(z) = \cos z, \quad Q(z_0) = \cos\left((2n+1)\frac{\pi}{2}\right) = 0$$

$$Q'(z_0) = -\sin z_0 \neq 0$$

i.e. $z_0 = (2n+1)\frac{\pi}{2}$ are simple pole

$$\operatorname{Res}_{z=z_0} f(z) = \frac{P(z_0)}{Q'(z_0)} = \frac{z_0}{-\sin z_0} =$$

$$\text{Res } f(z) = \frac{(2n+1)\pi}{2} - \sin\left(\frac{(2n+1)\pi}{2}\right), \quad n=0, \pm 1, \dots$$

(B) $f(z) = \frac{z}{\sin z}$

$$P(z) = z, \quad Q(z) = \sin z$$

$f(z)$ is not defined for $z_0 = n\pi$
 $n=0, \pm 1, \pm 2, \dots$

$$P(z_0) = n\pi \neq 0$$

$$Q(z_0) = 0$$

$$Q'(z_0) = \cos n\pi \neq 0$$

i.e. $z_0 = n\pi$ are simple poles.

$$\lim_{z \rightarrow z_0} f(z) = \lim_{z \rightarrow n\pi} \frac{P(z)}{Q(z)}$$

$$= \frac{P(z_0)}{Q'(z_0)} = \frac{n\pi}{\cos n\pi} = \frac{n\pi}{(-1)^n}$$

Q:- Residue of $\frac{e^{2z}}{1+e^z}$

Soln

$f(z)$ is not defined for

$$1 + e^z = 0$$

$$e^z = -1$$

$$e^z = e^{(2n+1)\pi i}, \quad n=0, \pm 1, \dots, \infty$$

$$z_0 = (2n+1)\pi i$$

$$P(z) = e^{2z}$$

$$P(z_0) = e^{2(2n+1)\pi i} = e^{(4n+2)\pi i} \neq 0$$

$$Q(z_0) = 0$$

$$Q'(z_0) = e^{(2n+1)\pi i} = -1$$

i.e. $z_0 = (2n+1)\pi i$ all are simple pole.

$$\text{Res}_{z=z_0} \frac{P(z)}{Q(z)} = \frac{P(z_0)}{Q'(z_0)} = \frac{1}{-1}$$

$$\boxed{\text{Res}_{z=z_0} \frac{P(z)}{Q(z)} = -1}$$

Q:- Find Residue of $\frac{1+e^z}{\sin z + z \cos z}$ at $z=0$.

Q:- Find Residue of $f(z) = \frac{e^{2z}}{(z-1)^2}$ at poles

Ans $2e^2$

Q:- Find Residue of $f(z) = \frac{\tan z}{z}$

Hint:- Singularities $z=0$ is removable.

g. $z=0, (2n+1)\frac{\pi}{2}$ $\star$ Res $f(z) = 0$

Find residue at $(2n+1)\frac{\pi}{2}$

Find residue of $f(z) = \frac{\cos z}{z^4}$ at its poles

$$\cos z = \frac{\cos z}{z^4 \sin z}$$

$f(z)$ is not defined for

$$z^4 \sin z = 0$$

$$z^4 = 0, \quad z = n\pi \quad n \in \mathbb{Z}$$

$z = 0, 0, 0, 0, 0$, is a pole of order 5
 $z = n\pi, \quad n = \pm 1, \pm 2, \dots$ are pole of order 1

at $z_0 = n\pi$ (simple pole)

$$P(n\pi) = (-1)^n, \quad Q(n\pi) = 0$$

$$Q'(n\pi) = (n\pi)^4 \cos n\pi + 4(n\pi)^3 \sin n\pi \neq 0$$

$$\text{Res } f(z)_{z=n\pi} = \frac{P(n\pi)}{Q'(n\pi)} = \frac{(-1)^n}{(n\pi)^4 (-1)^n} = \frac{1}{n^4 \pi^4}$$

at $z = 0$, pole of order 5

$$f(z) = \frac{\cos z}{z^4 \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} + \dots \right)}$$

$$f(z) = \frac{1}{z^5} \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \dots \right) \left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} + \dots \right)^{-1}$$

Residue at $z = 0$, coeff of $\frac{1}{z}$

$$\left[\text{Res } f(z)_{z=0} = -\frac{1}{20} \right]$$

Q:- Find the poles of $f(z) = \frac{1}{z^4 + 1}$

Soln:-

$$f(z) = \frac{1}{1 + z^4}$$

Poles of $f(z)$ are the points where

$$z^4 + 1 = 0$$

$$z^4 = -1$$

$$z^4 = \cos \pi + i \sin \pi$$

$$z^4 = e^{i(\pi + 2n\pi)}, \quad n = 0, 1, 2, 3,$$

$$z = e^{i\left(\frac{\pi}{4} + \frac{2n\pi}{4}\right)}, \quad n = 0, 1, 2, 3$$

$$z = e^{i\pi/4}, e^{i3\pi/4}, e^{i5\pi/4}, \dots \infty$$

all are simple poles.

Residue at $z = e^{i\pi/4}$

$$\text{Res } f(z) = \lim_{z \rightarrow z_1} (z - z_1) \cdot f(z)$$

$$= \lim_{z \rightarrow e^{i\pi/4}} (z - e^{i\pi/4}) \frac{1}{z^4 + 1}$$

$$= \lim_{z \rightarrow e^{i\pi/4}} \frac{1}{4z^3} = \frac{1}{4(e^{i\pi/4})^3}$$

$$= \frac{1}{4e^{3i\pi/4}} = \frac{e^{-3i\pi/4}}{4}$$

$$= \frac{1}{4} \left(\cos \frac{3\pi}{4} - i \sin \frac{3\pi}{4} \right)$$

$$= \frac{1}{4} \left(-\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}} \right)$$

$$= \frac{-1-i}{4\sqrt{2}}$$

Residue at $z_2 = e^{i3\pi/4}$

$$\text{Res } f(z) = \lim_{z \rightarrow z_2} (z - z_2) \cdot \frac{1}{z^4 + 1}$$

$$= \lim_{z \rightarrow e^{i3\pi/4}} \frac{1}{4z^3} = \frac{1}{4} e^{-9\pi i/4}$$

$$= \frac{1}{4} e^{-i\pi/4} = \frac{1-i}{4\sqrt{2}}$$

$$\left[e^{9\pi i/4} = e^{i\pi/4} \right]$$

Residue at $z_3 = e^{i5\pi/4}$