VIT

Vellore Institute of Technology

(Deemed to be University under section 3 of UGC Act, 1956)

SCHOOL OF ADVANCED SCIENCES
DEPARTMENT OF MATHEMATICS
CONTINUOUS ASSESSMENT TEST – II
FALL SEMESTER 2023-2024

Programme Name : B. Tech
Course Code : BMAT201L
Course Name : Complex Variables and Linear Algebra
Slot : C2+TC2+TCC2 (Common Question Paper)
Exam Duration: 90 minutes **Maximum Marks: 50**

General instruction(s): Answer all the following questions. **5 x 10 = 50 Marks**

Q.No	Question	Marks	(CO)	(BL)
1.	Evaluate $\int_0^{2\pi} \frac{1+2\cos\theta}{5+4\cos\theta} d\theta$ using contour integration.	10	3	3
2.	Find the eigen values and the corresponding eigen vectors of the matrix $A=\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$. Also, find the eigen values of A^T and A^{-1}.	10	5	2
3.	Apply the Gauss-Jordan method to solve the following system of equations. $2x+5y-3z=-10; -2x-2y+2z=2; -x-5y+z=17$.	10	5	3
4.	(a) Verify whether the set $H=\left\{\begin{bmatrix} 2a & b \\ 3a+b & 3b \end{bmatrix}, where\, a,b \in R\right\}$ forms a subspace of a vector space $M_{2X2}(R)$. (b)Verify whether the set $S=\{(2,1,4),(1,-1,2),(3,1,-2)\}$ forms a basis of a vector space $R^3(R)$.	10	4	4
5.	Find the dimension of the null space of $A=\begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -2 & 0 \\ 4 & 2 & 0 & 0 & 3 \\ 1 & 1 & 1 & -2 & 1 \\ 2 & 2 & 0 & 0 & 2 \\ 1 & 1 & 2 & -4 & 1 \end{bmatrix}$	10	4	2

Fall sem. 2023-24

BMAT201L-CAT-II Key

C_2-slot

---

(1) Let $z = e^{i\theta} \Rightarrow d\theta = \frac{dz}{iz}$ and $\cos\theta = \frac{z^2+1}{2z}$

$$\therefore \int_0^{2\pi} \frac{1+2\cos\theta}{5+4\cos\theta}\, d\theta = \frac{1}{i}\int_C \frac{z^2+z+1}{z(2z^2+5z+2)}\, dz$$

here C is $|z|=1$ and $z=0,-2,-1/2$ are simple poles. Also, $z=0,-1/2$ are inside C.

$\therefore$ By Cauchy's Residue theorem

$$\int_0^{2\pi} \frac{1+2\cos\theta}{5+4\cos\theta}\, d\theta = \frac{1}{i}\cdot 2\pi i\,[\text{Sum of Res.}]$$

$$= 2\pi\left[\frac{1}{2} - \frac{1}{2}\right] = 0$$

(2) Let $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

the characteristic eqn. of A: $|A - \lambda I| = 0$

$$\Rightarrow \lambda^3 - 3\lambda - 2 = 0$$

therefore, the eigen Values of A and A^T are:

$$\lambda = 2, -1, -1$$

$\boxed{\text{E.values of } A^{-1} \text{ are } 1/2, -1, -1}$

eigen vector for $\lambda = 2$: $(A - 2I)X = 0$

$$\Rightarrow X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

eigen vectors for $\lambda = -1$: $(A + I)X = 0$, when $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$\Rightarrow x_1 + x_2 + x_3 = 0$ Let $x_1 = k_1$ and $x_2 = k_2$

$\therefore x_3 = -k_1 - k_2$ and $X = k_1\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + k_2\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

$\Rightarrow X_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, $X_3 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

3) Given $2x+5y-3z=-10$; $-2x-2y+2z=2$;
$-x-5y+z=17$

Augmented matrix of the system:

$$\left[\begin{array}{ccc|c} 2 & 5 & -3 & -10 \\ -2 & -2 & 2 & 2 \\ -1 & -5 & 1 & 17 \end{array}\right]$$

after using row operations, the resulting row equivalent matrix is

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -4 \end{array}\right]$$

$\therefore x=-1,\ y=-4,\ z=-4$

4(a) Given $H = \left\{ \begin{bmatrix} 2a & b \\ 3a+b & 3b \end{bmatrix}, \text{ where } a, b \in \mathbb{R} \right\}$

since $\begin{bmatrix} 2a & b \\ 3a+b & 3b \end{bmatrix} = \begin{bmatrix} 2a & 0 \\ 3a & 0 \end{bmatrix} + \begin{bmatrix} 0 & b \\ b & 3b \end{bmatrix}$

$= a\begin{pmatrix} 2 & 0 \\ 3 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 1 & 3 \end{pmatrix}$

$H = \text{Span}\left\{ \begin{bmatrix} 2 & 0 \\ 3 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 3 \end{bmatrix} \right\}$

$\therefore$ H is a subspace of $M_{2\times 2}$

(or)

Let $c_1, c_2 \in \mathbb{R}$ and $A_1 = \begin{bmatrix} 2a_1 & b_1 \\ 3a_1+b_1 & 3b_1 \end{bmatrix}$, $A_2 = \begin{bmatrix} 2a_2 & b_2 \\ 3a_2+b_2 & 3b_2 \end{bmatrix} \in H$ where $a_1, b_1, a_2, b_2 \in \mathbb{R}$

then $c_1A_1 + c_2A_2 = \begin{bmatrix} 2(a_1c_1+a_2c_2) & (c_1b_1+c_2b_2) \\ 3(a_1c_1+a_2c_2)+(b_1c_1+b_2c_2) & 3(b_1c_1+b_2c_2) \end{bmatrix}$
$\in H$.

4(b) Let $S = \{(2,1,4),(1,-1,2),(3,1,-2)\}$ 3

for $a, b, c \in R$, such that

$$a(2,1,4)+b(1,-1,2)+c(3,1,-2)=0$$

$$\Rightarrow \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 4 & 2 & -2 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow a=b=c=0$$

$\therefore$ S is L.I.

Let $(x,y,z) \in R^3$

$$(x,y,z) = a(2,1,4)+b(1,-1,2)+c(3,1,-2)$$

$$\Rightarrow \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 4 & 2 & -2 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$R_2 \to 2R_2 - R_1$

$R_3 \to R_3 - 2R_1$

$$\sim \begin{bmatrix} 2 & 1 & 3 \\ 0 & -3 & -1 \\ 0 & 0 & -8 \end{bmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} x \\ 2y-x \\ z-2x \end{pmatrix}$$

$R_2 \to \frac{R_2}{-1}$

$R_3 \to \frac{R_3}{-8}$

$$\sim \begin{pmatrix} 2 & 1 & 3 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} x \\ x-2y \\ \frac{2x-z}{8} \end{pmatrix}$$

$$\left.\begin{aligned} &\Rightarrow c = \frac{2x-z}{8},\ b = \frac{8}{3}\left(\frac{x-2y}{2x-z}\right) \\ &a = \frac{x}{2} - \frac{4}{3}\left(\frac{x-2y}{2x-z}\right) - \frac{3}{2}\left(\frac{2x-z}{8}\right) \end{aligned}\right\} \in R$$

$\therefore L(S) = R^3$ and S forms a basis of R^3

(5) Given $A = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -2 & 0 \\ 4 & 2 & 0 & 0 & 3 \\ 1 & 1 & 1 & -2 & 1 \\ 2 & 2 & 0 & 0 & 2 \\ 1 & 1 & 2 & -4 & 1 \end{bmatrix}$

to find a basis of null space of A, consider

$$Ax = 0$$

using row operations we get the following form:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 0 & -1/2 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ \cancel{x_6} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow$ $x_1 + \frac{x_5}{2} = 0$

$x_2 - \frac{x_5}{2} = 0$

$x_3 - 2x_4 = 0$

for $x_4 = k_1$ and $x_5 = k_2$

$$\therefore X = k_1 \begin{pmatrix} 0 \\ 0 \\ 2 \\ 1 \\ 0 \end{pmatrix} + k_2 \begin{pmatrix} -\frac{1}{2} \\ -1/2 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\therefore N(A) = \text{Span}\left[\begin{pmatrix} 0 \\ 0 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1/2 \\ -1/2 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right]$$

$\therefore$ dim of $N(A) = 2$