DEFINITIONS The **tangent plane** at the point $P_0(x_0, y_0, z_0)$ on the level surface $f(x, y, z) = c$ of a differentiable function f is the plane through P_0 normal to $\nabla f|_{P_0}$.

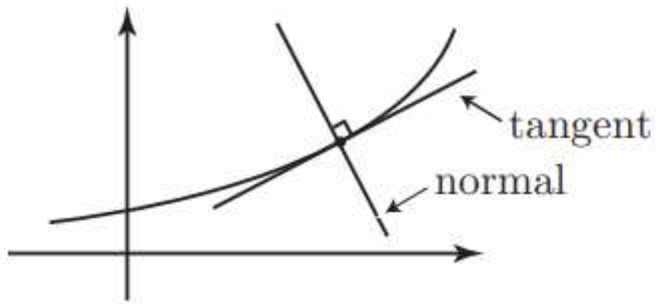
The **normal line** of the surface at P_0 is the line through P_0 parallel to $\nabla f|_{P_0}$.

Tangent Plane to $f(x, y, z) = c$ at $P_0(x_0, y_0, z_0)$

$$f_x(P_0)(x - x_0) + f_y(P_0)(y - y_0) + f_z(P_0)(z - z_0) = 0$$

Normal Line to $f(x, y, z) = c$ at $P_0(x_0, y_0, z_0)$

$$x = x_0 + f_x(P_0)t, \qquad y = y_0 + f_y(P_0)t, \qquad z = z_0 + f_z(P_0)t$$



The normal is a line at right angles to the tangent.

EXAMPLE Find the tangent plane and normal line of the surface

$$f(x, y, z) = x^2 + y^2 + z - 9 = 0 \qquad \text{A circular paraboloid}$$

at the point $P_0(1, 2, 4)$.

Solution

The tangent plane is the plane through P_0 perpendicular to the gradient of f at P_0. The gradient is

$$\nabla f|_{P_0} = (2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k})_{(1,2,4)} = 2\mathbf{i} + 4\mathbf{j} + \mathbf{k}.$$

The tangent plane is therefore the plane

$$2(x - 1) + 4(y - 2) + (z - 4) = 0, \qquad \text{or} \qquad 2x + 4y + z = 14.$$

The line normal to the surface at P_0 is

$$x = 1 + 2t, \qquad y = 2 + 4t, \qquad z = 4 + t.$$

Plane Tangent to a Surface $z = f(x, y)$ at $(x_0, y_0, f(x_0, y_0))$

The plane tangent to the surface $z = f(x, y)$ of a differentiable function f at the point $P_0(x_0, y_0, z_0) = (x_0, y_0, f(x_0, y_0))$ is

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0.$$

EXAMPLE Find the plane tangent to the surface $z = x \cos y - ye^x$ at $(0, 0, 0)$.

Solution We calculate the partial derivatives of $f(x, y) = x \cos y - ye^x$ and Then

$$f_x(0, 0) = (\cos y - ye^x)_{(0,0)} = 1 - 0 \cdot 1 = 1$$

$$f_y(0, 0) = (-x \sin y - e^x)_{(0,0)} = 0 - 1 = -1.$$

The tangent plane is therefore

$$1 \cdot (x - 0) - 1 \cdot (y - 0) - (z - 0) = 0,$$

or

$$x - y - z = 0.$$

Prove that div $(r^n\mathbf{R}) = (n+3)\,r^n$. *Hence show that* $\mathbf{R}/r^3$ *is solenoidal.*

We have $\mathbf{R} = x\mathbf{I} + y\mathbf{J} + z\mathbf{K}$ and $r = \sqrt{(x^2+y^2+z^2)}$

$$\therefore \quad div\,(r^n\mathbf{R}) = \nabla \,.\, (x^2+y^2+z^2)^{n/2}\,(x\mathbf{I} + y\mathbf{J} + z\mathbf{K})$$

$$= \frac{\partial}{\partial x}[x(x^2+y^2+z^2)^{n/2}] + \frac{\partial}{\partial y}[y(x^2+y^2+z^2)^{n/2}] + \frac{\partial}{\partial z}[z(x^2+y^2+z^2)^{n/2}]$$

$$= \Sigma\left\{1\,.\,(x^2+y^2+z^2)^{n/2} + x\,.\,\frac{n}{2}(x^2+y^2+z^2)^{\frac{n}{2}-1}\,.\,2x\right\}$$

$$= \Sigma r^n + n\,\Sigma x^2 (x^2+y^2+z^2)^{\frac{n}{2}-1} = 3r^n + nr^2\,.\,r^{n-2}$$

Thus $div\,(r^n\mathbf{R}) = (n+3)\,r^n$

When $n = -3$, $div\,(\mathbf{R}/r^3) = 0$ *i.e.*, $\mathbf{R}/r^3$ is solenoidal.

vector identities

(1) $\textit{div grad } f = \nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$

(2) $\textit{curl grad } f = \nabla \times \nabla f = \mathbf{0}$

(3) $\textit{div curl } \mathbf{F} = \nabla \cdot \nabla \times \mathbf{F} = 0$

(4) $\textit{curl curl } \mathbf{F} = \textit{grad div } \mathbf{F} - \nabla^2 \mathbf{F}$, *i.e.*, $\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2 \mathbf{F}$

(5) $\textit{grad div } \mathbf{F} = \textit{curl curl } \mathbf{F} + \nabla^2 \mathbf{F}$, *i.e.*, $\nabla(\nabla \cdot \mathbf{F}) = \nabla \times (\nabla \times \mathbf{F}) + \nabla^2 \mathbf{F}$.

Proofs. **(1)** $\nabla^2 f = \nabla \cdot \nabla f = \nabla \cdot \left(\mathbf{I}\frac{\partial f}{\partial x} + \mathbf{J}\frac{\partial f}{\partial y} + \mathbf{K}\frac{\partial f}{\partial z}\right)$

$$= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z}\right) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) f$$

$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is called the *Laplacian operator* and $\nabla^2 f = 0$ is called the *Laplace's equation*.

(2) $\nabla \times \nabla f = \nabla \times \left(\mathbf{I}\frac{\partial f}{\partial x} + \mathbf{J}\frac{\partial f}{\partial y} + \mathbf{K}\frac{\partial f}{\partial z}\right) = \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} = \Sigma \mathbf{I}\left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}\right) = \mathbf{0}$

$$(3)\quad \nabla \cdot \nabla \times \mathbf{F} = \left(\Sigma \mathbf{I}\,\frac{\partial}{\partial x}\right) \cdot \left(\mathbf{I} \times \frac{\partial \mathbf{F}}{\partial x} + \mathbf{J} \times \frac{\partial \mathbf{F}}{\partial y} + \mathbf{K} \times \frac{\partial \mathbf{F}}{\partial z}\right)$$

$$= \Sigma \mathbf{I} \cdot \left(\mathbf{I} \times \frac{\partial^2 \mathbf{F}}{\partial x^2} + \mathbf{J} \times \frac{\partial^2 \mathbf{F}}{\partial x \partial y} + \mathbf{K} \times \frac{\partial^2 \mathbf{F}}{\partial x \partial z}\right)$$

$$= \Sigma \left(\mathbf{I} \times \mathbf{I} \cdot \frac{\partial^2 \mathbf{F}}{\partial x^2} + \mathbf{I} \times \mathbf{J} \cdot \frac{\partial^2 \mathbf{F}}{\partial x \partial y} + \mathbf{I} \times \mathbf{K} \cdot \frac{\partial^2 \mathbf{F}}{\partial x \partial z}\right) = \Sigma \left(\mathbf{K} \cdot \frac{\partial^2 \mathbf{F}}{\partial x \partial y} - \mathbf{J} \cdot \frac{\partial^2 \mathbf{F}}{\partial x \partial z}\right) = 0.$$

$$(4)\quad \nabla \times (\nabla \times \mathbf{F}) = \left(\Sigma \mathbf{I}\,\frac{\partial}{\partial x}\right) \times \left(\mathbf{I} \times \frac{\partial \mathbf{F}}{\partial x} + \mathbf{J} \times \frac{\partial \mathbf{F}}{\partial y} + \mathbf{K} \times \frac{\partial \mathbf{F}}{\partial z}\right)$$

$$= \Sigma \mathbf{I} \times \left(\mathbf{I} \times \frac{\partial^2 \mathbf{F}}{\partial x^2} + \mathbf{J} \times \frac{\partial^2 \mathbf{F}}{\partial x \partial y} + \mathbf{K} \times \frac{\partial^2 \mathbf{F}}{\partial x \partial z}\right)$$

$$= \Sigma \left[\left\{\left(\mathbf{I} \cdot \frac{\partial^2 \mathbf{F}}{\partial x^2}\right)\mathbf{I} - (\mathbf{I} \cdot \mathbf{I})\frac{\partial^2 \mathbf{F}}{\partial x^2}\right\} + \left\{\left(\mathbf{I} \cdot \frac{\partial^2 \mathbf{F}}{\partial x \partial y}\right)\mathbf{J} - (\mathbf{I} \cdot \mathbf{J})\frac{\partial^2 \mathbf{F}}{\partial x \partial y}\right\} + \left\{\left(\mathbf{I} \cdot \frac{\partial^2 \mathbf{F}}{\partial x \partial z}\right)\mathbf{K} - (\mathbf{I} \cdot \mathbf{K})\frac{\partial^2 \mathbf{F}}{\partial x \partial z}\right\}\right]$$

$$= \Sigma \left[\left(\mathbf{I} \cdot \frac{\partial^2 \mathbf{F}}{\partial x^2}\right)\mathbf{I} + \left(\mathbf{I} \cdot \frac{\partial^2 \mathbf{F}}{\partial x \partial y}\right)\mathbf{J} + \left(\mathbf{I} \cdot \frac{\partial^2 \mathbf{F}}{\partial x \partial z}\right)\mathbf{K}\right] - \Sigma \frac{\partial^2 \mathbf{F}}{\partial x^2}$$

$$= \Sigma \mathbf{I}\,\frac{\partial}{\partial x}\left(\mathbf{I} \cdot \frac{\partial \mathbf{F}}{\partial x} + \mathbf{J} \cdot \frac{\partial \mathbf{F}}{\partial y} + \mathbf{K} \cdot \frac{\partial \mathbf{F}}{\partial z}\right) - \Sigma \frac{\partial^2 \mathbf{F}}{\partial x^2} = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2 \mathbf{F}.$$

To prove that

(1) grad $(fg) = f\,(\text{grad}\ g) + g(\text{grad}\ f)$ *i.e.* $\nabla(fg) = f\,\nabla g + g\nabla f.$

(2) *div* $(f\,\mathbf{G}) = (\text{grad}\ f)\ .\ \mathbf{G} + f\,(\text{div}\ \mathbf{G})$ *i.e.* $\nabla(f\,\mathbf{G}) = \nabla f\ .\ \mathbf{G} + f\,\nabla\ .\ \mathbf{G}$

(3) *curl* $(f\,\mathbf{G}) = (\textit{grad}\ f) \times \mathbf{G} + f\,(\textit{curl}\ \mathbf{G})$ *i.e.* $\nabla \times (f\,\mathbf{G}) = \nabla f \times \mathbf{G} + f\,\nabla \times \mathbf{G}$

(4) *grad* $(\mathbf{F}.\ \mathbf{G}) = (\mathbf{F}\ .\ \nabla)\ \mathbf{G} + (\mathbf{G}\ .\ \nabla)\ \mathbf{F} + \mathbf{F} \times \textit{curl}\ \mathbf{G} + \mathbf{G} \times \textit{curl}\ \mathbf{F}$

i.e., $\nabla\,(\mathbf{F}\ .\ \mathbf{G}) = (\mathbf{F}\ .\ \nabla)\ \mathbf{G} + (\mathbf{G}\ .\ \nabla)\ \mathbf{F} + \mathbf{F} \times (\nabla \times \mathbf{G}) + \mathbf{G} \times (\nabla \times \mathbf{F})$

(5) *div* $(\mathbf{F} \times \mathbf{G}) = \mathbf{G}\ .\ (\textit{curl}\ \mathbf{F}) - \mathbf{F}\ .\ (\textit{curl}\ \mathbf{G})$ *i.e.,* $\nabla\ .\ (\mathbf{F} \times \mathbf{G}) = \mathbf{G}\ .\ (\nabla \times \mathbf{F}) - \mathbf{F}\ .\ (\nabla \times \mathbf{G})$

(6) *curl* $(\mathbf{F} \times \mathbf{G}) = \mathbf{F}\,(\textit{div}\ \mathbf{G}) - \mathbf{G}\,(\textit{div}\ \mathbf{F}) + (\mathbf{G}\ .\ \nabla)\ \mathbf{F} - (\mathbf{F}\ .\ \nabla)\ \mathbf{G}$

i.e., $\nabla \times (\mathbf{F} \times \mathbf{G}) = \mathbf{F}(\nabla\ .\ \mathbf{G}) - \mathbf{G}\,(\nabla\ .\ \mathbf{F}) + (\mathbf{G}\ .\ \nabla)\ \mathbf{F} - (\mathbf{F}\ .\ \nabla)\ \mathbf{G}$

Proofs (**1**) $\nabla(fg) = \Sigma \mathbf{I} \,.\, \frac{\partial}{\partial x}(fg) = \Sigma \mathbf{I}\left(f\frac{\partial g}{\partial x} + g\frac{\partial f}{\partial x}\right)$

$$= f\,\Sigma \mathbf{I}\,\frac{\partial g}{\partial x} + g\,\Sigma \mathbf{I}\,\frac{\partial f}{\partial x} = f\,\nabla g + g\nabla f$$

(**2**) $\nabla \,.\, (f\,\mathbf{G}) = \Sigma \mathbf{I} \,.\, \frac{\partial}{\partial x}(f\,\mathbf{G}) = \Sigma \mathbf{I} \,.\, \left(\frac{\partial f}{\partial x}\mathbf{G} + f\,\frac{\partial \mathbf{G}}{\partial x}\right)$

$$= \left(\Sigma \mathbf{I}\,\frac{\partial f}{\partial x}\right) .\, \mathbf{G} + f\left(\Sigma \mathbf{I} \,.\, \frac{\partial \mathbf{G}}{\partial x}\right) = \nabla f \,.\, \mathbf{G} + f\,\nabla \,.\, \mathbf{G}$$

(**3**) $\nabla \times (f\,\mathbf{G}) = \Sigma \mathbf{I} \times \frac{\partial}{\partial x}(f\,\mathbf{G}) = \Sigma \mathbf{I} \times \left(f\,\frac{\partial \mathbf{G}}{\partial x} + \frac{\partial f}{\partial x}\,\mathbf{G}\right)$

$$= f\,\Sigma \mathbf{I} \times f\,\frac{\partial \mathbf{G}}{\partial x} + \Sigma \mathbf{I}\,\frac{\partial f}{\partial x} \times \mathbf{G} = f\,\nabla \times \mathbf{G} + \nabla f \times \mathbf{G}$$

(**4**) $\nabla(\mathbf{F} \,.\, \mathbf{G}) = \Sigma \mathbf{I}\,\frac{\partial}{\partial x}(\mathbf{F} \,.\, \mathbf{G}) = \Sigma \mathbf{I}\left(\frac{\partial \mathbf{F}}{\partial x} \,.\, \mathbf{G} + \mathbf{F} \,.\, \frac{\partial \mathbf{G}}{\partial x}\right) = \Sigma \mathbf{I}\,\frac{\partial \mathbf{F}}{\partial x} \,.\, \mathbf{G} + \Sigma \mathbf{I}\left(\mathbf{F} \,.\, \frac{\partial \mathbf{G}}{\partial x}\right)$ ...(*i*)

Now $\mathbf{G} \times \left(\mathbf{I} \times \frac{\partial \mathbf{F}}{\partial x}\right) = \left(\mathbf{G} \,.\, \frac{\partial \mathbf{F}}{\partial x}\right)\mathbf{I} - (\mathbf{G} \,.\, \mathbf{I})\,\frac{\partial \mathbf{F}}{\partial x}$

or $\left(\mathbf{G} \,.\, \frac{\partial \mathbf{F}}{\partial x}\right)\mathbf{I} = \mathbf{G} \times \left(\mathbf{I} \times \frac{\partial \mathbf{F}}{\partial x}\right) + (\mathbf{G} \,.\, \mathbf{I})\,\frac{\partial \mathbf{F}}{\partial x}$

$\therefore$ $\Sigma\left(\mathbf{G} \,.\, \frac{\partial \mathbf{F}}{\partial x}\right)\mathbf{I} = \mathbf{G} \times \Sigma \mathbf{I} \times \frac{\partial \mathbf{F}}{\partial x} + \Sigma\,(\mathbf{G} \,.\, \mathbf{I})\,\frac{\partial \mathbf{F}}{\partial x} = \mathbf{G} \times (\nabla \times \mathbf{F}) + (\mathbf{G} \,.\, \nabla)\mathbf{F}$ ...(*ii*)

Interchanging **F** and **G**, $\Sigma\left(\mathbf{F} \,.\, \frac{\partial \mathbf{G}}{\partial x}\right)\mathbf{I} = \mathbf{F} \times (\nabla \times \mathbf{G}) + (\mathbf{F} \,.\, \nabla)\,\mathbf{G}$ ...(*iii*)

Substituting in (*i*) from (*ii*) and (*iii*), we get

$$\nabla(\mathbf{F} \,.\, \mathbf{G}) = (\mathbf{F} \,.\, \nabla)\mathbf{G} + (\mathbf{G} \,.\, \nabla)\mathbf{F} + \mathbf{F} \times (\nabla \times \mathbf{G}) + \mathbf{G} \times (\nabla \times \mathbf{F})$$

$$(5) \quad \nabla \cdot (\mathbf{F} \times \mathbf{G}) = \Sigma \mathbf{I} \cdot \frac{\partial}{\partial x}(\mathbf{F} \times \mathbf{G}) = \Sigma \mathbf{I} \left(\frac{\partial \mathbf{F}}{\partial x} \times \mathbf{G} + F \times \frac{\partial \mathbf{G}}{\partial x} \right) = \Sigma \mathbf{I} \cdot \frac{\partial \mathbf{F}}{\partial x} \times \mathbf{G} - \Sigma \mathbf{I} \cdot \left(\frac{\partial \mathbf{G}}{\partial x} \times \mathbf{F} \right)$$

$$= \Sigma \left(I \times \frac{\partial \mathbf{F}}{\partial x} \right) \cdot \mathbf{G} - \Sigma \left(I \times \frac{\partial \mathbf{G}}{\partial x} \right) \cdot \mathbf{F} \qquad [\because \ \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = (A \times B) \cdot C]$$

$$= \mathbf{G} \cdot (\nabla \times \mathbf{F}) - \mathbf{F} \cdot (\nabla \times \mathbf{G})$$

$$(6) \quad \nabla \times (\mathbf{F} \times \mathbf{G}) = \Sigma \mathbf{I} \times \frac{\partial}{\partial x} (\mathbf{F} \times \mathbf{G}) = \Sigma \mathbf{I} \times \left(\frac{\partial \mathbf{F}}{\partial x} \times \mathbf{G} + \mathbf{F} \times \frac{\partial \mathbf{G}}{\partial x} \right)$$

$$= \Sigma \left[(\mathbf{I} \cdot \mathbf{G}) \frac{\partial \mathbf{F}}{\partial x} - \left(\mathbf{I} \cdot \frac{\partial \mathbf{F}}{\partial x} \right) \mathbf{G} \right] + \Sigma \left[\left(\mathbf{I} \cdot \frac{\partial \mathbf{G}}{\partial x} \right) \mathbf{F} - (\mathbf{I} \cdot \mathbf{F}) \frac{\partial \mathbf{G}}{\partial x} \right]$$

$$= \Sigma (\mathbf{G} \cdot \mathbf{I}) \frac{\partial \mathbf{F}}{\partial x} - \mathbf{G} \Sigma \mathbf{I} \cdot \frac{\partial \mathbf{F}}{\partial x} + \mathbf{F} \Sigma \mathbf{I} \cdot \frac{\partial \mathbf{G}}{\partial x} - \Sigma (\mathbf{F} \cdot \mathbf{I}) \frac{\partial \mathbf{G}}{\partial x}$$

$$= \mathbf{F} \left(\Sigma \mathbf{I} \cdot \frac{\partial \mathbf{G}}{\partial x} \right) - \mathbf{G} \Sigma \left(\mathbf{I} \cdot \frac{\partial \mathbf{F}}{\partial x} \right) + \Sigma (\mathbf{G} \cdot \mathbf{I}) \frac{\partial \mathbf{F}}{\partial x} - \Sigma (\mathbf{F} \cdot \mathbf{I}) \frac{\partial \mathbf{G}}{\partial x}$$

$$= \mathbf{F} (\nabla \cdot \mathbf{G}) - \mathbf{G} (\nabla \cdot \mathbf{F}) + (\mathbf{G} \cdot \nabla) \mathbf{F} - (\mathbf{F} \cdot \nabla) \mathbf{G}$$

If $u\mathbf{F} = \nabla v$, where u, v are scalar fields and $\mathbf{F}$ is a vector field, show that $\mathbf{F}$. curl $\mathbf{F} = 0$.

Solution. Since $\mathbf{F} = \frac{1}{u} \nabla v \quad \therefore \quad \text{curl } \mathbf{F} = \nabla \times \left(\frac{1}{u} \nabla v\right)$

or
$$\text{curl } \mathbf{F} = \nabla \frac{1}{u} \times \nabla v + \frac{1}{u} \nabla \times (\nabla v)$$

$$= \nabla \frac{1}{u} \times \nabla v \qquad [\because \nabla \times \nabla v = 0$$

Hence $\mathbf{F}$. curl $\mathbf{F} = \frac{1}{u} \nabla v \cdot \left(\nabla \frac{1}{u} \times \nabla v\right) = 0$, for it is a scalar triple product in which two factors are equal.

Properties of Scalar Triple Product:

i) If the vectors are cyclically permuted,then

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$$

ii) The product is cyclic in nature, i.e,

$$[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = [\mathbf{b}\ \mathbf{c}\ \mathbf{a}] = [\mathbf{c}\ \mathbf{a}\ \mathbf{b}] = -[\mathbf{b}\ \mathbf{a}\ \mathbf{c}] = -[\mathbf{c}\ \mathbf{b}\ \mathbf{a}] = -[\mathbf{a}\ \mathbf{c}\ \mathbf{b}]$$

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$