ECE 2006

Frequency Analysis of Signals and Systems-2 (DFT)

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Discrete-Time Fourier Series (DTFS)

- The (DT) Fourier transform (or spectrum) of x[n] is a representation of a signal in complex exponential sequence, $e^{j\omega n}$,where ω is the real frequency variable

$$X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

- x[n] can be reconstructed from its spectrum using the inverse Fourier transform

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X\left(e^{j\omega}\right) e^{j\omega n}\, d\omega$$

Notation:

$$X\left(e^{j\omega}\right) = \mathcal{F}\{x[n]\}$$
$$x[n] = \mathcal{F}^{-1}\{X\left(e^{j\omega}\right)\}$$

Discrete-Time Fourier Series (DTFS)

- In general DTFT of a sequence x[n] is a continuous function of ω and it can be written in a rectangular form as a combination of real and complex exponential part

$$X\left(e^{j\omega}\right) = X_{re}\left(e^{j\omega}\right) + jX_{im}\left(e^{j\omega}\right)$$

- Here both real and imaginary parts are real function of ω, and again it is given as

$$X_{re}\left(e^{j\omega}\right) = \frac{1}{2}\left\{X\left(e^{j\omega}\right) + X^{*}\left(e^{j\omega}\right)\right\}$$

$$X_{im}\left(e^{j\omega}\right) = \frac{1}{2j}\left\{X\left(e^{j\omega}\right) - X^{*}\left(e^{j\omega}\right)\right\}$$

Here, $X^{*}\left(e^{j\omega}\right)$ denotes the complex conjugate of $X\left(e^{j\omega}\right)$

- The polar form of $X\left(e^{j\omega}\right)$ can be expressed as

$$X\left(e^{j\omega}\right)=\left|X\left(e^{j\omega}\right)\right|e^{j\theta(\omega)}$$

$$\theta(\omega)=\arg\left\{X\left(e^{j\omega}\right)\right\}$$

- The quantity $\left|X\left(e^{j\omega}\right)\right|$ is called the magnitude function and the quantity $\theta(\omega)$ is called the phase function, with both functions again being real functions of ω.

  Again the relationship between the polar and rectangular form are given as

$$X_{re}\left(e^{j\omega}\right)=\left|X\left(e^{j\omega}\right)\right|\cos\theta(\omega)$$

$$X_{im}\left(e^{j\omega}\right)=\left|X\left(e^{j\omega}\right)\right|\sin\theta(\omega)$$

$$\left|X\left(e^{j\omega}\right)\right|^2=X\left(e^{j\omega}\right)X^*\left(e^{j\omega}\right)=X_{re}\left(e^{j\omega}\right)^2+X_{im}\left(e^{j\omega}\right)^2$$

$$\tan\theta(\omega)=\frac{X_{im}\left(e^{j\omega}\right)}{X_{re}\left(e^{j\omega}\right)}$$

Band-Limited Discrete Time Signal

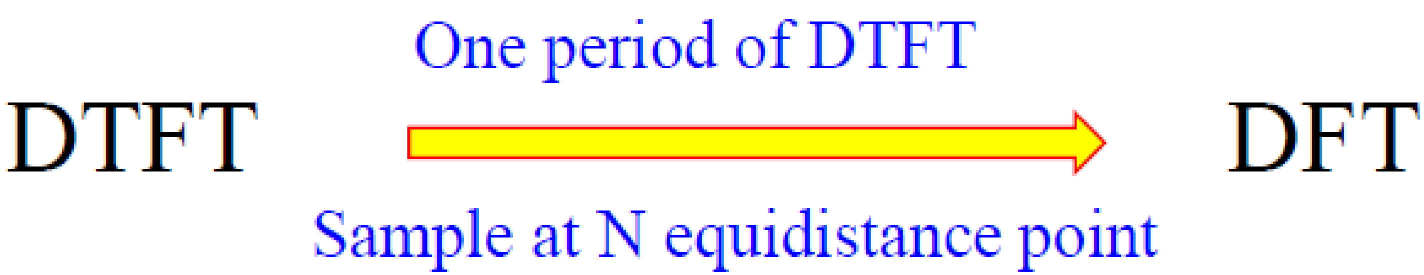
- Since the spectrum of the discrete time signal is a periodic function of ω with a period 2π (i.e., -π ≤ ω ≤ π, full band spectrum). A band limited discrete time signal has a spectrum that is limited to a portion of the above frequency range.
- An ideal band limited signal has a spectrum that is zero outside a finite frequency range, $0 \le \omega_a \le |\omega| \le \omega_b \le \pi$; that is,

$$X\left(e^{j\omega}\right) = \begin{cases} 0, & 0 \le |\omega| \le \omega_a \\ 0, & \omega_b \le |\omega| \le \pi \end{cases}$$

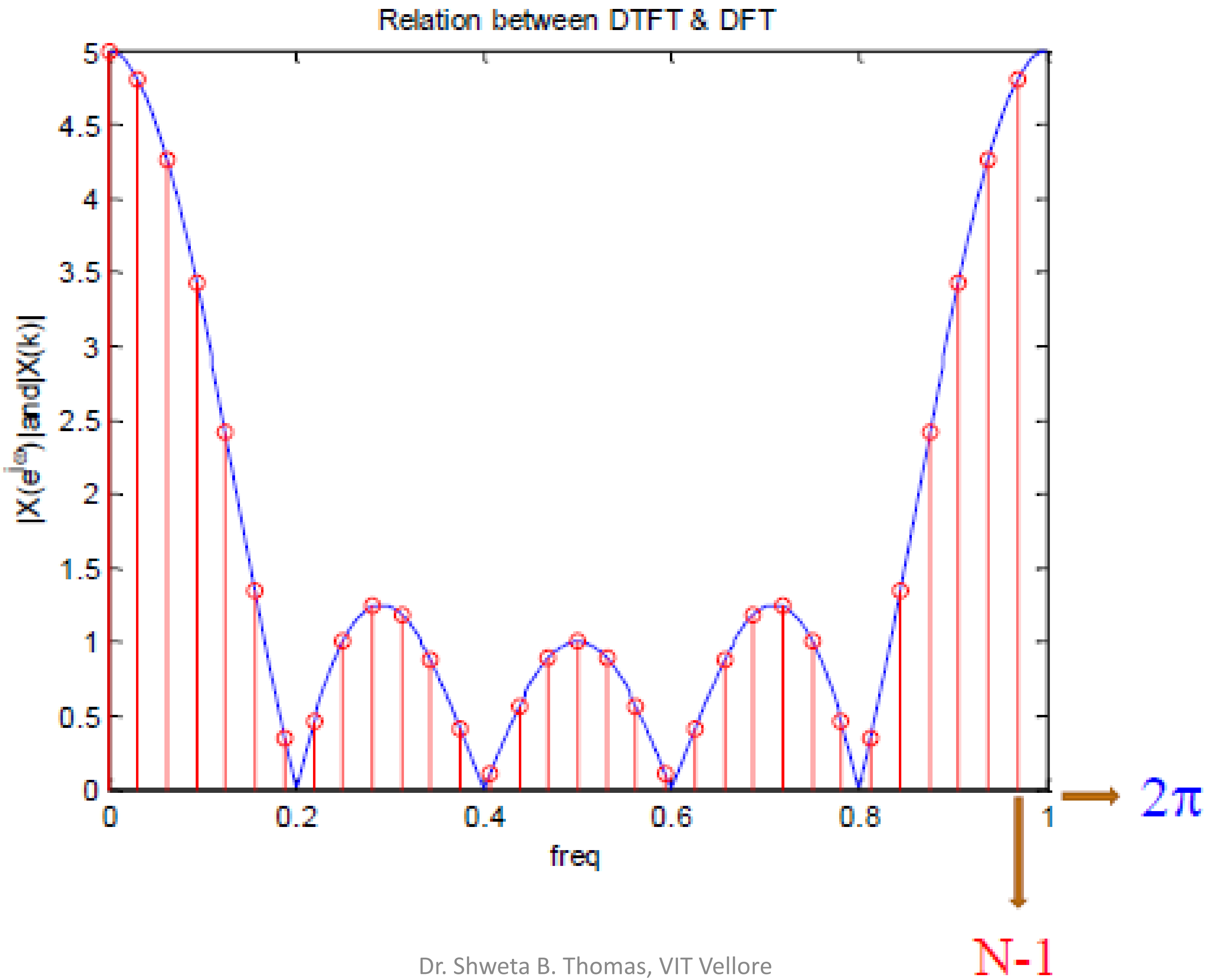
- However, in case of the continuous time signal, an ideal band limited signal can not be generated but, for practical purposes it is sufficient to insure that for a band limited signal, outside the specified frequency range, the signal energy is very small.

Discrete Fourier Transform

- The N point DFT sequence X[k] is precisely the set of frequency samples of Fourier Transform $X(e^{j\omega})$ of length N sequence x[n] at N equally spaced frequencies $\omega_k = 2\pi k/N$, $0 \le k \le N-1$.
- Since the computation of the DFT samples involve a finite sum, for time domain sequences with finite sample values, the DFT always exists.
- DTFT is not computationally convenient representation because here, ω is a continuous function of frequency and periodic with period 2π.



DTFT to DFT



Relation between DTFT & DFT
5
4.5
4
3.5
3
2.5
2
1.5
1
0.5
0
0
0.2
0.4
0.6
0.8
1
freq
2π
N-1

DFT representation

Let $x(n)$ is a finite duration sequence of length $L \le N$.

$i.e., x(n) = 0$ for $n < 0$ & also for $n \ge L$.

$$X(\omega) = \sum_{n=0}^{L-1} x(n) e^{-j\omega n}$$

$$X(k) = X(\omega)\Big|_{\omega=\frac{2\pi k}{N}} = \sum_{n=0}^{L-1} x(n) e^{-j\omega n}\Bigg|_{\omega=\frac{2\pi k}{N}}$$

$$= \sum_{n=0}^{L-1} x(n) e^{\frac{-j2\pi kn}{N}}$$

N - point DFT of $x(n)$ for $N \geq L$ is

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{\frac{-j2\pi kn}{N}}$$

where $k = 0, 1, 2, \ldots\ldots, N-1$

If $N > L$, then Spectral display will be better due to zero padding.

If $N < L$, then time domain aliasing takes place.

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{\frac{-j2\pi kn}{N}} \qquad \text{where } k = 0, 1, 2, \ldots\ldots, N-1$$

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \qquad \text{where } k = 0, 1, 2, \ldots\ldots, N-1$$

Where by defination

$$W_N = e^{\frac{-j2\pi}{N}}$$

Phasefactor or Twiddle factor

Symmetry property

$$W_N^{k+\frac{N}{2}} = -W_N^k$$

Periodicity property

$$W_N^{k+N} = W_N^k$$

Inverse DFT (IDFT)

IDFT of $X(k)$ is

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k) e^{\frac{j2\pi kn}{N}}$$

where $n = 0, 1, 2, \ldots\ldots, N-1$

IDFT of $X(k)$ is

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k) W_N^{-kn}$$

where $n = 0, 1, 2, \ldots\ldots, N-1$

Properties of DFT

Periodicity

If a sequence $x(n)$ is periodic with period of N samples then N-point DFT, $X(k)$ is also periodic period of N samples.

$$If \quad x(n) \overset{DFT}{\underset{N}{\longleftrightarrow}} X(k)$$

$$then \quad x(n+N) = x(n) \;\; for\ all\ n$$

$$and \quad X(k+N) = X(k) \;\; for\ all\ k$$

Proof

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N} \qquad 0 \le n \le N-1$$

$$X(k+N) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi(k+N)n/N}$$

$$= \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N} e^{-j2\pi Nn/N}$$

$$= \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N} e^{-j2\pi n} \qquad \Big| e^{-j2\pi n} = \cos 2\pi n - j \sin 2\pi n = 1$$

$$= \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}$$

$$= X(k)$$

Linearity

$$If \quad x_1(n) \xleftrightarrow[N]{DFT} X_1(k) \quad and \quad x_2(n) \xleftrightarrow[N]{DFT} X_2(k)$$

$$then \quad a_1 x_1(n) + a_2 x_2(n) \xleftrightarrow[N]{DFT} a_1 X_1(k) + a_2 X_2(k)$$

where a_1 and a_2 are constants.

Proof

$$X(k)=\sum_{n=0}^{N-1}x(n)e^{-j2\pi kn/N} \qquad 0\le n\le N-1$$

$$DFT\left\{a_1x_1(n)+a_2x_2(n)\right\}=\sum_{n=0}^{N-1}\left\{a_1x_1(n)+a_2x_2(n)\right\}e^{-j2\pi kn/N}$$

$$=\sum_{n=0}^{N-1}\left\{a_1x_1(n)e^{-j2\pi kn/N}+a_2x_2(n)e^{-j2\pi kn/N}\right\}$$

$$=a_1\sum_{n=0}^{N-1}x_1(n)e^{-j2\pi kn/N}+a_2\sum_{n=0}^{N-1}x_2(n)e^{-j2\pi kn/N}$$

$$=a_1X_1(k)+a_2X_2(k)$$

Circular Time Shift

$$If \quad x(n) \xleftrightarrow[N]{DFT} X(k)$$

$$then \quad x((n-m))_N \xleftrightarrow[N]{DFT} X(k)e^{-j2\pi km/N}$$

Proof

If a discrete time signal is circularly shifted in time by 'm' units then its DFT

$$DFT\left\{x((n-m))_N\right\} = \sum_{n=0}^{N-1} x((n-m))_N \, e^{-j2\pi kn/N} \quad \Big| Let\ p = n-m$$

$$= \sum_{n=0}^{N-1} x(p) e^{-j2\pi k(p+m)/N}$$

$$= \left[\sum_{n=0}^{N-1} x(p) e^{-j2\pi kp/N}\right] e^{-j2\pi km/N}$$

$$= X(k) e^{-j2\pi km/N}$$

Time Reversal

$$If \quad x(n) \xleftrightarrow[N]{DFT} X(k)$$

$$then \quad x((-n))_N = x(N-n) \xleftrightarrow[N]{DFT} X((-k))_N = X(N-k)$$

Time reversal property of DFT says that, reversing the N-point sequence in time is equivalent to reversing DFT sequence.

Proof

$$DFT\{x(N-n)\} = \sum_{n=0}^{N-1} x(N-n) e^{-j2\pi kn/N} \quad \Big| Let\, m = N-n$$

$$= \sum_{n=0}^{N-1} x(m) e^{-j2\pi k(N-m)/N} \quad = \sum_{n=0}^{N-1} x(m) e^{-j2\pi kN/N} e^{-j2\pi k(-m)/N}$$

$$= \sum_{n=0}^{N-1} x(m) e^{j2\pi km/N} e^{-j2\pi k} \quad \Big| e^{-j2\pi k} = 1, \text{since } k \text{ is integer}$$

$$= \sum_{n=0}^{N-1} x(m) e^{j2\pi km/N}$$

$$= \sum_{n=0}^{N-1} x(m) e^{j2\pi km/N} e^{-j2\pi m} \quad \Big| e^{-j2\pi m} = 1, \text{since } m \text{ is integer}$$

$$= \sum_{n=0}^{N-1} x(m) e^{j2\pi km/N} e^{-j2\pi mN/N} = \sum_{n=0}^{N-1} x(m) e^{-j2\pi m(N-k)/N}$$

$$= X(N-k)$$

Circular Convolution

$$If \quad x_1(n) \xleftrightarrow[N]{DFT} X_1(k) \quad and \quad x_2(n) \xleftrightarrow[N]{DFT} X_2(k)$$

$$then \quad x_1(n) \otimes x_2(n) \xleftrightarrow[N]{DFT} X_1(k).X_2(k)$$

$$x_1(n) \otimes x_2(n) = \sum_{m=0}^{N-1} x_1(m) x_2((n-m))_N$$

Proof

$$X_1(k)=\sum_{n=0}^{N-1}x_1(n)e^{-j2\pi kn/N}=\sum_{m=0}^{N-1}x_1(m)e^{-j2\pi km/N} \quad \Big| Let\ n=m$$

$$k=0,1,\ldots\ldots,N-1$$

$$X_2(k)=\sum_{n=0}^{N-1}x_2(n)e^{-j2\pi kn/N}=\sum_{p=0}^{N-1}x_1(p)e^{-j2\pi kp/N} \quad \Big| Let\ n=p$$

$$k=0,1,\ldots\ldots,N-1$$

$$DFT^{-1}\{X_1(k).X_2(k)\}=\frac{1}{N}\sum_{k=0}^{N-1}\{X_1(k).X_2(k)\}e^{-j2\pi kn/N}$$

$$=\frac{1}{N}\sum_{k=0}^{N-1}\left[\sum_{m=0}^{N-1}x_1(m)e^{-j2\pi km/N}\right]\left[\sum_{p=0}^{N-1}x_2(p)e^{-j2\pi kp/N}\right]e^{-j2\pi kn/N}$$

$$=\frac{1}{N}\sum_{m=0}^{N-1}x_1(m)\sum_{p=0}^{N-1}x_2(p)\sum_{k=0}^{N-1}e^{j2\pi k(n-m-p)/N} \quad ----------(1)$$

Proof

Let $n-m-p=qN$, where q is an integer

$$\sum_{k=0}^{N-1} e^{j2\pi k(n-m-p)/N} = \sum_{k=0}^{N-1} e^{j2\pi kqN/N} = \sum_{k=0}^{N-1}\left(e^{j2\pi q}\right)^k$$

Since q is an integer, $e^{j2\pi q}=1$

$$=\sum_{k=0}^{N-1}(1)^k = N \qquad -------------(2)$$

Since $n-m-p=qN$, $\quad p=n-m-qN$

$$\sum_{p=0}^{N-1} x_2(p) = \sum_{m=0}^{N-1} x_2(n-m-qN)$$

$$=\sum_{m=0}^{N-1} x_2(n-m, \bmod N) = \sum_{m=0}^{N-1} x_2\left((n-m)\right)_N \qquad -------(3)$$

Using equation (2) & (3), equation (1) can be written as

$$DFT^{-1}\{X_1(k).X_2(k)\} = \frac{1}{N}\sum_{m=0}^{N-1} x_1(m).\sum_{m=0}^{N-1} x_2\left((n-m)\right)_N .N$$

$$=\sum_{m=0}^{N-1} x_1(m).x_2\left((n-m)\right)_N = x_1(n)\otimes x_2(n)$$

Circular Frequency Shift

$$If \quad x(n) \xleftrightarrow[N]{DFT} X(k)$$

$$then \quad x(n)e^{j2\pi mn/N} \xleftrightarrow[N]{DFT} X((k-m))_N$$

Proof

$$DFT\left\{x(n)e^{j2\pi mn/N}\right\} = \sum_{n=0}^{N-1} x(n)e^{j2\pi mn/N}e^{-j2\pi kn/N}$$

$$= \sum_{n=0}^{N-1} x(n)e^{-j2\pi(k-m)n/N} = X((k-m))_N$$

Conjugate

Let $x(n)$ be a complex N-point discrete sequence and $x^*(n)$ be its conjugate sequence.

$$If \quad x(n) \xleftrightarrow[N]{DFT} X(k)$$

$$then \quad x^*(n) \xleftrightarrow[N]{DFT} X^*(N-k)$$

Proof

$$DFT\{x^*(n)\} = \sum_{n=0}^{N-1} x^*(n) e^{-j2\pi kn/N} = \left[\sum_{n=0}^{N-1} x(n) e^{j2\pi kn/N}\right]^*$$

$$= \left[\sum_{n=0}^{N-1} x(n) e^{j2\pi kn/N} e^{-j2\pi n}\right]^* \qquad \Big| e^{-j2\pi n} = 1$$

$$= \left[\sum_{n=0}^{N-1} x(n) e^{j2\pi kn/N} e^{-j2\pi nN/N}\right]^*$$

$$= \left[\sum_{n=0}^{N-1} x(n) e^{-j2\pi n(N-k)/N}\right]^* = [X(n-k)]^* = X^*(n-k)$$

Perceval's relation

$$If \quad x_1(n) \xleftrightarrow[N]{DFT} X_1(k) \quad and \quad x_2(n) \xleftrightarrow[N]{DFT} X_2(k)$$

$$\sum_{n=0}^{N-1} x_1(n) x_2^*(n) = \frac{1}{N} \sum_{k=0}^{N-1} X_1(k) X_2^*(k)$$

Proof

$$\frac{1}{N}\sum_{k=0}^{N-1} X_1(k)X_2^*(k) = \frac{1}{N}\sum_{k=0}^{N-1}\left[\sum_{n=0}^{N-1} x_1(n)e^{-j2\pi kn/N}\right]X_2^*(k)$$

$$= \sum_{n=0}^{N-1} x_1(n)\left[\frac{1}{N}\sum_{k=0}^{N-1} X_2^*(k)e^{-j2\pi kn/N}\right]$$

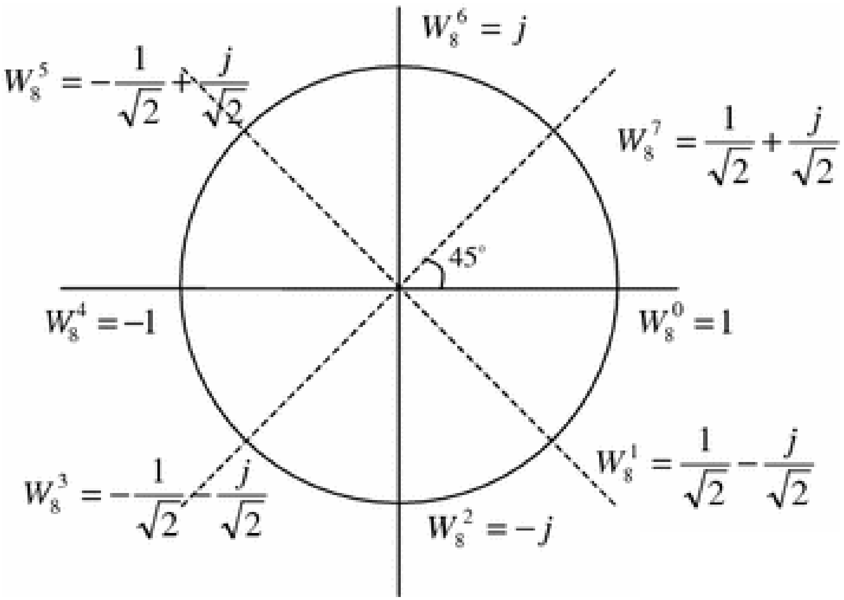
$$= \sum_{n=0}^{N-1} x_1(n)\left[\frac{1}{N}\sum_{k=0}^{N-1} X_2(k)e^{j2\pi kn/N}\right]^*$$

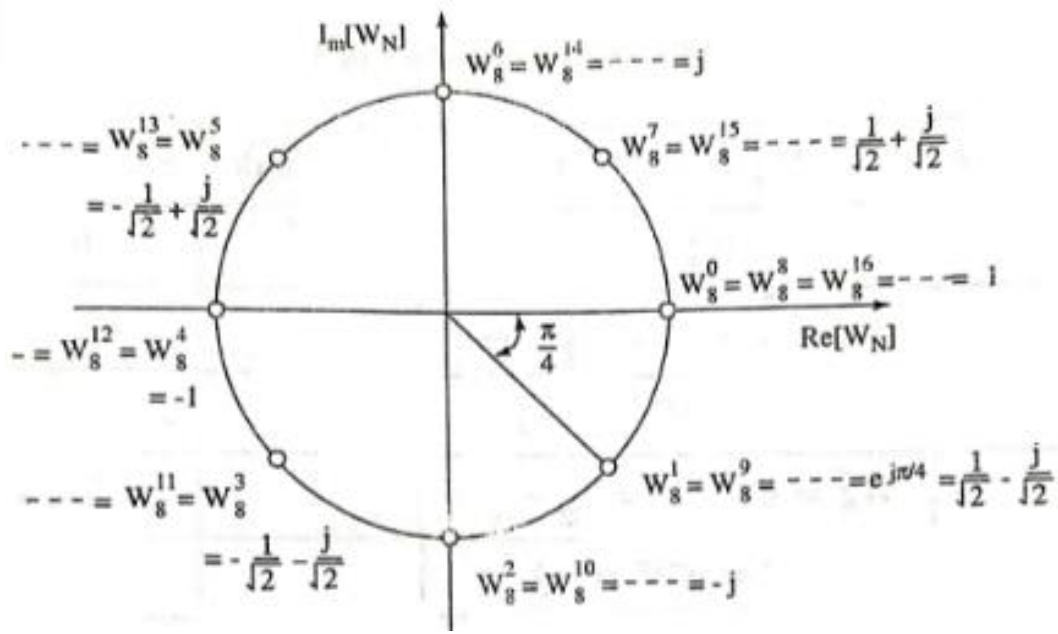
$$= \sum_{n=0}^{N-1} x_1(n)x_2^*(n)$$

Twiddle factor Calculation for N = 8

n_k	W_N^{nk} (N=8)	$e^{-\frac{j2\pi nk}{N}}$	$\cos\frac{2\pi nk}{N} - j\sin\frac{2\pi nk}{N}$	$\cos\frac{2\pi nk}{N} - j\sin\frac{2\pi nk}{N}$
0	W_8^0	$e^{-\frac{j2\pi 0}{8}}$	$\cos\frac{2\pi 0}{8} - j\sin\frac{2\pi 0}{8}$	1
1	W_8^1	$e^{-\frac{j2\pi 1}{8}}$	$\cos\frac{2\pi 1}{8} - j\sin\frac{2\pi 1}{8}$	$0.707 - j0.707$
2	W_8^2	$e^{-\frac{j2\pi 2}{8}}$	$\cos\frac{2\pi 2}{8} - j\sin\frac{2\pi 2}{8}$	$0 - j1$
3	W_8^3	$e^{-\frac{j2\pi 3}{8}}$	$\cos\frac{2\pi 3}{8} - j\sin\frac{2\pi 3}{8}$	$-.707 - j.707$
4	W_8^4	$e^{-\frac{j2\pi 4}{8}}$	$\cos\frac{2\pi 4}{8} - j\sin\frac{2\pi 4}{8}$	-1
5	W_8^5	$e^{-\frac{j2\pi 5}{8}}$	$\cos\frac{2\pi 5}{8} - j\sin\frac{2\pi 5}{8}$	$-.707 + j.707$
6	W_8^6	$e^{-\frac{j2\pi 6}{8}}$	$\cos\frac{2\pi 6}{8} - j\sin\frac{2\pi 6}{8}$	$0 + j.1$
7	W_8^7	$e^{-\frac{j2\pi 7}{8}}$	$\cos\frac{2\pi 7}{8} - j\sin\frac{2\pi 7}{8}$	$.707 + j.707$

Symmetricity and periodicity properties of Twiddle factor for N=8





Numerical Problem

Problem 1. Determine the DFT of the sequence

$$x(n) = \begin{cases} \frac{1}{4}, & for\ 0 \le n \le 2 \\ 0 & otherwise \end{cases}$$

Solution The N-point DFT of the sequence $x(n)$ is defined as

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi nk/N}, k = 0, 1, \ldots, N-1.$$

$$x(n) = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$$

Therefore, $$X(k) = \frac{1}{4}\left[1 + e^{-j\omega} + e^{-j2\omega}\right]\bigg|_{\omega=\frac{2\pi k}{N}}$$

$$= \frac{1}{4} e^{-j\omega}[1 + 2\cos\omega] \quad \text{where } \omega = \frac{2\pi k}{N} \text{ and } N = 3$$

$$= \frac{1}{4} e^{-j2\pi k/3}\left[1 + 2\cos\frac{2\pi k}{3}\right]$$

Hence, $X(k) = \frac{1}{4} e^{-j2\pi nk/3}\left[1 + 2\cos\left(\frac{2\pi k}{3}\right)\right]$ where $k = 0, 1, \ldots, N-1$

Problem 2. Derive the DFT of the sample data sequence *x(n)* = {1, 1, 2, 2, 3, 3} and also compute the corresponding amplitude and phase spectrum.

Solution The N-point DFT of a finite duration sequence $x(n)$ is defined as

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi nk/N},\ k = 0, 1, ..., N-1.$$

For $k = 0$

$$X(0) = \sum_{n=0}^{5} x(n)e^{-j2\pi(0)n/6} = \sum_{n=0}^{5} x(n) = 1+1+2+2+3+3 = 12$$

For $k = 1$

$$X(1) = \sum_{n=0}^{5} x(n)\,e^{-j2\pi(1)n/6}$$

$$= \sum_{n=0}^{5} x(n)e^{-j\pi n/3}$$

$$= 1 + e^{-j\pi/3} + 2\ e^{-j2\pi/3} + 2\ e^{-j\pi} + 3\ e^{-j4\pi/3} + 3\ e^{-j5\pi/3}$$

$$= 1 + 0.5 - j0.866 + 2(-0.5 - j0.866) + 2(-1)$$
$$+ 3(-0.5 + j0.866) + 3(0.5 + j0.866)$$
$$= -1.5 + j2.598$$

For $k = 2$

$$X(2) = \sum_{n=0}^{5} x(n) e^{-j2\pi(2)n/6}$$

$$= \sum_{n=0}^{5} x(n) e^{-2j\pi n/3}$$

$$= 1 + e^{-j2\pi/3} + 2\, e^{-j4\pi/3} + 2e^{-j2\pi} + 3e^{-j8\pi/3} + 3e^{-j10\pi/3}$$

$$= 1 + (-0.5) - j0.866 + 2(-0.5 + j0.866) + 2(1)$$

$$+ 3(-0.5 - j0.866) + 3(-0.5 + j0.866)$$

$$= -1.5 + j0.866$$

For $k = 3$

$$X(3) = \sum_{n=0}^{5} x(n) e^{-j2\pi(3)n/6}$$

$$= \sum_{n=0}^{5} x(n) e^{-j\pi n}$$

$$= 1 + e^{-j\pi} + 2e^{-j2\pi} + 2e^{-j3\pi} + 3e^{-j4\pi} + 3e^{-j5\pi}$$

$$= 1 - 1 + 2(1) + 2(-1) + 3(1) + 3(-1) = 0$$

For $k = 4$

$$X(4) = \sum_{n=0}^{5} x(n) e^{-j2\pi(4)n/6}$$

$$= \sum_{n=0}^{5} x(n) e^{-j4\pi n/3}$$

$$= 1 + e^{-j4\pi/3} + 2e^{-j8\pi/3} + 2e^{-j4\pi} + 3e^{-j16\pi/3} + 3e^{-j20\pi/3}$$

$$= 1 + (-0.5 + j0.866) + 2(-0.5 - j0.866) + 2(1)$$
$$+ 3(-0.5 + j0.866) + 3(-0.5 - j0.866)$$

$$= -1.5 - j0.866$$

For $k = 5$

$$X(5) = \sum_{n=0}^{5} x(n) e^{-j2\pi(5)n/6}$$

$$= \sum_{n=0}^{5} x(n) e^{-j5\pi n/3}$$

$$= 1 + e^{-j5\pi/3} + 2e^{-j10\pi/3} + 2e^{-j5\pi} + 3e^{-j20\pi/3} + 3e^{-j25\pi/3}$$

$$= 1 + (-0.5 + j\,0.866) + 2(-0.5 + j\,0.866) + 2(-1)$$
$$+ 3(-0.5 - j\,0.866) + 3(0.5 - j\,0.866)$$

$$= -1.5 - j2.598$$

$$X(k) = \{12, -1.5 + j2.598, -1.5 + j0.866, 0, -1.5 - j\,0.866,$$
$$-1.5 - j\,2.598\}$$

The corresponding amplitude spectrum is given by

$$|x(k)| = \left\{\sqrt{12\times12}, \sqrt{(-1.5)^2+(-2.598)^2}, \sqrt{(-1.5)^2+(0.866)^2}, 0, \sqrt{(-1.5)^2+(-0.866)^2}, \sqrt{(-1.5)^2+(-2.598)^2}\right\}$$

$$= \{12,\ 2.999,\ 1.732,\ 0,\ 1.732,\ 2.999\}$$

and the corresponding phase spectrum is given by

$$\angle X(k) = \left\{\tan^{-1}(0), \tan^{-1}\left(\frac{2.598}{-1.5}\right), \tan^{-1}\left(\frac{0.866}{-1.5}\right), \tan^{-1}(0), \tan^{-1}\left(\frac{-0.866}{-1.5}\right), \tan^{-1}\left(\frac{-2.598}{-1.5}\right)\right\}$$

$$= \left\{0, -\frac{\pi}{3}, -\frac{\pi}{6}, 0, \frac{\pi}{6}, \frac{\pi}{3}\right\}$$

Problem 3. Determine the IDFT of X(k) = {3, (2+j), 1, (2-j)}

Solution The IDFT is defined as

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)e^{j2\pi nk/N}, 0 \le n \le N-1.$$

Given $N=4$, $x(n) = \frac{1}{4}\sum_{k=0}^{3} X(k)e^{j\pi nk/2}, 0 \le n \le 3$

When $n = 0$

$$x(0) = \frac{1}{4}\sum_{k=0}^{3} X(k)e^{0}$$

$$= \frac{1}{4}[3+(2+j)+1+(2-j)] = 2$$

When $n = 1$

$$x(1) = \frac{1}{4}\sum_{k=0}^{3} X(k)e^{j\pi k/2}$$

$$= \frac{1}{4}\left[3+(2+j)e^{j\pi/2} + e^{j\pi} + (2-j)e^{j3\pi/2}\right]$$

$$= \frac{1}{4}[3+(2+j)j-1+(2-j)(-j)] = 0$$

When $n=2$

$$x(2)=\frac{1}{4}\sum_{k=0}^{3}X(k)e^{j\pi k}$$
$$=\frac{1}{4}[3+(2+j)e^{j\pi}+e^{j2\pi}+(2-j)e^{j3\pi}]$$
$$=\frac{1}{4}[3+(2+j)(-1)+1+(2-j)(-1)]=0$$

When $n=3$

$$x(3)=\frac{1}{4}\sum_{k=0}^{3}X(k)e^{j3\pi k/2}$$
$$=\frac{1}{4}[3+(2+j)e^{j3\pi/2}+e^{j3\pi}+(2-j)e^{j9\pi/2}]$$
$$=\frac{1}{4}[3+(2+j)(-j)-1+(2-j)(j)]=1$$

Therefore, the IDFT of the given DFT produces the following original sequence values

$$x(n)=[2,\ 0,\ 0,\ 1]$$

Circular Convolution (Periodic Convolution)

Matrix Multiplication Method

- The convolution of two sequences *x*(*n*) and *h*(*n*) can be obtained by representing these sequences in matrix form as given below:

$$\begin{bmatrix} x(0) & x(N-1) & x(N-2) & \cdots\cdots & x(2) & x(1) \\ x(1) & x(0) & x(N-1) & \cdots\cdots & x(3) & x(2) \\ x(2) & x(1) & x(0) & \cdots\cdots & x(4) & x(3) \\ \vdots & \vdots & \vdots & \cdots\cdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \cdots\cdots & \vdots & \vdots \\ x(N-2) & x(N-3) & x(N-4) & \cdots & x(0) & x(N-1) \\ x(N-1) & x(N-2) & x(N-3) & \cdots & x(1) & x(0) \end{bmatrix} \begin{bmatrix} h(0) \\ h(1) \\ h(2) \\ \vdots \\ \vdots \\ h(N-2) \\ h(N-1) \end{bmatrix} = \begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ \vdots \\ \vdots \\ y(N-2) \\ y(N-1) \end{bmatrix}$$

- The sequence *x*(*n*) is repeated via circular path shift of samples and represented in *N X N* matrix form. *The* sequence *h*(*n*) is represented as column matrix. The multiplication of these two matrices gives the sequence *y*(*n*).

Problem

Compute (a) linear and (b) circular periodic convolutions of the two sequences $x_1(n)$ = {1, 1, 2, 2} and $x_2(n)$ = {1, 2, 3, 4}. Also find circular convolution using the DFT and IDFT.

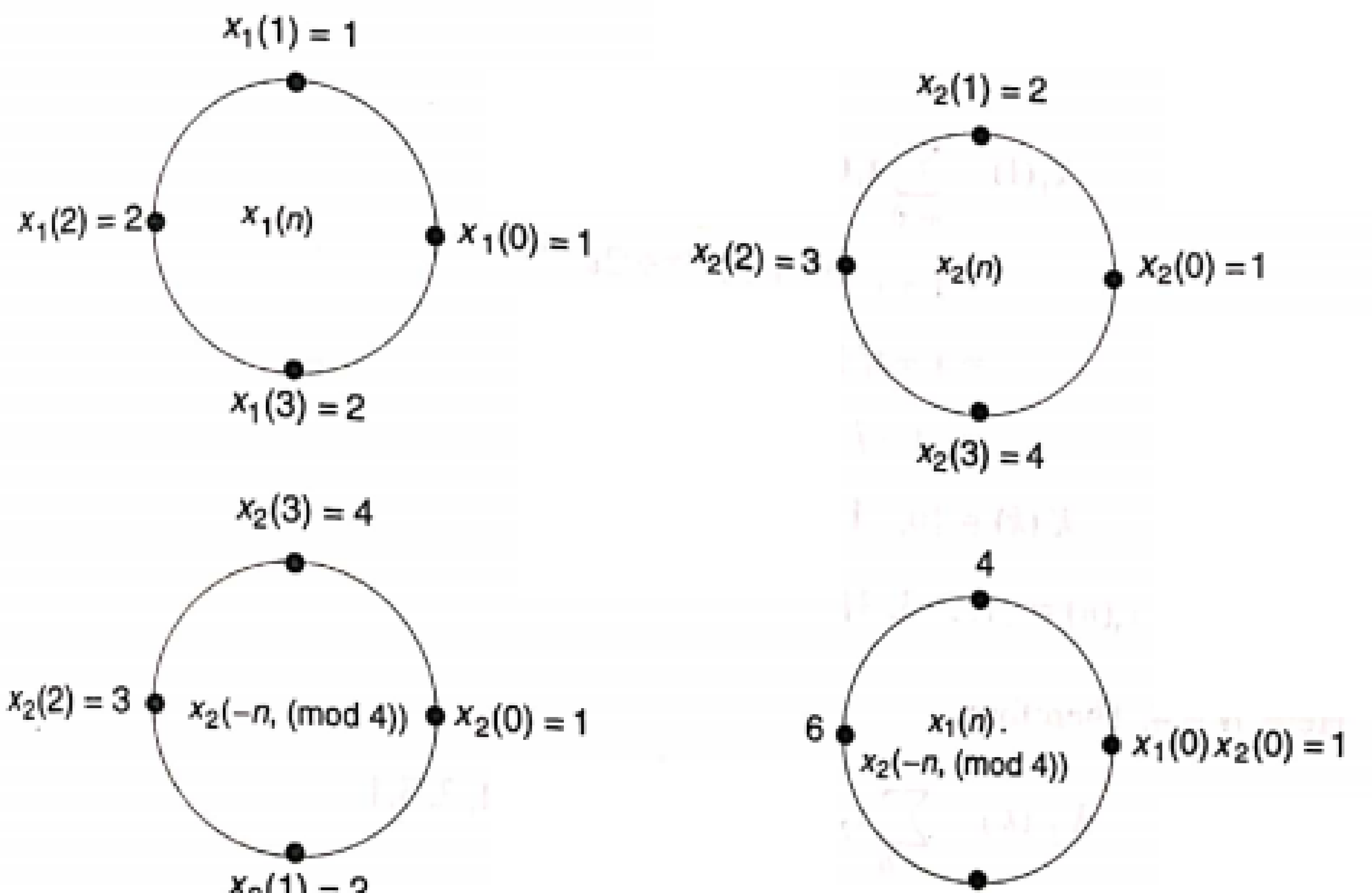
Solution

(a)Using matrix representation, the linear convolution of the two sequences can be determined as

$$\begin{bmatrix} 1 & 2 & 2 & 1 \\ 1 & 1 & 2 & 2 \\ 2 & 1 & 1 & 2 \\ 2 & 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1+4+6+4=15 \\ 1+2+6+8=17 \\ 2+2+3+8=15 \\ 2+4+3+4=13 \end{bmatrix}$$

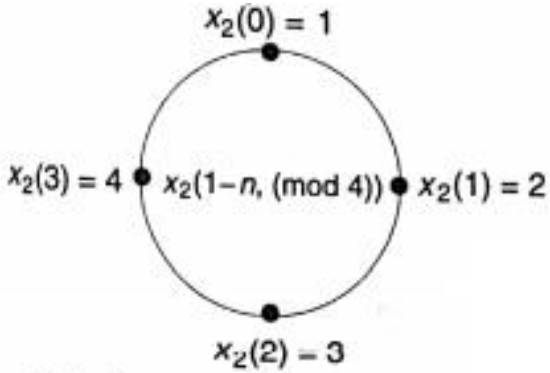
$x_3(n)$ = {15, 17, 15, 13}

(b) Circular periodic convolutions of the two sequences x1(n) = {1, 1, 2, 2} and x2(n) = {1, 2, 3, 4} is given as

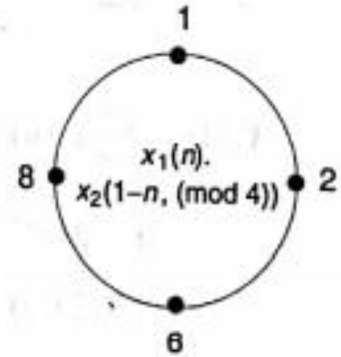


Folded Sequence

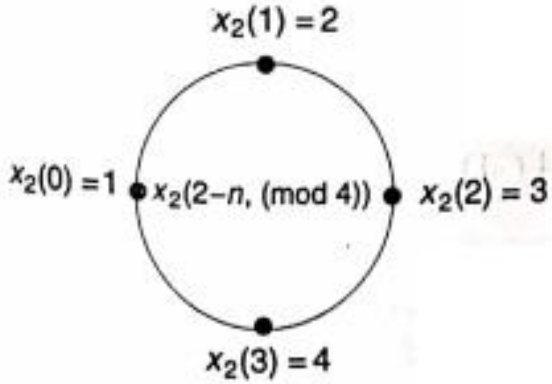
Product Sequence $x_3(0) = 1 + 4 + 6 + 4 = 15$



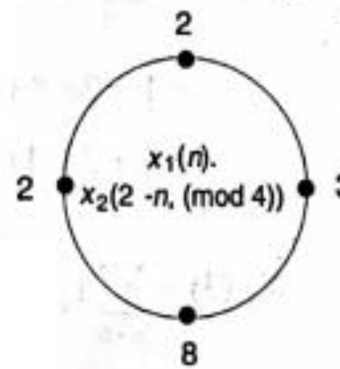
Folded sequence rotated by one unit in time



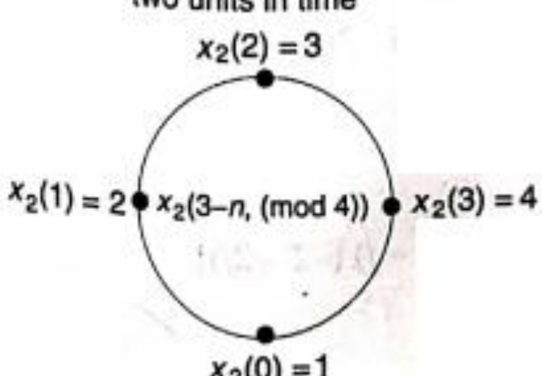
Product sequence, $x_3(1)$
$= 2 + 1 + 8 + 6 = 17$



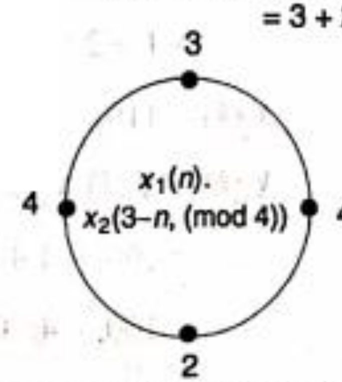
Folded sequence rotated by two units in time



Product sequence, $x_3(2)$
$= 3 + 2 + 2 + 8 = 15$



Folded sequence rotated by three units in time



Product sequence, $x_3(3)$
$= 4 + 3 + 4 + 2 = 13$

(c) Circular convolutions by DFT and IDFT

$$X3(k) = X1(k)\, X2(k)$$

$$x_3(n) = \sum_{k=0}^{N-1} X_3(k) e^{j\frac{2\pi nk}{N}}$$

(i) When $x_1(n) = [1, 1, 2, 2]$

$$X_1(k) = \sum_{n=0}^{N-1} x_1(n) e^{-j2\pi nk/N}, \; k = 0, 1, \ldots, N-1$$

For $N = 4$

$$X_1(k) = \sum_{n=0}^{3} x_1(n) e^{-j2\pi nk/4}, k = 0, 1, 2, 3$$

For $k = 0$

$$X_1(0) = \sum_{n=0}^{3} x_1(n) e^{-j2\pi n(0)/4} = 6$$

For $k=1$

$$X_1(1)=\sum_{n=0}^{3}x_1(n)e^{-j\pi n/2}$$

$$=1+e^{-j\pi/2}+2e^{-j\pi}+2e^{-j3\pi/2}$$

$$=1-j+2(-1)+2(j)$$

$$=-1+j$$

For $k=2$

$$X_1(2)=\sum_{n=0}^{3}x_1(n)e^{-jn\pi}$$

$$=1+e^{-j\pi}+2e^{-2\pi}+2e^{-j3\pi}$$

$$=1-1+2(1)+2(-1)=0$$

For $k=3$

$$X_1(1)=\sum_{n=0}^{3}x_1(n)e^{-j3n\pi/2}$$

$$=1+e^{-j3\pi/2}+2e^{-j3\pi}+2e^{-j9\pi/2}$$

$$=1+(j)+2(-1)-j(2)$$

$$=-1-j$$

$$X_1(k)=\{6,-1+j,0,-1-j\}$$

(ii) When $x_2(n) = \{1, 2, 3, 4\}$

Here, $N = 4$. Therefore,

$$X_2(k) = \sum_{n=0}^{3} x_2(n) e^{-j2\pi nk/4}, \; k = 0, 1, 2, 3.1$$

For $k = 0$

$$X_2(0) = \sum_{n=0}^{3} x_2(n) e^{-j2\pi n(0)/4} = 10$$

For $k = 1$

$$X_2(1) = \sum_{n=0}^{3} x_2(n) e^{-j\pi n/2}$$

$$= 1 + 2e^{-j\pi/2} + 3e^{-j\pi} + 4e^{-j3\pi/2}$$

$$= 1 + 2(-j) + 3(-1) + 4(j) = -2 + j2$$

For $k=2$

$$X_2(2)=\sum_{n=0}^{3}x_2(n)e^{-j\pi n}$$

$$=1+2e^{-j\pi}+3e^{-j\pi}+4e^{-j3\pi}$$

$$=1+2\,(-1)+3\,(1)+4\,(-1)=-2$$

For $k=3$

$$X_2\;(3)=\sum_{n=0}^{3}x_2(n)e^{-j(3\pi/2)n}$$

$$=1+2e^{-j3\pi/2}+3e^{-j3\pi}+4e^{-j9\pi/2}$$

$$=1+2\,(j)+3\,(-1)+4\,(-j)=-2-2j$$

$$X_2(k)=\{10,-2+2j,-2,-2-2j\}$$

$$X_3(k)=X_1(k)\;X_2\;(k)$$

$$=\{60,(-1+j)\,(-2+2j),0,(-1-j)\,(-2-2j)\}$$

$$=\{60,-4j,0,4j\}$$

We know that $x_3(n)=\text{IDFT}\;\{X_3(k)\}$

$$x_3(n) = \frac{1}{4}\sum_{k=0}^{3} X_3(k)e^{j2\pi nk/4}, n = 0, 1, 2, 3$$

$$= \frac{1}{4}\left[60 + (-4j)e^{j\pi n/2} + (4j)e^{j3\pi n/2}\right]$$

$$x_3(0) = \frac{1}{4}\left[60 + (-4j) + (4j)\right] = 15$$

$$x_3(1) = \frac{1}{4}\left[60 - 4je^{j\pi/2} + 4je^{j3\pi/2}\right] = \frac{1}{4}\left[60 - 4j(+j) + 4j(-j)\right]$$

$$= \frac{1}{4}\left[60 + 4 + 4\right] = 17$$

$$x_3(2) = \frac{1}{4}\left[60 - 4j\,e^{j\pi} + 4j\,e^{j3\pi}\right]$$

$$= \frac{1}{4}\left[60 - 4j(-1) + 4j(-1)\right] = 15$$

$$x_3(3) = \frac{1}{4}\left[60 + (-4j)e^{j3\pi/2} + 4je^{j9\pi/2}\right]$$

$$= \frac{1}{4}\left[60 + (-4j)(-j) + 4j(j)\right]$$

$$= \frac{1}{4}\left[60 - 4 - 4\right] = 13$$

Therefore, $x_3(n) = [15, 17, 15, 13]$

THANK YOU