Inverse Laplace Transforms :-

<u>Definition</u> If $L\{f(t)\} = F(s)$, then we write $L^{-1}\{F(s)\} = f(t)$ and it is called as inverse Laplace transform of $F(s)$.

Ex. ① $L^{-1}\left\{\frac{1}{s}\right\} = 1$

② $L^{-1}\left\{\frac{1}{s+3}\right\} = e^{-3t}$

③ $L^{-1}\left\{\frac{1}{s^2+4}\right\} = \frac{1}{2}\sin 2t$

④ $L^{-1}\left\{\frac{s+2}{(s+2)^2+9}\right\} = e^{-2t}\cos 3t$

⑤ $L^{-1}\left\{e^{-\pi s}\cdot\frac{2}{s^3}\right\} = (t-\pi)^2 u(t-\pi)$

⑥ $L^{-1}\{1\} = \delta(t)$

Methods for finding inverse Laplace Transforms:

① Partial fractions method:

(i) Find $L^{-1}\left\{\frac{s}{(s-1)(s+1)(s+3)}\right\}$

Sol. Resolving $\frac{s}{(s-1)(s+1)(s+3)}$ into partial fractions

$$\frac{s}{(s-1)(s+1)(s+3)} = \frac{A}{s-1} + \frac{B}{s+1} + \frac{C}{s+3}$$

$$\Rightarrow \quad s = A(s+1)(s+3) + B(s-1)(s+3) + C(s-1)(s+1)$$

Taking $s=1$, we get $A = \frac{1}{8}$

Taking $s=-3$, we get $C = \frac{-3}{8}$

Taking $s=-1$, we get $B = \frac{1}{4}$

Therefore, $\frac{s}{(s-1)(s+1)(s+3)} = \frac{1}{8} \cdot \frac{1}{s-1} + \frac{1}{4}\frac{1}{s+1} - \frac{3}{8}\frac{1}{s+3}$

and hence $L^{-1}\left\{\frac{s}{(s-1)(s+1)(s+3)}\right\} = \frac{1}{8}L^{-1}\left\{\frac{1}{s-1}\right\} + \frac{1}{4}L^{-1}\left\{\frac{1}{s+1}\right\} - \frac{3}{8}L^{-1}\left\{\frac{1}{s+3}\right\}$

$$= \frac{1}{8}e^{t} + \frac{1}{4}e^{-t} - \frac{3}{8}e^{-3t}$$

(ii) $L^{-1}\left\{\frac{s^2}{(s+1)(s^2+4)}\right\}$

Sol. Resolving $\frac{s^2}{(s+1)(s^2+4)}$ into partial fractions

$$\frac{s^2}{(s+1)(s^2+4)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+4}$$

$$\Rightarrow \quad s^2 = A(s^2+1) + (Bs+C)(s+1)$$

Equating the coeff's of s^2, s and constant terms on both sides we get $A = \frac{1}{2}$; $B = \frac{1}{2}$; $C = \frac{-1}{2}$

Therefore, $\dfrac{s^2}{(s+1)(s^2+4)} = \dfrac{1}{2}\dfrac{1}{s+1} + \dfrac{1}{2}\dfrac{s}{s^2+4} - \dfrac{1}{2}\dfrac{1}{s^2+4}$

and hence $L^{-1}\left\{\dfrac{s^2}{(s+1)(s^2+4)}\right\} = \dfrac{1}{2}L^{-1}\left\{\dfrac{1}{s+1}\right\} + \dfrac{1}{2}L^{-1}\left\{\dfrac{s}{s^2+4}\right\} - \dfrac{1}{2}L^{-1}\left\{\dfrac{1}{s^2+4}\right\}$

$$= \frac{1}{2}e^{-t} + \frac{1}{2}\cos 2t - \frac{1}{4}\sin 2t.$$

(iii) $L^{-1}\left\{\dfrac{s}{s^4+4a^4}\right\}$

Sol. We can write $\dfrac{s}{s^4+4a^4}$ as $\dfrac{s}{(s^2+2a^2)^2-(2as)^2}$

(or) $\cancel{\left[\dfrac{s}{(s^2+2as+2a^2)(s^2+2as}\right]}$ $\dfrac{s}{(s^2+2a^2+2as)(s^2+2a^2-2as)}$

Now, resolving $\dfrac{s}{(s^2+2as+2a^2)(s^2-2as+2a^2)}$ into

Partial fractions, we get

$$\frac{s}{(s^2+2as+2a^2)(s^2-2as+2a^2)} = \frac{1}{4a}\left\{\frac{1}{s^2-2as+2a^2} - \frac{1}{s^2+2as+2a^2}\right\}$$

Therefore,

$$L^{-1}\left\{\frac{s}{(s^2+2as+2a^2)(s^2-2as+2a^2)}\right\} = \frac{1}{4a}L^{-1}\left\{\frac{1}{(s-a)^2+a^2}\right\} - \frac{1}{4a}L^{-1}\left\{\frac{1}{(s+a)^2+a^2}\right\}$$

Hence $L^{-1}\left\{\dfrac{s}{s^4+4a^4}\right\} = \dfrac{1}{4a}\left\{\dfrac{1}{a}e^{at}\sin at\right\} - \dfrac{1}{4a}\left\{\dfrac{1}{a}e^{-at}\sin at\right\}.$

$$= \frac{1}{2a^2}\sin at\left\{\frac{e^{at}-e^{-at}}{2}\right\}$$

$$= \frac{1}{2a^2}\sin at \sinh at.$$

Exercise :-

Find the following using partial fractions method.

① $L^{-1}\left\{\dfrac{s-2}{s^2+5s+6}\right\}$ ② $L^{-1}\left\{\dfrac{s^2+2s-4}{(s^2+4)(s-1)}\right\}$

③ $L^{-1}\left\{\dfrac{s^2+2}{(s+1)(s+2)}\right\}$ (Hint: $\frac{a}{b} = q + \frac{r}{b}$) $0 \leq r < b$

④ $L^{-1}\left\{\dfrac{s-1}{(s^2+4s)(s^2+1)}\right\}$ ⑤ $L^{-1}\left\{\dfrac{s^2+s-2}{s(s-2)(s+3)}\right\}$

Convolution theorem :-

If $L^{-1}\{F(s)\} = f(t)$ and $L^{-1}\{G(s)\} = g(t)$,

then $L^{-1}\{F(s).G(s)\} = f(t) * g(t)$

where $f(t) * g(t) = \int_0^t f(u)\, g(t-u)\, du$

(Convolution Product)

Note: $f(t) * g(t) = g(t) * f(t)$.

Examples :

① Evaluate $L^{-1}\left\{\frac{1}{(s+1)(s^2+4)}\right\}$

Sol. Let $F(s) = \frac{1}{(s+1)}$ and $G(s) = \frac{1}{s^2+4}$

then $L^{-1}\{F(s)\} = e^{-t} = f(t)$ say

and $L^{-1}\{G(s)\} = \frac{1}{2}\sin 2t = g(t)$ say

By the Convolution theorem, we have

$$L^{-1}[F(s).G(s)] = f(t) * g(t)$$

$$= g(t) * f(t)$$

$$= \frac{1}{2}\int_0^t \sin 2u \,.\, e^{-(t-u)}\, du$$

$$= \frac{1}{2} e^{-t} \int_0^t e^{u} \sin 2u \, du$$

$$= \frac{e^{-t}}{2}\left\{\left[\frac{e^{u}}{1^2+2^2}\left(1.\sin 2u - 2\cos 2u\right)\right]_0^t\right\}$$

$$= \frac{e^{-t}}{2}\left\{\frac{e^{t}}{5}\left(\sin 2t - 2\cos 2t\right) - \frac{1}{5}(0-2)\right\}$$

$$= \frac{1}{10}(\sin 2t - 2\cos 2t) + \frac{e^{-t}}{5}.$$

② $L^{-1}\left\{\dfrac{s}{(s^2+1)(s^2+9)}\right\}$

Sol: Let $F(s) = \dfrac{1}{s^2+1}$ and $G(s) = \dfrac{s}{s^2+9}$

Then $L^{-1}\{F(s)\} = \sin t = f(t)$ say

and $L^{-1}\{G(s)\} = \cos 3t = g(t)$ say

By Convolution theorem, we have

$$L^{-1}\{F(s).G(s)\} = f(t) * g(t)$$

$$= \int_0^t \sin u \cos 3(t-u)\, du$$

$$= \frac{1}{2}\int_0^t \sin(3t-2u)\, du + \frac{1}{2}\int_0^t \sin(4u-3t)\, du$$

$$= \frac{1}{2}\left[\frac{-\cos(3t-2u)}{(-2)}\right]_0^t + \frac{1}{2}\left[\frac{-\cos(4u-3t)}{4}\right]_0^t$$

$$= \frac{1}{4}\left[\cos t - \cos 3t\right] + \left(\frac{-1}{8}\right)\left[\cos t - \cos 3t\right]$$

$$= \frac{1}{8}\left[\cos t - \cos 3t\right].$$

Exercise : Find the following using Convolution theorem.

① $L^{-1}\left\{\frac{1}{(s-1)(s+2)}\right\}$

② $L^{-1}\left\{\frac{s}{(s^2+1)^2}\right\}$

③ $L^{-1}\left\{\frac{1}{(s+3)(s^2+16)}\right\}$

④ $L^{-1}\left\{\frac{s^2}{(s^2+1)(s^2+4)}\right\}$

⑤ $L^{-1}\left\{\frac{s^2}{(s^2+4)^2}\right\}$