Transformation or Mapping :-

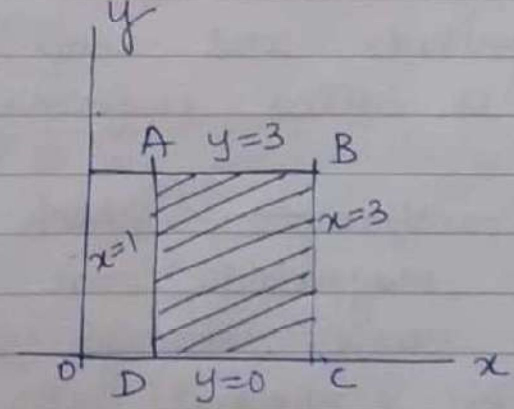
for every point (x, y) in z-plane, the relation $w = f(z)$ defines a corresponding point (u, v) in w plane. This is called transformation or mapping of z-plane into w-plane. If a point z_0 maps to w_0, then w_0 is known as image of z_0. or in other words a curve C in z-plane is mapped to corresponding curve C' in w-plane by relation $w = f(z)$.

Ex:- Transform rectangular region ABCD in z-plane bounded by $x=1$, $x=3$, $y=0$, $y=3$, under the transformation $w = z + (2+i)$.

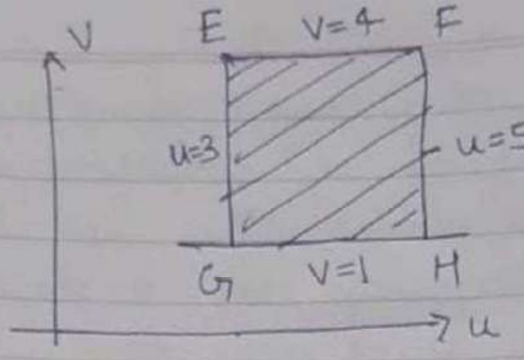
Sol^n

$$w = z + (2+i)$$
$$w = (x+iy) + (2+i)$$
$$w = (x+2) + i(y+1)$$
$$w = u + iv$$
$$u = x+2 \qquad v = y+1$$

z-plane :-



z plane	mapped to in w plane
$y=0$	$v=1$
$x=1$	$u=3$
$x=3$	$u=5$
$y=3$	$v=4$



w plane

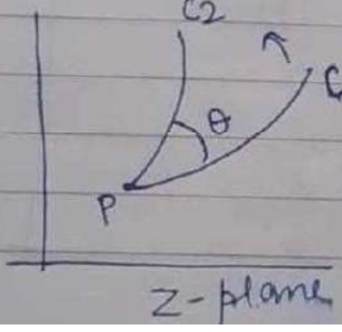
the region ABCD in z-plane is mapped to region EFGH.

<u>Conformal transformation or Conformal mapping :-</u>

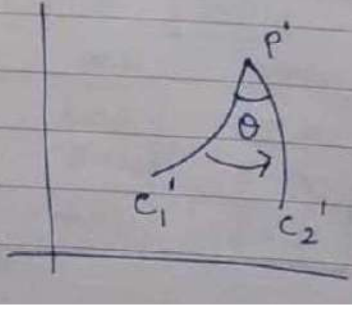
Suppose two curves c_1 and c_2 in the z-plane intersects at the point P and corresponding curves c_1' and c_2' in w-plane intersects at point P' under the transformation $w = f(z)$.

If the angle of intersection of curve at P in z-plane <u>is same as</u> angle of intersection of curves at P' in w-plane in magnitude and sense (rotation), then the mapping is called <u>conformal mapping</u>.

i.e A transformation which preserves angles both in magnitude and sense b/w every pair of curves through a point is said to be conformal at that point.



z-plane



If only magnitude of angle is preserved, then mapping is called isogonal.

Condition under which transformation $w=f(z)$ is conformal:-

Theo:- If $f(z)$ is analytic and $f'(z)\neq 0$ in region R then mapping $w=f(z)$ is conformal.

Note:-

(1) The point at which $f'(z)=0$ is called critical point of transformation.

(2) An analytic function ceases to be conformal at points where $f'(z)=0$.

Coefficient of magnification:-

Coefficient of magnification of conformal mapping $w=f(z)$ at $z=\alpha+i\beta$ is given by $|f'(\alpha+i\beta)|$

Angle of rotation:-

Angle of rotation of conformation mapping $w=f(z)$ at $z=\alpha+i\beta$ is $\operatorname{amp}(f'(\alpha+i\beta))$

Ex:- for the conformal mapping $w=z^2$ show that

(1) coeff. of magnification at $z=1+i$ is $2\sqrt{2}$

(2) The angle of rotation at $z=1+i$ is $\pi/4$.

Sol^n
$f(z)=z^2$, $f'(z)=2z$, $f'(1+i)=2+2i$

coeff. of magnification $=|2+2i|=\sqrt{4+4}=2\sqrt{2}$

angle of rotation $=\operatorname{amp}(2+2i)=\tan^{-1}(1)=\pi/4$.

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Ex:1 Determine the points (critical points) at which the following mappings are not conformal.

(1) $w = (z-a)^3$

(2) $w = z^2 + (1/z^2)$ June-2014 with def.ⁿ

(3) $w = z^2 + bz + c$

(4). $w = z + 1/z$

(5) $w = \cosh z$

(6) $w = \sinh z$

(7) $w = 2z^3 - 9z^2 - 60z + 5$

Sol.ⁿ ① $w = (z-a)^3$

According to thm-1 the mapping is not conformal at points where $f'(z) = 0$

$w = f(z) = (z-a)^3$

$\Rightarrow \frac{dw}{dz} = 3(z-a)^2$

$\therefore \frac{dw}{dz} = 3(z-a)^2 = 0$

$\Rightarrow (z-a) = 0$

$\Rightarrow \boxed{z = a}$ critical pt

Thus, at $z = a$, given mapping is not conformal.

② $w = z^2 + (1/z^2) = f(z)$

$$\frac{dw}{dz} = 2z - \frac{2}{z^3} = \frac{2z^4 - 2}{z^3}$$

$$= \frac{2(z^4 - 1)}{z^3}$$

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$$\frac{dw}{dz} = f'(z) = 0$$

$$\Rightarrow \frac{2(z^4-1)}{z^3} = 0$$

$$\Rightarrow z^4 - 1 = 0$$

$$\Rightarrow \boxed{z = \pm 1, \pm i} \text{ critical point}$$

Thus at $z = \pm 1, \pm i$ the given mapping is not conformal.

(3) $w = z^2 + bz + c$

$w = f(z) = z^2 + bz + c$

$$\frac{dw}{dz} = 2z + b$$

$$\therefore \frac{dw}{dz} = f'(z) = 0 \Rightarrow 2z + b = 0$$

$$\Rightarrow \boxed{z = -b/2} \text{ critical pt}$$

at $z = -b/2$ mapping is not conformal.

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(4) $w = z + 1/z$ (Joukowski airfoil)

$$\frac{dw}{dz} = 1 - 1/z^2$$

$$\therefore \frac{dw}{dz} = f'(z) = 0 \Rightarrow 1 - \frac{1}{z^2} = 0$$

$$\Rightarrow \frac{z^2-1}{z^2} = 0$$

$$\Rightarrow z^2 - 1 = 0$$

$$\Rightarrow \boxed{z = \pm 1} \text{ critical pt.}$$

$\therefore$ at $z = \pm 1$ mapping is not conformal.

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⑤ $w = \cosh z$

$\frac{dw}{dz} = f'(z) = \sinh z$

$\frac{dw}{dz} = f'(z) = 0 \Rightarrow \sinh z = 0$

$\Rightarrow \sinh^2 x + \sin^2 y = 0$

$\Rightarrow x = 0$ & $y = k\pi$; $k \in Z$

$\Rightarrow \boxed{z = ik\pi}$; $k \in Z$

at $z = ik\pi$; $k \in Z$; mapping is not conformal

⑥ $w = \sinh z$

$\frac{dw}{dz} = f'(z) = \cosh z$

$\frac{dw}{dz} = f'(z) = 0 \Rightarrow \cosh z = 0$

$\Rightarrow \sinh^2 x + \cos^2 y = 0$

$\Rightarrow \sinh^2 x = 0$ & $\cos^2 y = 0$

$\Rightarrow x = 0$ & $y = (2k+1)\frac{\pi}{2}$; $k \in Z$

$\Rightarrow z = i\frac{\pi}{2}(2k+1)$; $k \in Z$

at $z = i\frac{\pi}{2}\cdot(2k+1)$, mapping is not conformal

⑦ $w = 2z^3 - 9z^2 - 60z + 5$

$\frac{dw}{dz} = f'(z) = 6z^2 - 18z - 60$

$\Rightarrow f'(z) = 0$

$\Rightarrow 6(z^2 - 3z - 10) = 0$

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$\therefore (z-5)(z+2) = 0$

$\therefore z = 5$, or $z = -2$

at $z = 5, -2$ mapping is not conformal

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Some elementary transformations:-

(1) Translation:- $w = z + c$, where c is complex constant.

Let $z = x + iy$, $c = a + ib$

$$w = (x+iy) + (a+ib)$$

$$w = (x+a) + i(y+b)$$

i.e $u + iv = (x+a) + i(y+b)$

$$u = x + a$$

$$v = y + b$$

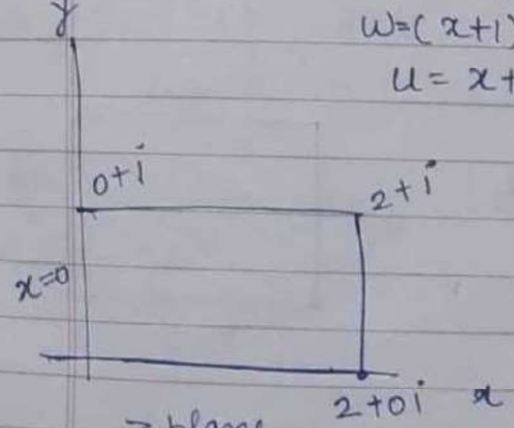
Thus this transformation is a translation of the axes and preserves the shape and size.

Ex:- Rectangle $x=0, y=0, x=2, y=1$ in z plane is transformed to another rectangle under the transformation $w = z + (1+2i)$

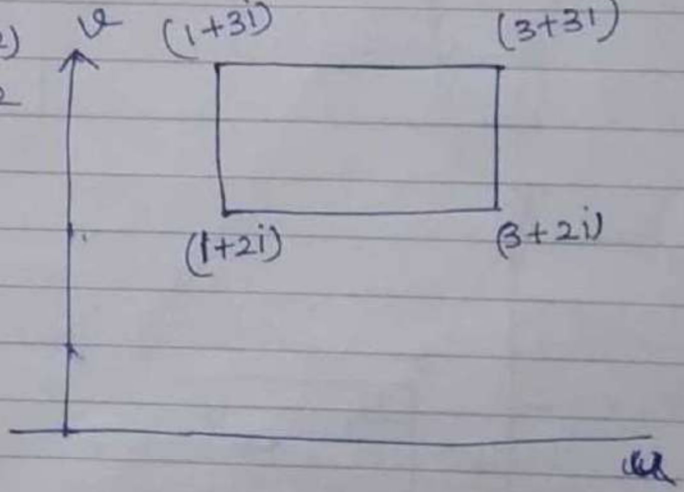
$$w = x + iy + 1 + 2i$$

$$w = (x+1) + i(y+2)$$

$$u = x+1,\ v = y+2$$



z plane



w-plane

i.e

2) **Magnification :-** $w = cz$, where c is real no

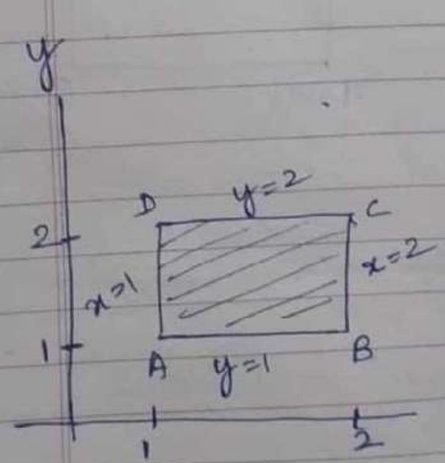
The figure in w-plane is magnified c times the size of figure in z-plane.
If $0 < c < 1$ then size of image will get shrink.

Ex:- Determine the region in w plane on the transformation of rectangular region enclosed by $x=1, y=1, x=2, y=2$ in z-plane. The transformation is $w = 3z$.

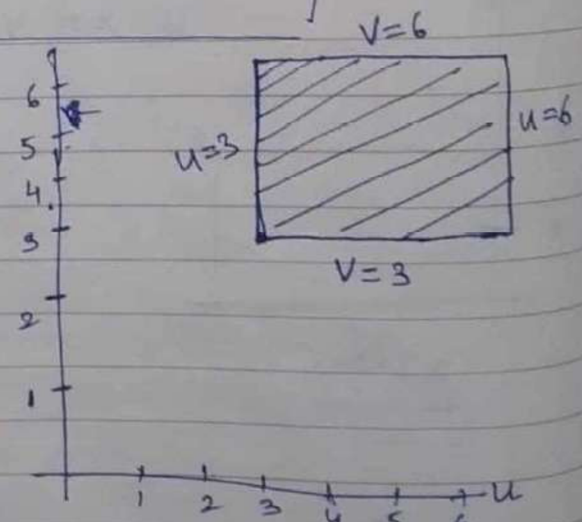
Sol^n

$$w = 3z$$
$$w = 3(x+iy)$$
$$w = 3x + 3iy$$
$$w = u + iv = 3x + 3iy$$
$$u = 3x, \quad v = 3y$$

Z plane	W plane
$x = 1$	$u = 3$
$x = 2$	$u = 6$
$y = 1$	$v = 3$
$y = 2$	$v = 6$



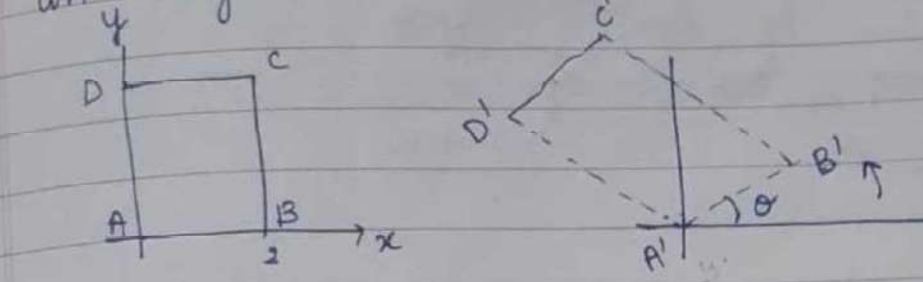
z-plane



w-plane

Rotation:- $w = z e^{i\theta}$

The figure in z plane rotates through an angle θ in antidockwise in wplane.



z plane

w-plane

$\theta \rightarrow$ always antidock

$z = r e^{i\alpha}$

$w = z\, e^{i\theta}$

$= r e^{i\alpha} e^{i\theta}$

$w = r e^{i(\alpha+\theta)}$

$r \rightarrow$ magnitude

$\alpha+\theta \rightarrow$ Amplitude

mag remains same, but amp changes to $\theta+\alpha$

Ex:- construct the transformation ~~to~~ $w = z e^{i\pi/4}$ and determine region R' in wplane corresponding to triangular region R bounded by lines $x=0$, $y=0$ and $x+y=1$ in z plane.

Soln

$w = z\, e^{i\pi/4}$

$w = z\,(\cos\pi/4 + i\sin\pi/4)$

$w = (x+iy)\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)$

$w = \frac{1}{\sqrt{2}}\left((x-y) + i(x+y)\right)$

$u = \frac{x-y}{\sqrt{2}}, \quad v = \frac{x+y}{\sqrt{2}}$

$u = \frac{x-y}{\sqrt{2}}, \; V = \frac{x+y}{\sqrt{2}}$

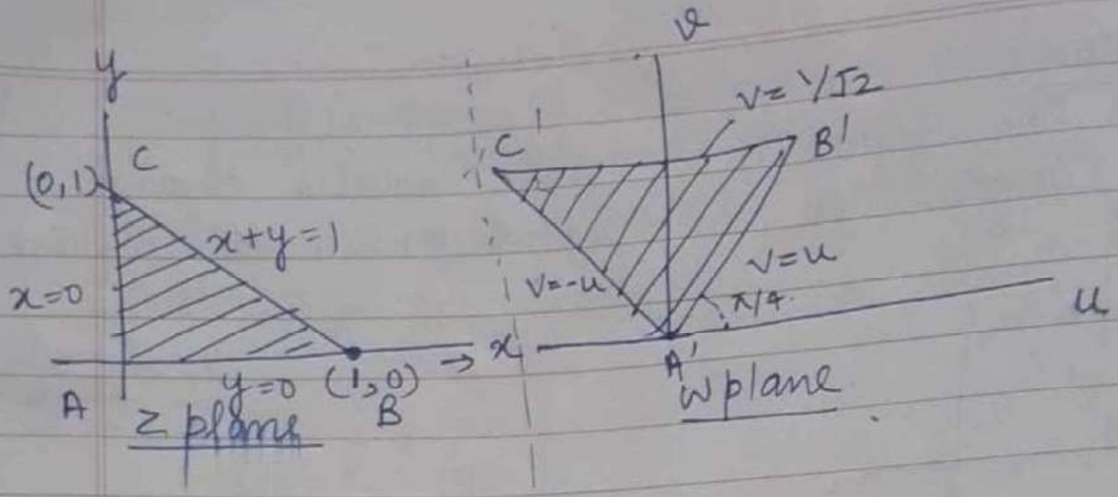
adding

$u+v = 2\frac{x}{\sqrt{2}}$

$x = \frac{u+v}{\sqrt{2}}$

& $y = \frac{v-u}{\sqrt{2}}$

z plane	w plane
$x=0$	$u = -\frac{y}{\sqrt{2}},\ v = \frac{y}{\sqrt{2}}$ or $u = -v$
$y=0$	$u = \frac{x}{\sqrt{2}},\ v = \frac{y}{\sqrt{2}}$ or $u = v$
$x+y=1$	$v = 1/\sqrt{2}$



Hence triangular region ABC is mapped to triangular region $A'B'C'$ in wplane, bounded by lines $v=-u$, $v=u$ and $v=\frac{1}{\sqrt{2}}$.

angle of rotation = $\arg(f'(z))$

$$f'(z) = \frac{1}{\sqrt{2}}(1+i)$$

$$\arg(f'(z)) = \tan^{-1} 1 = \pi/4$$

The mapping $w = z e^{i\pi/4}$ performs a rotation by an angle $\pi/4$.

Linear transformation = Translation + magnification + rotation

Note $w = cz$ where c is complex no. then its rotation + Magnification

If $\arg(c) > 0$, rotation is anticlockwise

$\arg(c) < 0$ rotation is clockwise

$\arg(c) = 0$ no rotation

Questions on translations, magnification and rotation

(a) Inversion :-

(1) what is the image of triangular region of z-plane bounded by $x=0$, $y=0$, $x+y=1$, under the transformation $w = z+(1+i)$ in w-plane.

Sol^n

$$w = z + (1+i)$$

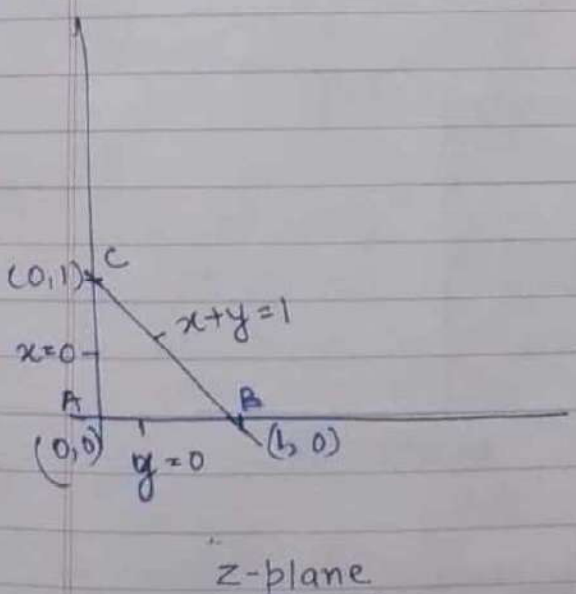
$$w = u+iv = (x+iy) + (1+i)$$

$$u+iv = (x+1) + i(y+1)$$

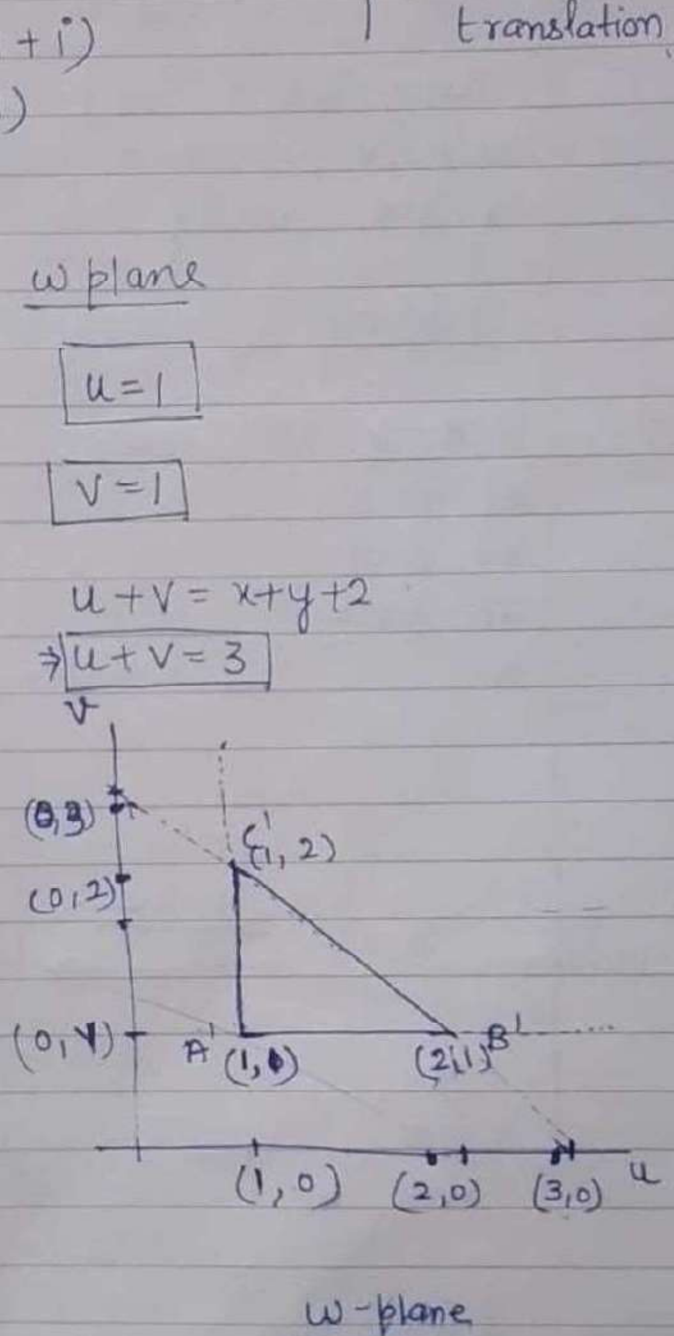
$$u = x+1, \quad v = y+1$$

| $w = z+c$ translation

z plane	w plane
(1) $x=0$	$\boxed{u=1}$
(2) $y=0$	$\boxed{v=1}$
(3) $x+y=1$.	$u+v = x+y+2$ $\Rightarrow \boxed{u+v=3}$



z-plane



w-plane

image of triangular region in z-plane is a triangular region A'B'C' in w-plane.

Q(2) what is image of ~~rect~~angular region of z-plane bounded by lines $x=0$, $y=0$, $x=1$, $y=2$ under the transformation, $w=2z$ in w plane

($w=2z$ magnificatn ?

Soln

$w = 2z$

$w = 2(x+iy)$

$w = 2x + i2y$

$u + iv = 2x + i2y$.

$u = 2x$, $v = 2y$

z-plane	w-plane
1) $x=0$	$u=0$
2) $y=0$	$v=0$
3) $x=1$	$u=2$
4) $y=2$	$v=4$

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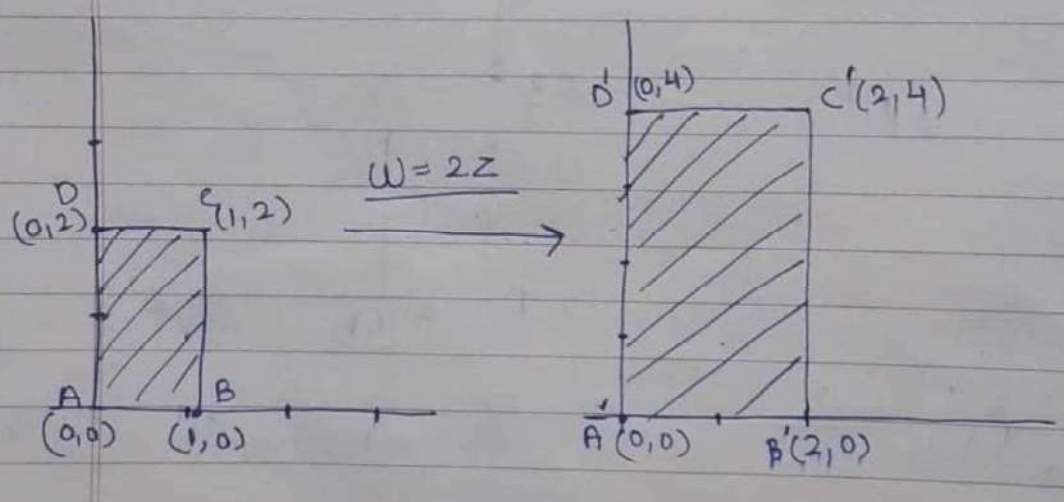


z-plane   w-plane

rectangle ABCD is z plane, mapped to rectangle A'B'C'D' in w-plane under $w=2z$.

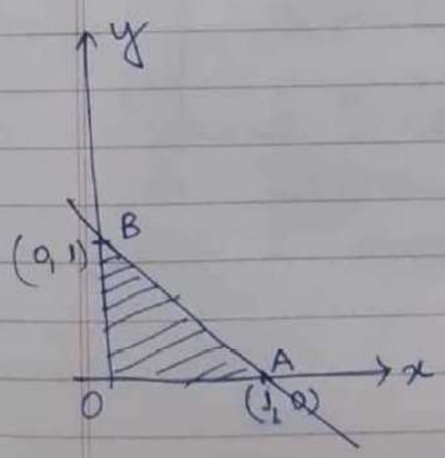
(Q3) What is image of triangular region of z plane $x=0,\ y=0,\ x+y=1$ under the transformation $w = \frac{1}{2}z$, in the w-plane.

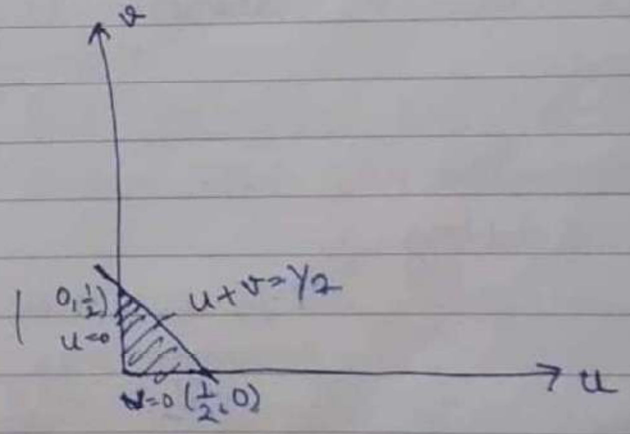
Soln: $w = \frac{z}{2} = \frac{x}{2} + i\frac{y}{2}$

$$u + iv = \frac{x}{2} + i\frac{y}{2}$$

$$u = \frac{x}{2}, \quad v = y/2$$

z-plane	w-plane
1) $x = 0$	$u = 0$
2) $y = 0$	$v = 0$
(3) $x + y = 1$	$u + v = \frac{1}{2}$





Q Find the image of semiinfinite strip $x > 0,\ 0 < y < 2$ under the transformation $\underline{w = iz + i}$, also draw the graph.

Rotation examples $\quad w = e^{i\alpha} z$ (α anticlock wise)

Q:- what is image of triangular region of z-plane bounded by lines $x=0$, $y=0$, $x+y=1$, under the transformatn $w = ze^{i\frac{\pi}{3}}$.

Solⁿ:- $w = e^{i\frac{\pi}{3}} z$

$$u+iv = \left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)(x+iy)$$

$$u+iv = \left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)(x+iy)$$

$$u+iv = \frac{1}{2}\left[(x-\sqrt{3}y) + i(\sqrt{3}x+y)\right]$$

$$u = \frac{1}{2}(x-\sqrt{3}y), \quad v = \frac{\sqrt{3}x+y}{2} \quad \text{— (1)}$$

from (1)

adding

$$\left.\begin{aligned} u &= \frac{x}{2} - \frac{\sqrt{3}}{2}y \\ v &= \frac{\sqrt{3}x}{2} + \frac{y}{2} \end{aligned}\right\} \text{— (1)}$$

adding

$$\sqrt{3}u + v = 2x$$

$$\boxed{x = \frac{\sqrt{3}u+v}{2}}$$

Substrating

$$\boxed{y = \frac{v-\sqrt{3}u}{2}}$$

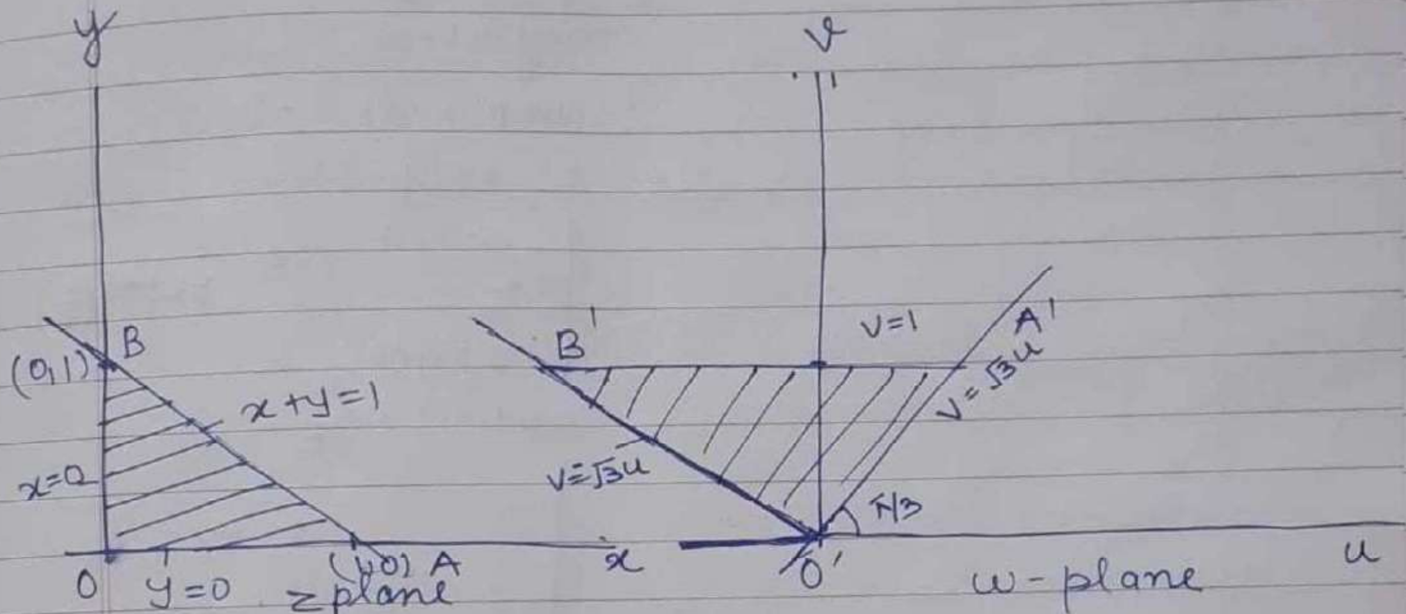
z - plane $\qquad\qquad$ w plane

$x = 0 \quad \xrightarrow{w = e^{i\pi/3}\cdot z} \quad \sqrt{3}u + v = 0$

$y = 0 \qquad\qquad v - \sqrt{3}u = 0$

$x + y = 1 \qquad\qquad \cancel{\frac{\sqrt{3}u}{2}} + \frac{v}{2} + \frac{v}{2} - \cancel{\frac{\sqrt{3}u}{2}} = 1$

$$\frac{2v}{2} = 1 \Rightarrow v = 1$$



Triange OAB in zplane get rolated by 60° in w plane under the transformation $w = e^{i\pi/3} z$.

Linear Transformation :-

translation + magnification + rotation

$w = \alpha z + \beta$, α, β are cx

Ex:- $w = (1+i)z + (2-i)$

$w = \alpha z + \beta$

$\alpha = 1+i = r e^{i\theta}$

ie $\boxed{w = r e^{i\theta} z + B}$

Q: Determine and sketch the image of ~~xxxx~~ $x>0, 0<y<2$ under the transformation $w = iz + 1$.

Q:- find the image of circle $|z| = 2$ under the transformation $w = \sqrt{2}\, z\, e^{i\pi/4}$

~~Binare~~

sol^n $|z| = 2$

$\sqrt{x^2+y^2} = 2$

$\boxed{x^2+y^2 = 4}$

$w = \sqrt{2}\, z\, e^{i\pi/4}$

$u + iv = \sqrt{2}(x+iy)\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)$

$u + iv = (1+i)(x+iy)$

$u + iv = (x-y) + i(x+y)$

$u = x - y,\quad v = x + y$

adding subst.

$x = \frac{u+v}{2},\quad y = \frac{v-u}{2}$

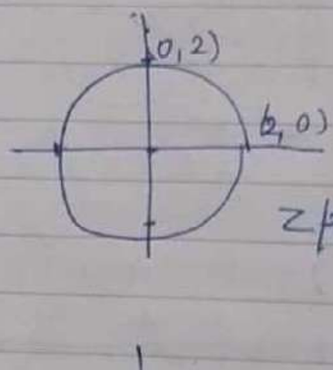
Now $x^2 + y^2 = 4$

$$\left(\frac{u+v}{2}\right)^2 + \left(\frac{v-u}{2}\right)^2 = 4$$

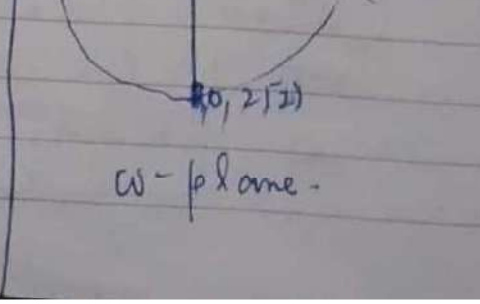
$$u^2 + v^2 + \cancel{2uv} + v^2 + u^2 - \cancel{2uv} = 16$$

$$u^2 + v^2 = 8$$

circle $(x^2+y^2=4)$ in z plane transformed to $u^2+v^2 = 8$ in w-plane.



Z plane



w-plane.

(4) <u>Inversion</u>:- or (Reflection)

$w = \frac{1}{z}$

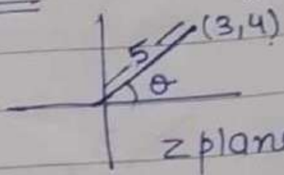
$w = \frac{1}{z}$, $z = x+iy$
$z = re^{i\theta}$

$w = \frac{1}{re^{i\theta}}$

$w = \frac{1}{r} e^{-i\theta}$

<u>In w plane</u> :-
megnitude $= \frac{1}{r}$
amplitude $= -\theta$

<u>Ex</u> $z = x+iy = 3+i4$

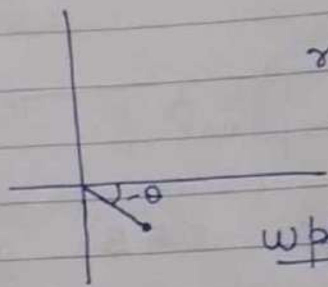


$r = 5$ $\theta = \tan^{-1}\left(\frac{4}{3}\right)$

z plane

under $w = \frac{1}{z}$

$r' = \frac{1}{r}$



<u>w plane</u>

In polar coordinates:-

$w = \frac{1}{z}$

Let $z = re^{i\theta}$, $w = Re^{i\phi}$

then $Re^{i\phi} = \frac{1}{r} e^{-i\theta}$

i.e in w plane $R = \frac{1}{r}$, $\phi = -\theta$

1 find the image of circle $x^2+y^2=4y$ under the transformation $w=\frac{1}{z}$.

Solⁿ:- $w = \frac{1}{z}$

$$\cancel{u+iv} = \frac{\cancel{1}}{x+iy} \times \frac{\cancel{x-iy}}{x+iy}$$

$\cancel{u+iv} =$

$$u+iv = \frac{1}{x+iy}$$

$$\frac{1}{u+iv} = x+iy$$

$$x+iy = \frac{1}{u+iv}\,\frac{u-iv}{(u-iv)}$$

$$x+iy = \frac{u-iv}{u^2+v^2}$$

$\cancel{x+iv}$ $\boxed{x = \frac{u}{u^2+v^2}}$, $\boxed{y = \frac{-v}{u^2+v^2}}$

Now z plane is

$$x^2+y^2 = 4y$$

$$\left(\frac{u}{u^2+v^2}\right)^2 + \left(\frac{-v}{u^2+v^2}\right)^2 = 4\left(\frac{-v}{u^2+v^2}\right)$$

$$\frac{u^2+v^2}{(u^2+v^2)^2} = -\frac{4v}{u^2+v^2}$$

$$\frac{(u^2+v^2)^2}{(u^2+v^2)^2} = -4v$$

$\Rightarrow (1+4v)(u^2+v^2)^2 = 0$

i.e. $u^2+v^2 \neq 0$

$\Rightarrow \boxed{-4v = 1}$

i.e

z plane	w plane
$x^2+y^2=4y$	$-4v=1$
	or $v=-\frac{1}{4}$
	$\downarrow$ straight line

i.e circle $x^2+y^2=4y$ in z plane is mapped to straight line $v=-\frac{1}{4}$ in w-plane under the transformation $w=1/z$.

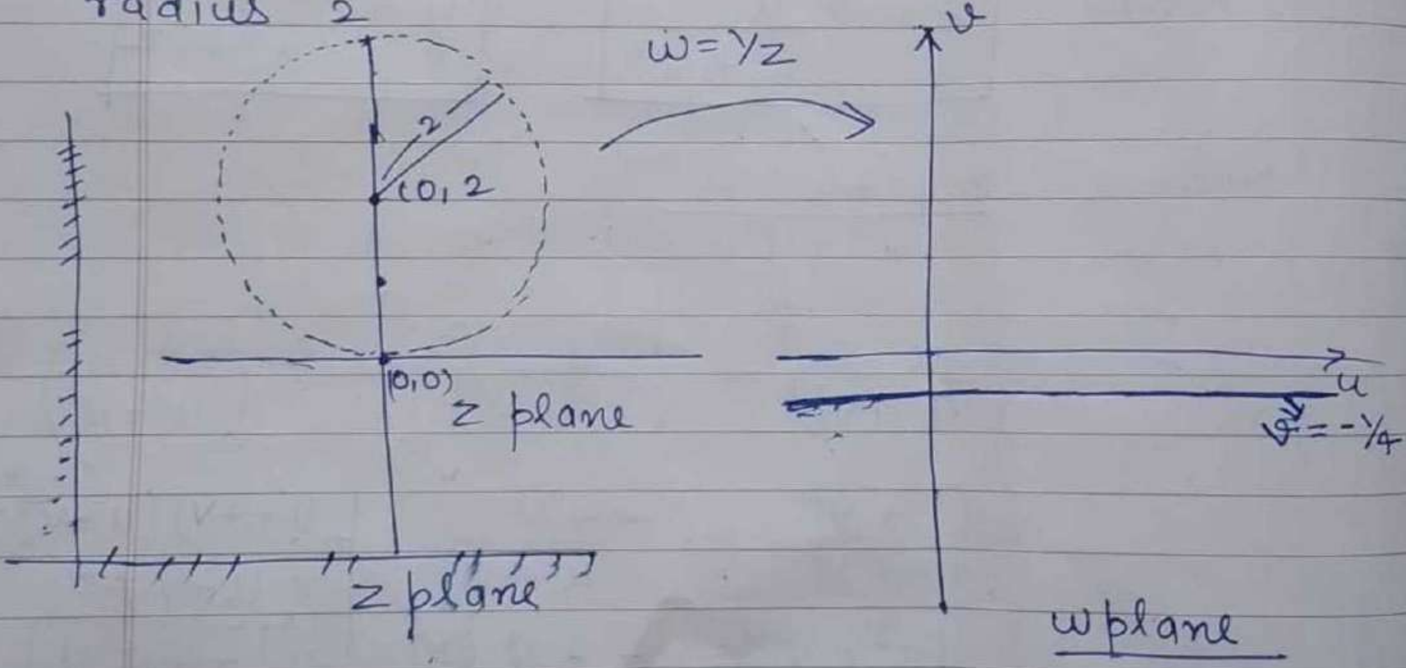
$x^2+y^2=4y$

$x^2+y^2-4y+4-4=0$

$x^2+(y-2)^2=4$

center $(0,2)$

radius 2



Ex 2 find the image of line $y-x+1=0$ under the transformation $w = 1/z$.

$$w = \frac{1}{z}$$

$$u+iv = \frac{1}{x+iy}$$

$$x+iy = \frac{1}{u+iv}$$

$$x+iy = \frac{u-iv}{u^2+v^2}$$

$$x = \frac{u}{u^2+v^2}, \quad y = \frac{-v}{u^2+v^2}$$

Now $y-x+1=0$ is a straight line in z-plane

$$\frac{-v}{u^2+v^2} - \frac{u}{u^2+v^2} + 1 = 0$$

$$-v-u+u^2+v^2 = 0$$

$$u^2+v^2-u-v=0$$

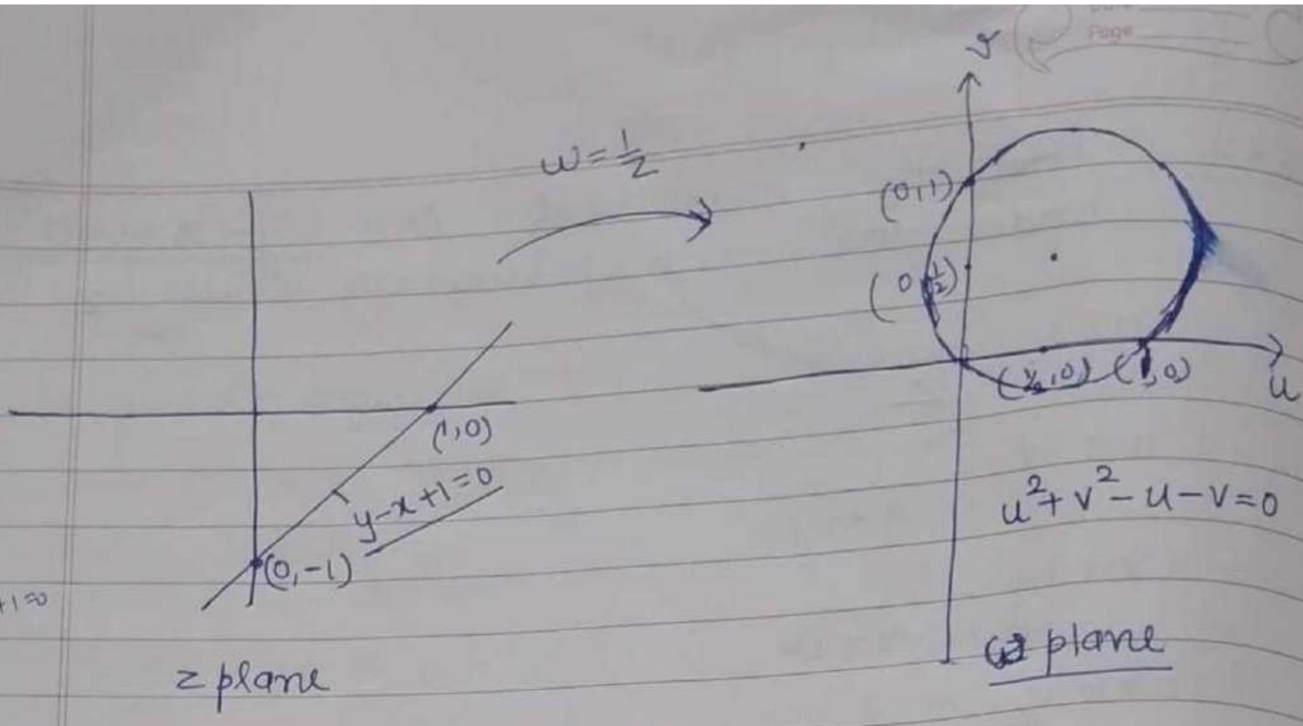
radius $= \sqrt{\frac{1}{4}+\frac{1}{4}} = \frac{1}{\sqrt{2}}$

center $\left(\frac{1}{2}, \frac{1}{2}\right)$

[standard eq^n of circle
$x^2+y^2+2gx+2fy+c=0$
radius $= \sqrt{g^2+f^2-c}$
center $= (-g, -f)$

or

$$\left(u-\frac{1}{2}\right)^2 + \left(v-\frac{1}{2}\right)^2 = \frac{1}{2}$$



$y-x+1=0$

ie line $y-x+1=0$ in z plane is mapped to circle $u^2+v^2-u-v=0$ in w plane.

Q (3) find the image of the ∞ strip $\frac{1}{4} \le y \le \frac{1}{2}$ under the transformation $w=\frac{1}{z}$.

$$w = \frac{1}{z}$$

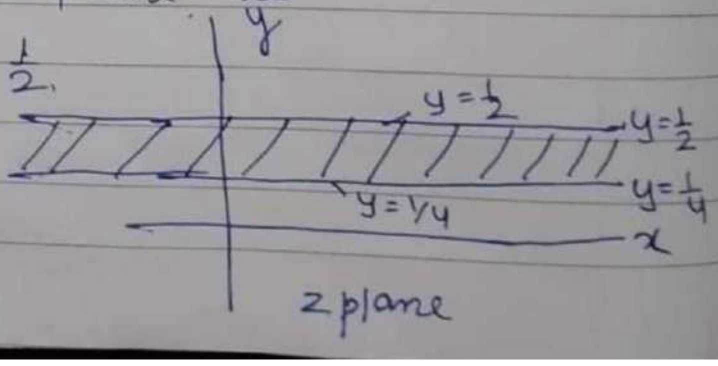
$$u+iv = \frac{1}{x+iy}$$

$$x+iy = \frac{u-iv}{u^2+v^2}$$

$$x = \frac{u}{u^2+v^2}, \quad y = \frac{-v}{u^2+v^2}$$

∞ strip in z plane is

$$\frac{1}{4} \le y \le \frac{1}{2}.$$



$$\underline{y \geq \frac{1}{4}}$$

$$\frac{-v}{u^2+v^2} \geq \frac{1}{4} \Rightarrow u^2+v^2 \leq -4v$$

$$\cancel{u^2+v^2+4v \geq 0} \Rightarrow u^2+v^2+4v \leq 0$$

$$\cancel{u^2+(v+2)^2 \geq 4}$$

$$\cancel{u^2+(v+2)^2 \geq 2^2} \Rightarrow u^2+(v+2)^2 \leq 4$$

circle : center $(0,-2)$, radius 2 — represents inner portion of circle center $(0,-2)$, radius 2.

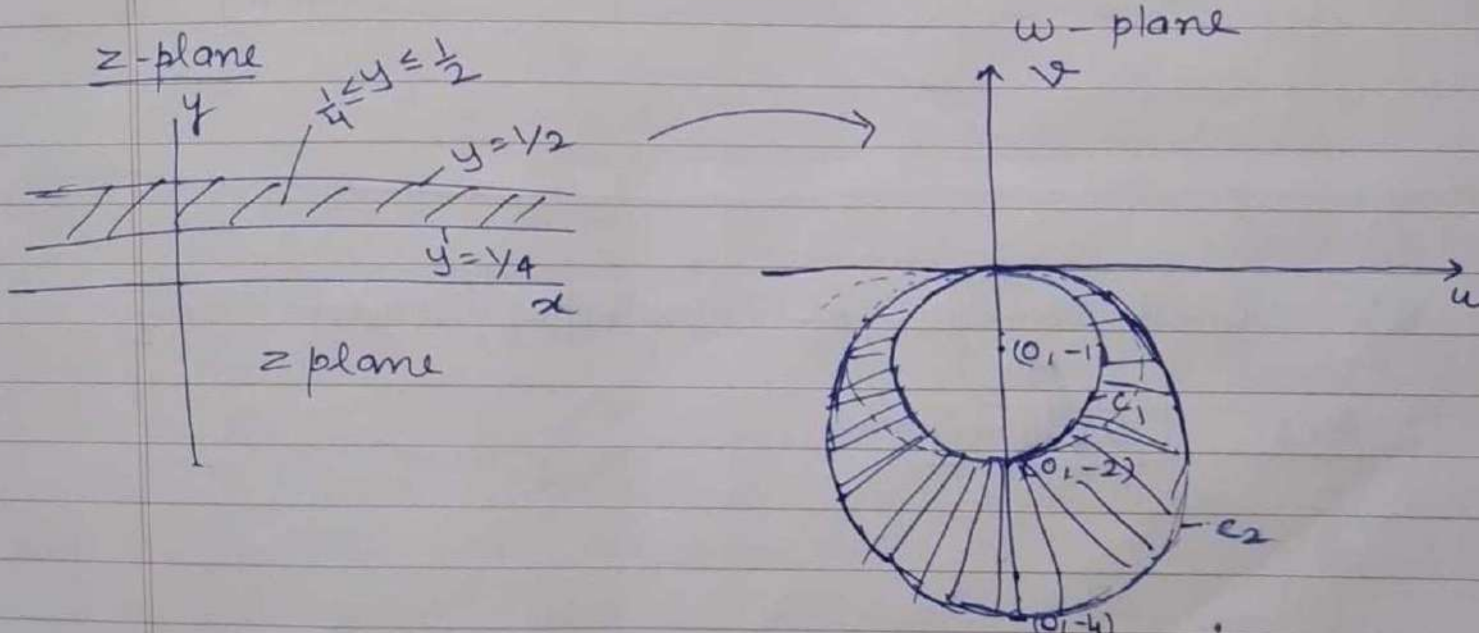
Now $y \leq \frac{1}{2}$

$$\frac{-v}{u^2+v^2} \leq \frac{1}{2}$$

$$u^2+v^2+2v \geq 0$$

$$u^2+(v+1)^2 \geq 1 \longrightarrow$$ represent on and outer portion of circle center $(0,-1)$ and radius 1.

circle :- center $(0,-1)$, radius 1.



area in z plane mapped to shaded area in w plane.

Note:- under the transformation $w = \frac{1}{z}$

(1) Circle C_1 in z plane is mapped to

either

straight line (if C_1 passes through origin) or circle (If C_1 does not passes through origin)

(2) a straight line (L_1) in z plane is mapped to.

straight line (If L_1 passes through origin) or circle (If L_1 does not passes through origin)

Q:- find image of $|z+1| = 1$ under $w = \frac{1}{z}$

Ans $\boxed{2u+1=0}$

Q: find the image of $|z-3i|=3$ under the transformation $w = 1/z$.

Sol^n. ~~(Hint)~~ we have $|z-3i| = 3$

$$|x+iy-3i|=3$$

$$|x+i(y-3)|=3$$

$$\sqrt{x^2+(y-3)^2}=3$$

$$x^2+(y-3)^2=9$$

$|z-3i|=3$ is $x^2+(y-3)^2=9$ is a circle with center $(0,3)$ & radius 3.

Now $w = \frac{1}{z}$

$$\Rightarrow x = \frac{u}{u^2+v^2}, \quad y = \frac{-v}{u^2+v^2}$$

we have $x^2+(y-3)^2=9$

$$\frac{u^2}{(u^2+v^2)^2} + \left(\frac{-v}{u^2+v^2} - 3\right)^2 = 9$$

$$\frac{u^2}{(u^2+v^2)^2} + \frac{v^2}{(u^2+v^2)^2} + 9 + \frac{6v}{u^2+v^2} = 9$$

$$(u^2+v^2) + \cancel{9(u^2+v^2)^2} + 6v(u^2+v^2) = \cancel{9(u^2+v^2)^2}$$

$$\cancel{(u^2+v^2) + 9(u^4+v^4+2u^2v^2) + 6vu^2 + 6v^3} = \cancel{9(u^4+v^4+2u^2v^2)}$$

$$(u^2+v^2) + 6v(u^2+v^2) = 0$$

$$\cancel{u^2+v^2+6u^2v+6v^3=0}$$

$$(u^2+v^2) + 6v(u^2+v^2) = 0$$

$$(u^2+v^2)(1+6v)=0$$

$\Rightarrow 1+6v=0$ since u^2+v^2 cannot be equal to

$$\Rightarrow \boxed{v = -\frac{1}{6}}$$ straight line.

Inversion & Reflection:

Ex:5 Find & sketch the image of region $x \geq 1$ under the transformation $w = \frac{1}{z}$

147 145 150 EF RC $l \to c$

The given transformation is

$$w = \frac{1}{z}$$

$$\therefore \quad z = \frac{1}{w}$$

$$\Rightarrow x + iy = \frac{1}{u+iv} \times \left(\frac{u-iv}{u-iv}\right)$$

$$= \frac{u-iv}{u^2+v^2}$$

$$\Rightarrow x + iy = \left(\frac{u}{u^2+v^2}\right) + i\left(\frac{-v}{u^2+v^2}\right)$$

comparing real & imaginary part,

$$\Rightarrow x = \frac{u}{u^2+v^2}, \quad y = \frac{-v}{u^2+v^2}$$

Now $x \geq 1$

$$\Rightarrow \frac{u}{u^2+v^2} \geq 1$$

$$\Rightarrow u \geq u^2+v^2$$

$$\Rightarrow u^2+v^2 \leq u$$

$$\Rightarrow u^2+v^2-u \leq 0$$

$$\Rightarrow \left(u-\frac{1}{2}\right)^2 + v^2 \leq \frac{1}{4}$$

which represents on & inner portion of the circle with center $\left(\frac{1}{2}, 0\right)$ & radius $\frac{1}{2}$

$x^2+y^2+2gx+2fy+c=0$ eq of circle center: $(-g, -f)$

$r = \sqrt{g^2+f^2-c}$

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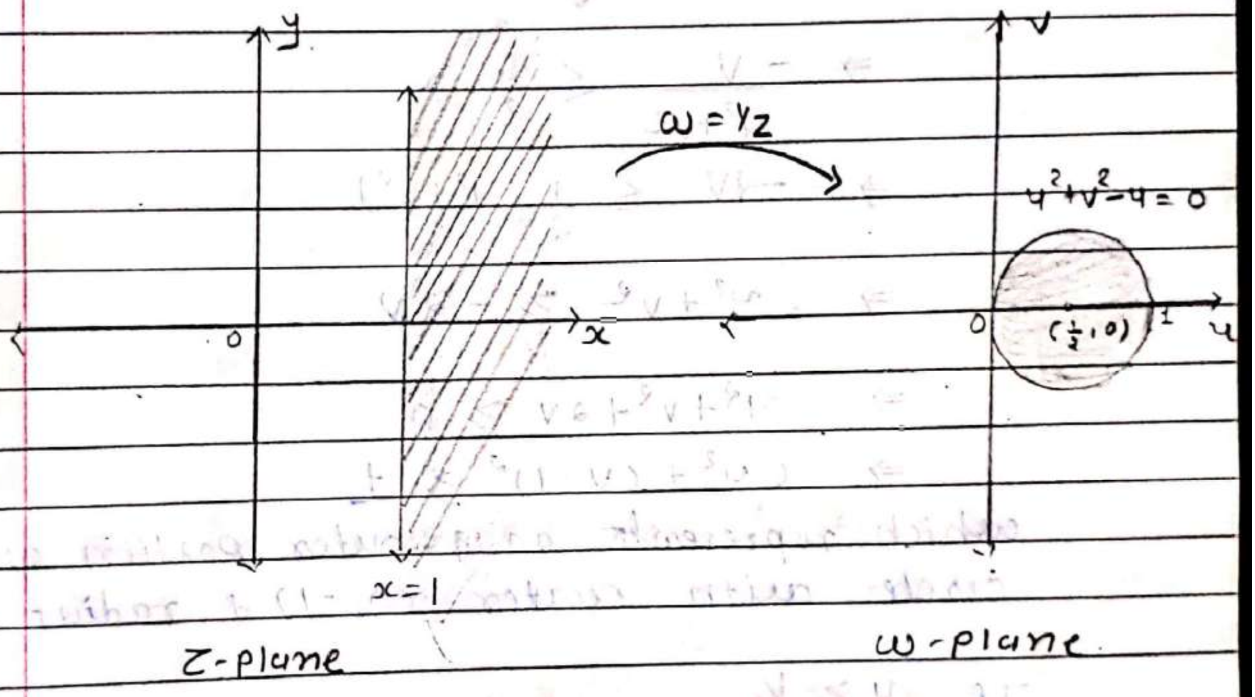
$(u-a)^2 + (v-b)^2 = r^2$

$u^2 - 2ua + a^2 + v^2 - 2vb + b^2 = r^2$

Now $u^2 + v^2 - u \leq 0$ $\quad \therefore a = 1/2, b = 0, r = 1/2$

Hence the image of region $x \geq 1$ under trans. $\omega = \frac{1}{z}$ is on & inner portion of the circle with center $(1/2, 0)$ & radius $1/2$.



Ex-5

Ex: 7 Find the image of the infinite strip $\frac{1}{4} \leq y \leq 1/2$ under the transformation $\omega = \frac{1}{z}$.

146 GP

148 RC Also show the region graphically in both

156 A the planes. $l \to c$

111 K.

$\omega = \frac{1}{z}$

$\Rightarrow z = \frac{1}{\omega}$

$\Rightarrow x + iy = \frac{1}{u+iv} \times \frac{u-iv}{u-iv}$

$= \frac{u - iv}{u^2+v^2}$

$= \frac{u}{u^2+v^2} + i\left(\frac{-v}{u^2+v^2}\right)$

$$\Rightarrow x = \frac{u}{u^2+v^2} \quad \& \quad y = \frac{-v}{u^2+v^2} \quad \text{—} ①$$

we have $y \le 1/2$ then ① becomes,

$$\Rightarrow \frac{-v}{u^2+v^2} \le 1/2$$

$$\Rightarrow -2v \le 1\,(u^2+v^2)$$

$$\Rightarrow u^2+v^2 \geq -2v$$

$$\Rightarrow u^2+v^2+2v \geq 0$$

$$\Rightarrow u^2+(v+1)^2 \geq 1$$

which represents on & outer portion of the circle with center $(0,-1)$ & radius 1.

If $y \geq 1/4$ $(1/4 \le y)$ then ① becomes,

$$\Rightarrow \frac{-v}{u^2+v^2} \geq 1/4$$

$$\Rightarrow -4v \geq u^2+v^2$$

$$\Rightarrow u^2+v^2 \le -4v$$

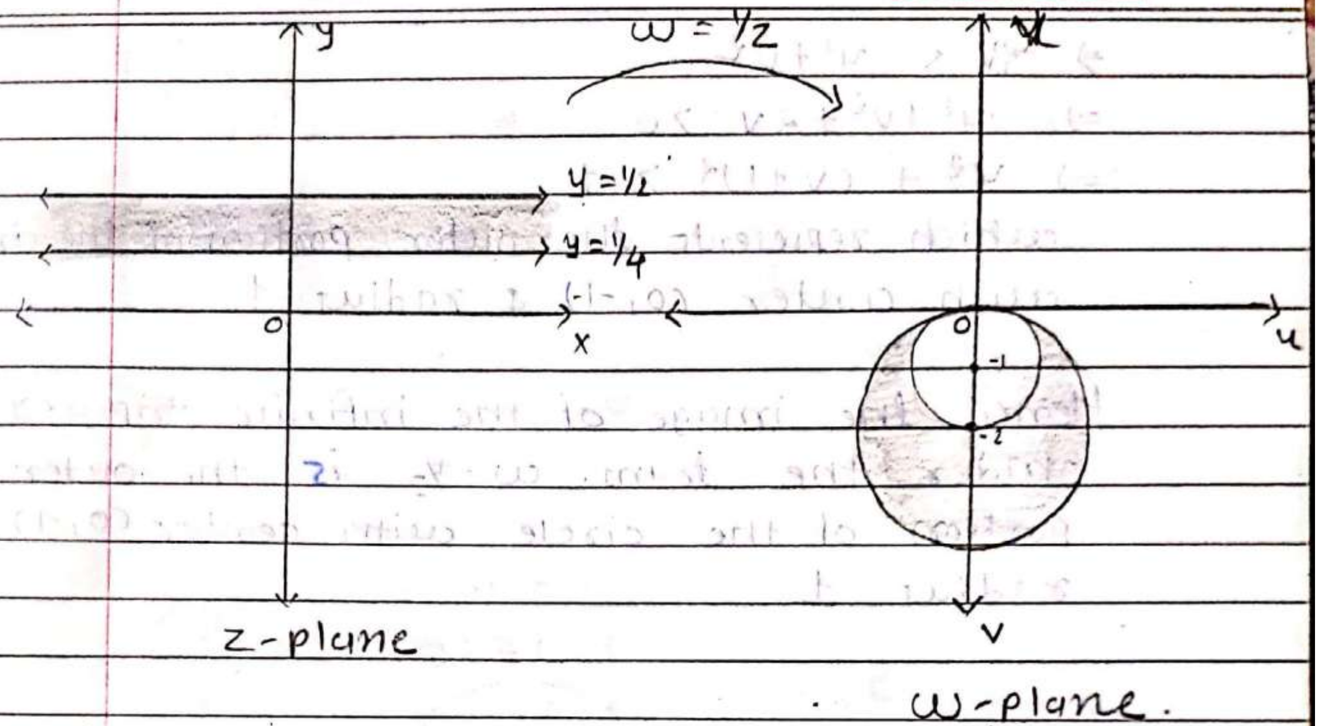
$$\Rightarrow u^2+v^2+4v \le 0$$

$$\Rightarrow u^2+(v+2)^2 \le 4$$

which represents on & inner portion of the circle center $(0,-2)$ & radius 2

Hence, the image of infinite strip $1/4 \le y \le 1/2$ under $w = 1/z$ is lies between the two circle $u^2+v^2+2v \geq 0$ & $u^2+v^2+4v \le 0$

The graphically representation of the above region are shown in figure.



Ex-6

EX: 8 find the image of the infinite strip $0 < y < 1/2$ under the transformation $w = \frac{1}{z}$
$l \to c$

$w = \frac{1}{z} \Rightarrow z = 1/w$

$x + iy = \frac{1}{u+iv} \times \frac{u-iv}{u-iv}$

$= \frac{u}{u^2+v^2} + i\left(\frac{-v}{u^2+v^2}\right)$

$\Rightarrow x = \frac{u}{u^2+v^2}$ & $y = \frac{-v}{u^2+v^2}$ —— ①

If $y = 0$, ① becomes

$\frac{-v}{u^2+v^2} = 0$

$\Rightarrow \boxed{v = 0}$

$0 < y$

$0 < -v/_{u^2+v^2}$

$0 < -v$ $\quad -v > 0$

$\boxed{v < 0}$

$0 < y < 1/2$

If $y < 1/2$,

$\Rightarrow \frac{-v}{u^2+v^2} < 1/2$

$0 < \frac{-v}{u^2+v^2} < \frac{1}{2}$

$\underbrace{0 < -v}$ $< \frac{u^2+v^2}{2}$

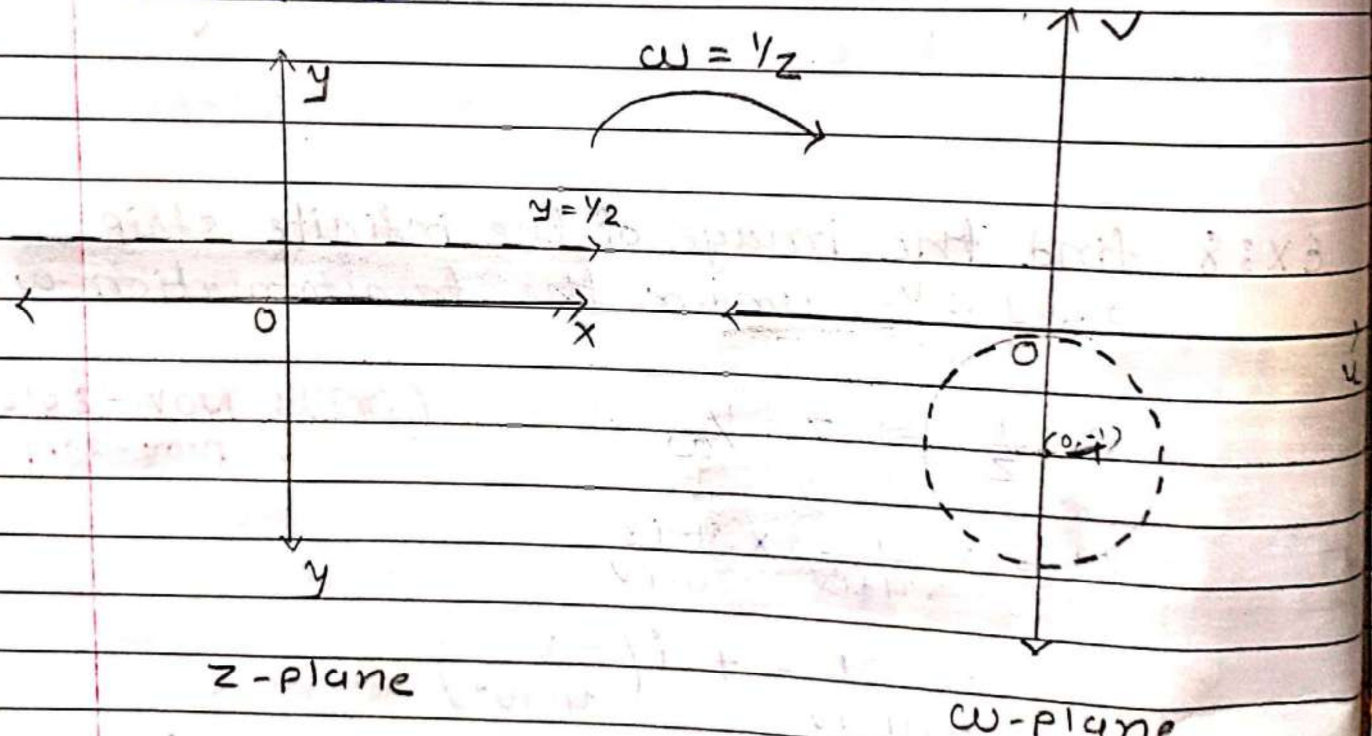
$\Rightarrow -2v < u^2+v^2$

$\Rightarrow u^2+v^2+2v > 0$

$\Rightarrow u^2+(v+1)^2 > 1$

which represents the outer portion of the circle with center $(0,-1)$ & radius 1

Hence, the image of the infinite strip $0<y<\frac{1}{2}$ under the trans. $w=\frac{1}{z}$ is the outer portion of the circle with center $(0,-1)$ & radius 1



Ex: 9 Find the images of the following under the transformation $w=\frac{1}{z}$

148 EP.

(i) $|z+2i|=3$ $C \to C$

(ii) $|z-1|=1$ $C \to L$

(iii) $y=-\frac{1}{2}$ $L \to C$

(iv) $2<x<4$ $L \to C$

(v) $x-y-1=0$ $L \to C$

Also sketch the region.

solution:

$$\omega = \frac{1}{z} \Rightarrow z = \frac{1}{\omega} = \frac{1}{u+iv} \times \frac{u-iv}{u-iv}$$

$$\Rightarrow x+iy = \frac{u}{u^2+v^2} + i\left(\frac{-v}{u^2+v^2}\right)$$

$$\therefore x = \frac{u}{u^2+v^2} \;\; \& \;\; y = \frac{-v}{u^2+v^2} \quad \text{—(1)}$$

(1) $|z+2i| = 3$

$|x+iy+2i| = 3$

$\therefore |x+i(y+2)| = 3$

$\therefore x^2 + (y+2)^2 = 9$

which rep. circle with center $(0,-2)$ & radius 3 in z-plane.

Putting the values from (1) & (2),

$$\frac{u^2}{(u^2+v^2)^2} + \left(2 - \frac{v}{u^2+v^2}\right)^2 = 9$$

$$\therefore \frac{u^2}{(u^2+v^2)^2} + 4 - \frac{4v}{u^2+v^2} + \frac{v^2}{(u^2+v^2)^2} = 9$$

$$\therefore \frac{u^2+v^2}{(u^2+v^2)^2} - \frac{4v}{u^2+v^2} = 9-4$$

$$\therefore \frac{1}{u^2+v^2} - \frac{4v}{u^2+v^2} - 5 = 0$$

$$\therefore 1 - 4v - 5(u^2+v^2) = 0$$

$$\therefore 5(u^2+v^2) + 4v - 1 = 0$$

$$\therefore u^2+v^2 + \frac{4}{5}v - \frac{1}{5} = 0$$

$2g = 4/5$ $\quad$ $\sqrt{g^2+f^2-c}$

$g = 2/5$ $\quad$ $\sqrt{\frac{4}{25}+\frac{1}{5}}$

$(-g, -f) = (0, -2/5)$ $\quad$ $\sqrt{\frac{4+5}{25}}$

$\sqrt{9/25}$

which repre. circle with center at $\left(0, -\frac{2}{5}\right)$ & radius $3/5$ in the w-plane.

$x^2+y^2+2gx+2fy+c=0$, eqn of circle

centre: $(-g,-f)$

radius: $\sqrt{g^2+f^2-c}$

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The image of circle is a circle. Hence circle $|z+2i|=3$ in the z-plane is transformed on to circle $\left|z+\frac{2i}{5}\right| = \frac{3}{5}$ in the w-plane

$(u-a)^2+(v-b)^2=r^2$

$u^2-2ua+a^2+v^2-2vb+b^2=r^2$ $\qquad u^2+v^2+\frac{4}{5}v-\frac{1}{5}=0$

Here $-2ua=0$ $\qquad -2vb=\frac{4}{5}v$

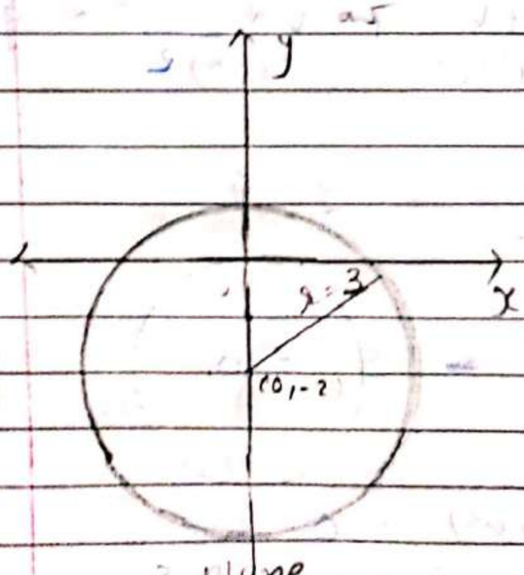
$\Rightarrow a=0$ $\qquad -2b=4/5$

$b=-4/2\cdot5$

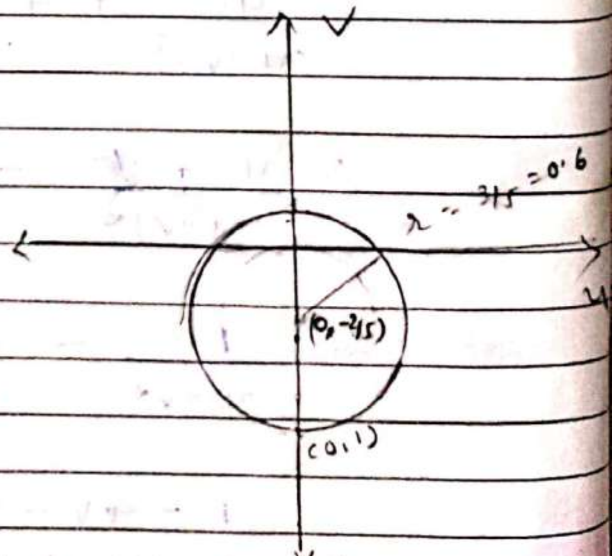
$\boxed{b=-2/5}$

$r^2=\frac{1}{5}+\frac{4}{25}$

$=5+4/25 = 9/25 = \boxed{3/5=r}$



z plane



w-plane

$2g=0$, $\quad 2f=4/5$ $\qquad r=\sqrt{g^2+f^2-c}$

$g=0$ $\quad f=4/10=2/5$ $\qquad =\sqrt{0+\frac{4}{25}+\frac{1}{5}}=\sqrt{9/25}$

$\therefore(-g,-f)=(0,-2/5)$ $\qquad =3/5$

$2/5 = 0.4$
$3/5 = 0.6$

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(2) $|z-1| = 1$

The eqⁿ. $|z-1|=1$ represents a circle with centre at $(1,0)$ & radius $r=1$ in the z-plane.

putting $z = \frac{1}{\omega}$

$\therefore |z-1| = 1$

$\therefore \left|\frac{1}{\omega} - 1\right| = 1$

$\therefore |1 - \omega| = |\omega|$

$\therefore |1-(u+iv)|^2 = |u+iv|^2$

$\therefore |(1-u)+iv|^2 = |u+iv|^2$

$\therefore (1-u)^2 + v^2 = u^2+v^2$

$\therefore 1 - 2u = 0$

$\therefore \boxed{u = 1/2}$

$|z-1| = 1$

$|x+iy-1| = 1$

$(x-1)^2 + y^2 = 1$

$x = \frac{u}{u^2+v^2}$, $y = \frac{-v}{u^2+v^2}$ put

$\left(\frac{u}{u^2+v^2} - 1\right)^2 + \frac{v^2}{(u^2+v^2)^2} = 1$

$\frac{u^2}{(u^2+v^2)^2} - \frac{2u}{u^2+v^2} + 1 + \frac{v^2}{(u^2+v^2)^2} = 1$

$\frac{1}{u^2+v^2} - \frac{2u}{u^2+v^2} = 0$

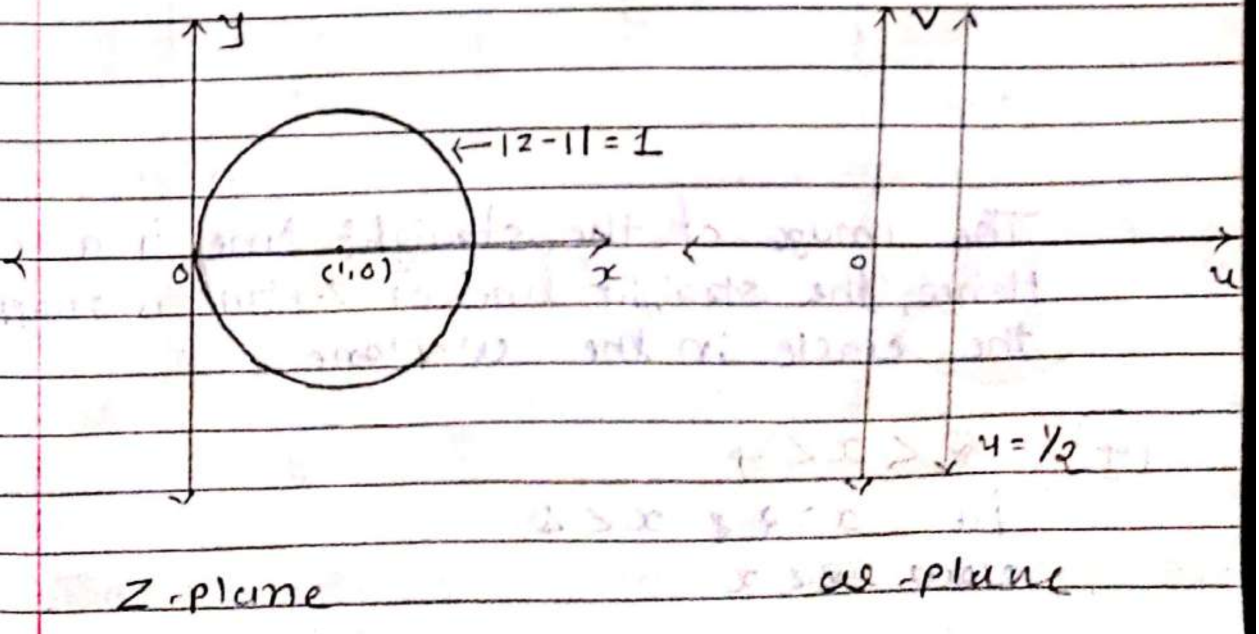
$1 - 2u = 0$

$\boxed{u = 1/2}$

which represents a straight line in the w-plane.

$\therefore$ The image of the circle is a straight line.

Hence, circle $|z-1|=1$ in the z-plane is tran. on to the straight line $u=1/2$ in the w-plane



(3) $y = -\frac{1}{2}$

From (2) putting $y = \dfrac{-v}{u^2+v^2}$

Now $y = -\dfrac{1}{2}$

$$\therefore \frac{-v}{u^2+v^2} = -\frac{1}{2}$$

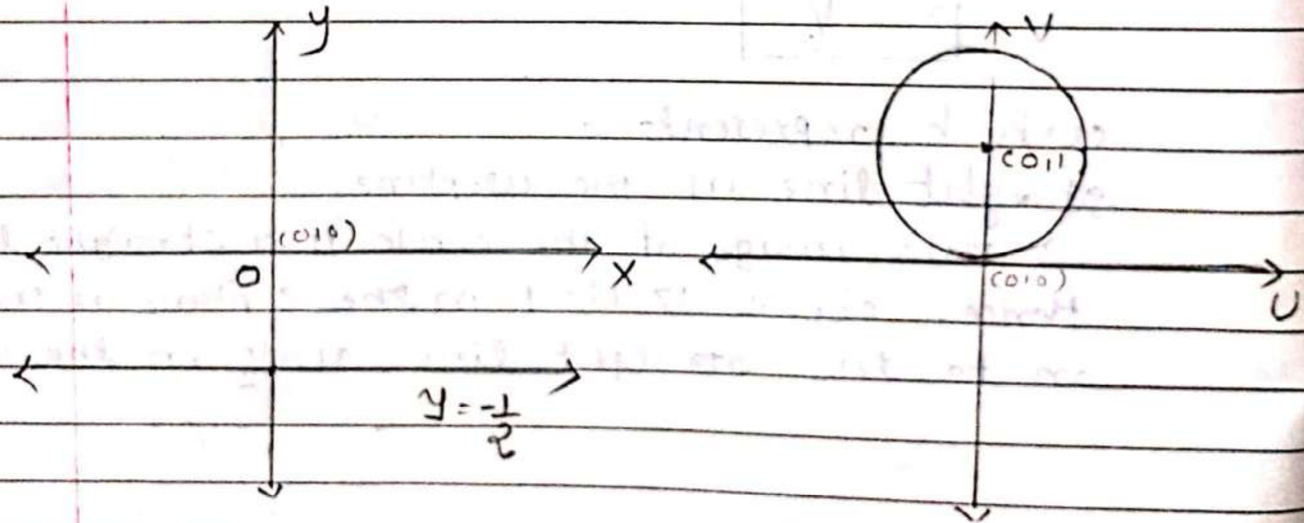
$$\therefore 2v = u^2+v^2$$

$$\therefore u^2+v^2-2v = 0$$

$$\therefore (u-0)^2 + (v-1)^2 = 1$$

which is circle in the ω-plane with centre at $(0,1)$ & radius 1

$\left(\begin{array}{lll} 2g = 0, & 2f = -2 & \therefore (-g,-f) = (0,1) \\ g = 0 & f = -1 & r = \sqrt{g^2+f^2-c} = \sqrt{1} = 1 \end{array} \right)$



z-plane $\qquad\qquad$ ω-plane

The image of the straight line is a circle. Hence, the straight line of z-plane is mapped on to the circle in the ω-plane

(4) $2 < x < 4$

i.e. $x > 2$ & $x < 4$

Now $2 < x$

$\therefore\ 2 < \dfrac{u}{u^2+v^2}$ $\quad \left(\because\ x = \dfrac{u}{u^2+v^2}\right)$

$\therefore\ 2u^2+2v^2 < u$

$\therefore\ u^2+v^2-\dfrac{u}{2} < 0$ —— (1) $\qquad r = \sqrt{g^2+f^2-c}$ $= \sqrt{1/16+0-0} = 1/4$

This is a circle with centre at $\left(\frac{1}{4}, 0\right)$ & radius $\frac{1}{4}$ $\quad \left(\begin{array}{lll} 2g = -1/2 & 2f = 0 & \therefore (-g,-f) = \\ g = -1/4 & f = 0 & (1/4, 0) \end{array}\right.$

Hence $x>2$ is mapped on the interior of the circle $u^2+v^2-\frac{1}{2}u = 0$

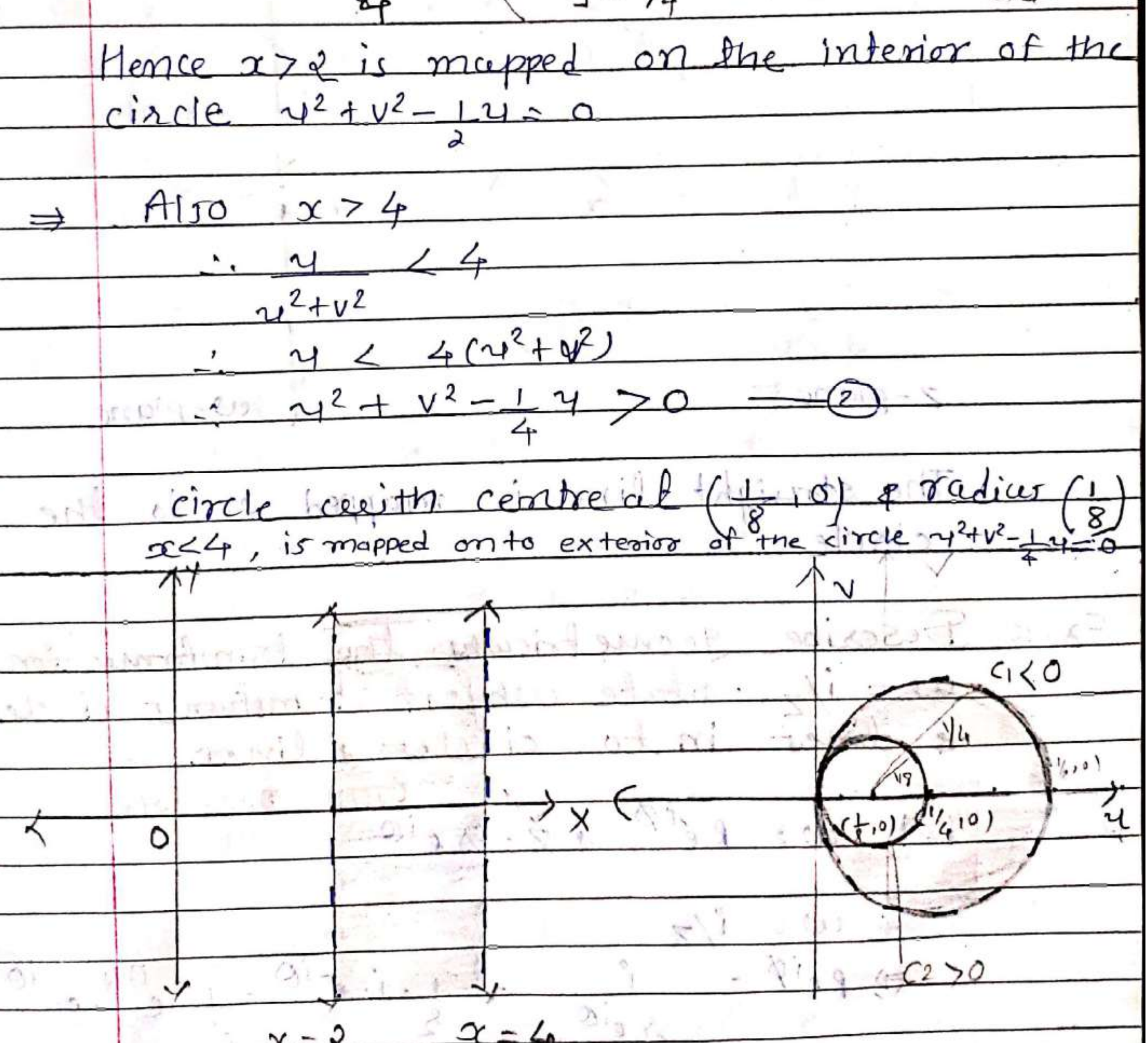
⇒ Also $x > 4$

$\therefore\ \dfrac{u}{u^2+v^2} < 4$

$\therefore\ u < 4(u^2+v^2)$

$\therefore\ u^2+v^2-\dfrac{1}{4}u > 0$ —— (2)

circle with centre at $\left(\frac{1}{8}, 0\right)$ & radius $\left(\frac{1}{8}\right)$
$x<4$, is mapped onto exterior of the circle $u^2+v^2-\frac{1}{4}u = 0$



Thus the infinite vertical strip of z-plane is mapped on to the region between two circle C1<0 & C2>0

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(5) $x - y - 1 = 0$

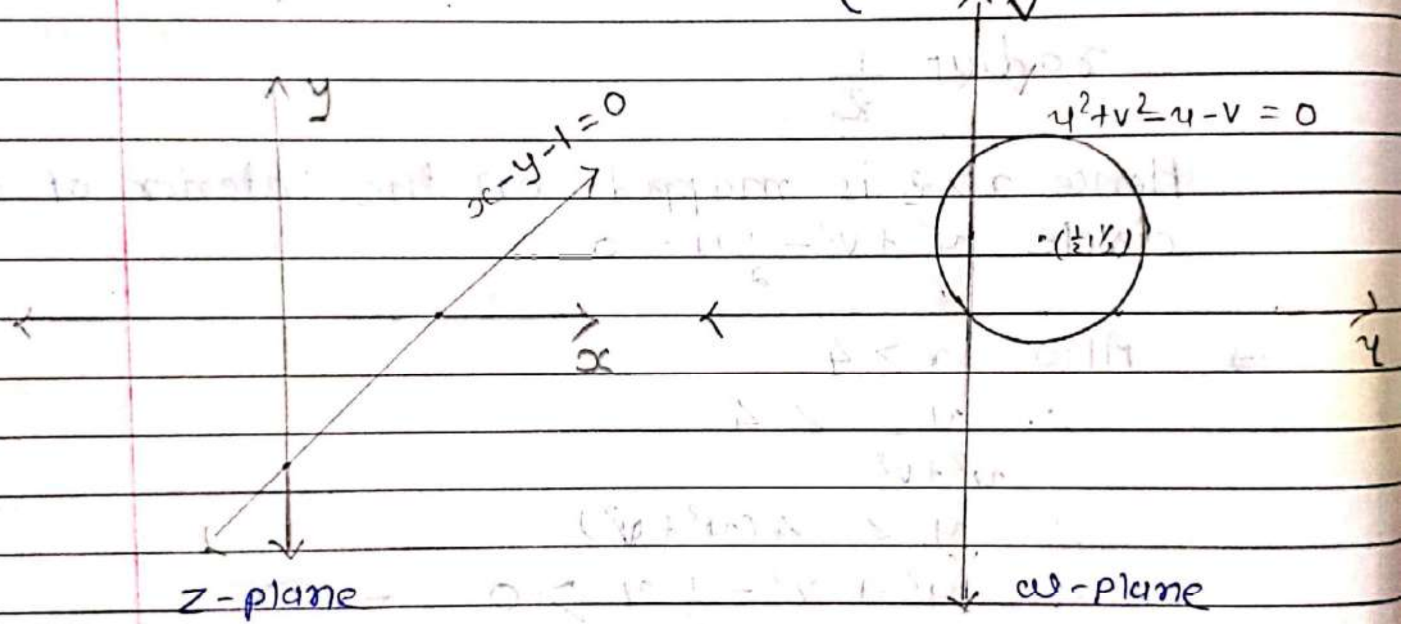
From (1) & (2),

$$\frac{u}{u^2+v^2} + \frac{v}{u^2+v^2} - 1 = 0$$

$$\therefore \quad u + v - u^2 - v^2 = 0$$

$$\therefore \quad u^2 + v^2 - u - v = 0$$

circle with centre at $\left(\frac{1}{2}, \frac{1}{2}\right)$ & radius is $\sqrt{\frac{1}{2}}$



The straight line is mapped on to the circle.

Ex:10 Describe geometrically the transformation $w = i/z$. State why it transforms circles & lines in to circles & lines.

187 265
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Let $w = Re^{i\phi}$ & $z = re^{i\theta}$

$$\therefore w = i/z$$

$$\Rightarrow Re^{i\phi} = \frac{i}{re^{i\theta}} = \frac{1}{r} \cdot i \cdot e^{-i\theta} = \frac{1}{r} e^{i\pi/2} \cdot e^{-i\theta}$$

$$\left(\because i = e^{i\pi/2}\right)$$

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$$= \frac{1}{r} e^{i(\pi/2 - \theta)}$$

$$\therefore R = \frac{1}{r} \quad \& \quad \phi = \frac{\pi}{2} - \theta$$

By this transformation, a point $P(R, \phi)$ in the z-plane is mapped in to the point $Q\left(\frac{1}{r}, \frac{\pi}{2} - \theta\right)$ in the w-plane.

Now, $w = u + iv$, $z = x + iy$

$$w = \frac{i}{z}$$

$$z = \frac{i}{w}$$

$$x + iy = \frac{i}{u+iv} \times \left(\frac{u-iv}{u-iv}\right)$$

$$= \frac{iu + v}{u^2 + v^2} = \frac{v}{u^2+v^2} + i\left(\frac{u}{u^2+v^2}\right)$$

$$\therefore x = \frac{v}{u^2+v^2} \quad \& \quad y = \frac{u}{u^2+v^2}$$

The general eqn. of any circle in the z-plane is $x^2 + y^2 + 2gx + 2fy + c = 0$

$$\therefore \frac{v^2}{(u^2+v^2)^2} + \frac{u^2}{(u^2+v^2)^2} + \frac{2gv}{u^2+v^2} + \frac{2fu}{u^2+v^2} + c = 0$$

$$\therefore \frac{u^2+v^2}{(u^2+v^2)^2} + \frac{2g \cdot v}{u^2+v^2} + \frac{2f \cdot u}{u^2+v^2} + c = 0$$

$$\therefore c(u^2+v^2) + 2fu + 2gv + 1 = 0 \quad \text{--- (1)}$$

If $c \neq 0$, eqn (1) is a circle & if $c = 0$ then (1) is a straight line.

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Again, the eqⁿ of straight line is,

$$ax+by+c=0$$

$$\Rightarrow a\left(\frac{v}{u^2+v^2}\right)+b\left(\frac{u}{u^2+v^2}\right)+c=0$$

$$\Rightarrow c(u^2+v^2)+av+bu=0 \quad \text{—(2)}$$

If $c\neq 0$, eqⁿ (2) is a circle & if $c=0$ then eqⁿ (2) is a straight line.

Thus, the given transformation transforms circles and lines into circles and lines.

Ex: 11 Translation:—

140 EP Find the image of the rectangular region whose vertices are $0, 1, 1+2i$ & $2i$ under the transformation $w=(1+i)z+2$ sketch these regions.

If $Z_1=(1+i)\cdot z$ —(1)

then $w=(1+i)\cdot z+2$ takes the form

$$w=Z_1+2 \quad \text{—(2)}$$

As $1+i=\sqrt{2}\cdot e^{i\pi/4}$ $(r=\sqrt{2},\ \theta=\pi/4)$

The first transformation $Z_1=(1+i)z$ is an expansion by the factor $\sqrt{2}$ & a rotation through the angle $\frac{\pi}{4}$ $[\arg(1+i)]$

The second trans. $w=Z_1+2$ is a translation one unit to the right.

$$Z_1=(1+i)z$$
$$=(1+i)\cdot(x+iy)$$
$$=x+iy+ix-y$$
$$X+iY=(x-y)+i(x+y)$$

Table

	Transformation	eq^n	
1.	Identity transformation	$w = z$	
2.	Translation	$w = z + c$	z is cx. no
3	Rotation	$w = e^{i\theta} z$	θ is real const $\theta > 0$, rotation is anticlockwise $\theta < 0$, rotation is clockwise. $\theta = 0$ no rotation.
4.	Magnification	$w = cz$	c is real If $0 < c < 1$, its stretching.
5.	Rotation and Magnification	$w = cz$	c is complex constant
6.	Linear transformation = translation + Magnification + rotation	$w = az + b$	a & b both are complex
7.	Inversion (Reflection)	$w = \frac{1}{z}$, $z \neq 0$	

classmate

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Example :- Find image of the following regions under the transformation $w = \frac{1}{z}$.

(1) half plane $x > c$ when $c > 0$.

(2) half plane $y > c$ when $c < 0$

(3) the infinity strips $\frac{1}{4} \leq y \leq \frac{1}{2}$.

Sol^n :- $w = \frac{1}{z}$.

$$z = \frac{1}{w}$$

$$z = \frac{u - iv}{u^2 + v^2}$$

$$x = \frac{u}{u^2+v^2}, \quad y = \frac{-v}{u^2+v^2}$$

(1) image of the region $x > c$ when $c > 0$

$$\frac{u}{u^2+v^2} > c$$

$$u > c(u^2+v^2)$$

$$c(u^2+v^2) < u$$

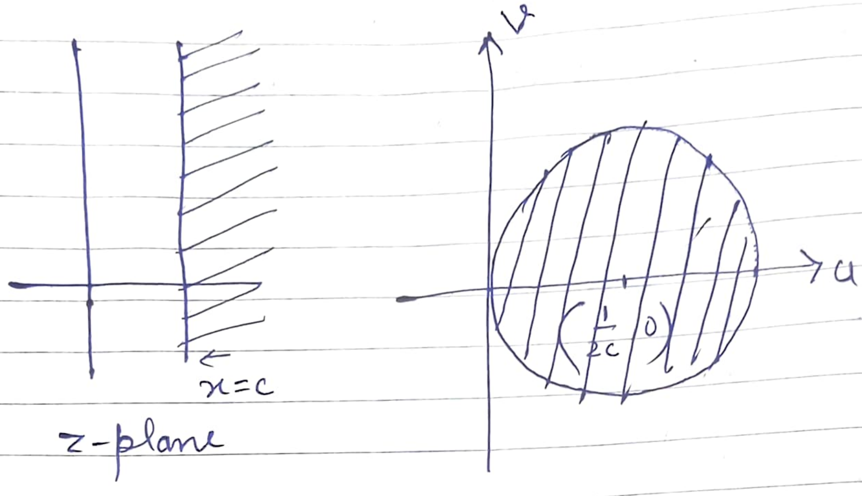
$$u^2+v^2 < \frac{u}{c}$$

$$u^2+v^2 - \frac{u}{c} < 0 \qquad (c > 0)$$

$$u^2 - \frac{u}{c} + \left(\frac{1}{2c}\right)^2 + v^2 < \left(\frac{1}{2c}\right)^2$$

$$\left(u-\frac{1}{2c}\right)^2+v^2<\left(\frac{1}{2c}\right)^2 \quad \text{—— (i)}$$

circle with center $\left(\frac{1}{2c}, 0\right)$
and radius $\left(\frac{1}{2c}\right)$



(2) image of region $y>0$, when $c>0$

$$\frac{-v}{u^2+v^2}>c$$

$$-v>c(u^2+v^2)$$

$$c(u^2+v^2)<-v$$

$$u^2+v^2+\frac{v}{c}>0$$

$$u^2+v^2+\frac{v}{c}+\left(\frac{1}{2c}\right)^2>\left(\frac{1}{2c}\right)^2$$

$$u^2+\left(v+\frac{1}{2c}\right)^2>\left(\frac{1}{2c}\right)^2$$

$$u^2 + \left(v + \frac{1}{2c}\right)^2 = \left(\frac{1}{2c}\right)^2$$

y is circle $\left(0, -\frac{1}{2c}\right)$, radius $\left(\frac{1}{2c}\right)$