Module:5 Multiple integrals

- Evaluation of double integrals
- Change of order of integration
- Change of variables between Cartesian and polar co-ordinates
- Evaluation of triple integrals
- Change of variables between Cartesian and cylindrical and spherical co-ordinates
- Evaluation of multiple integrals using gamma and beta functions

Evaluation of double integrals

double integral of f over R, written as

$$\iint_R f(x, y)\, dA \qquad \text{or} \qquad \iint_R f(x, y)\, dx\, dy.$$

THEOREM If $f(x, y)$ is continuous throughout the rectangular region R: $a \leq x \leq b$, $c \leq y \leq d$, then

$$\iint_R f(x, y)\, dA = \int_c^d \int_a^b f(x, y)\, dx\, dy = \int_a^b \int_c^d f(x, y)\, dy\, dx.$$

EXAMPLE Calculate $\iint_R f(x, y)\, dA$ for

$$f(x, y) = 100 - 6x^2y \qquad \text{and} \qquad R:\quad 0 \le x \le 2, \quad -1 \le y \le 1.$$

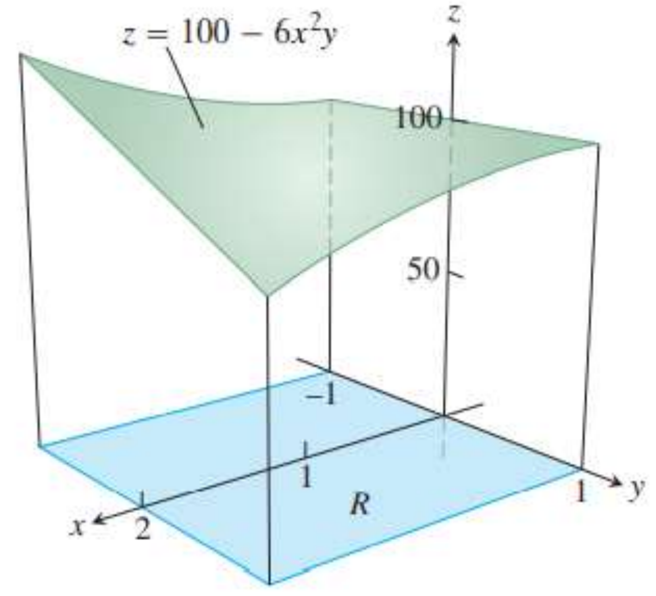
Solution

$$\iint_R f(x, y)\, dA = \int_{1}^{1}\int_0^2 (100 - 6x^2y)\, dx\, dy$$

$$= \int_{1}^{1} \Big[100x - 2x^3y\Big]_{x=0}^{x=2}\, dy$$

$$= \int_{-1}^{1} (200 - 16y)\, dy$$

$$= \Big[200y - 8y^2\Big]_{-1}^{1} = 400.$$



The double integral $\iint_R f(x, y)\, dA$ gives the volume under this surface over the rectangular region R

Reversing the order of integration gives the same answer:

$$\int_0^2\int_{-1}^{1} (100 - 6x^2y)\, dy\, dx = \int_0^2 \Big[100y - 3x^2y^2\Big]_{y=-1}^{y=1}\, dx$$

$$= \int_0^2 [(100 - 3x^2) - (-100 - 3x^2)]\, dx$$

$$= \int_0^2 200\, dx = 400.$$

Evaluate the following integrals

Answer 1)16 and 2) 14

1) $\int_0^3 \int_0^2 (4 - y^2)\, dy\, dx$

2) $\iint_R (6y^2 - 2x)\, dA, \quad R: \quad 0 \leq x \leq 1, \quad 0 \leq y \leq 2$

Find the volume of the region bounded above by the paraboloid $z = x^2 + y^2$ and below by the square $R: -1 \leq x \leq 1$, $-1 \leq y \leq 1$.

Answer 8/3

THEOREM Let $f(x, y)$ be continuous on a region R.

1. If R is defined by $a \leq x \leq b$, $g_1(x) \leq y \leq g_2(x)$, with g_1 and g_2 continuous on $[a, b]$, then

$$\iint_R f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx.$$

2. If R is defined by $c \leq y \leq d$, $h_1(y) \leq x \leq h_2(y)$, with h_1 and h_2 continuous on $[c, d]$, then

$$\iint_R f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy.$$

Finding Limits of Integration

Using Vertical Cross-sections When faced with evaluating $\iint_R f(x, y)\, dA$, integrating first with respect to y and then with respect to x, do the following three steps:

1. *Sketch.* Sketch the region of integration and label the bounding curves
2. *Find the y-limits of integration.* Imagine a vertical line L cutting through R in the direction of increasing y. Mark the y-values where L enters and leaves. These are the y-limits of integration and are usually functions of x (instead of constants)
3. *Find the x-limits of integration.* Choose x-limits that include all the vertical lines through R. The integral

$$\iint_R f(x, y)\, dA$$

Using Horizontal Cross-sections To evaluate the same double integral as an iterated integral with the order of integration reversed, use horizontal lines instead of vertical lines in Steps 2 and 3. The integral is

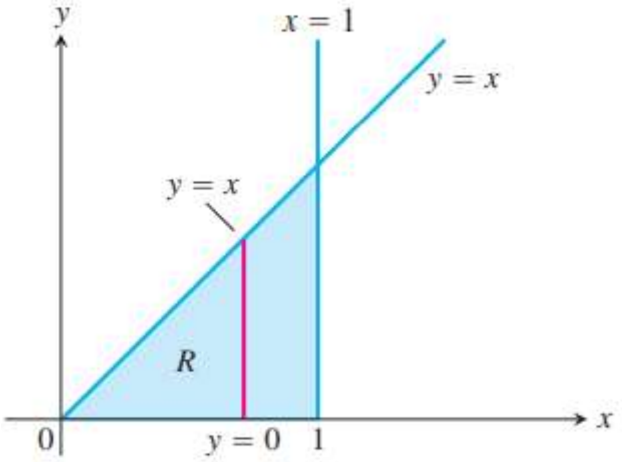
$$\iint_R f(x, y)\, dA = \iint_R f(x, y)\, dx\, dy.$$

EXAMPLE Find the volume of the prism whose base is the triangle in the xy-plane bounded by the x-axis and the lines $y = x$ and $x = 1$ and whose top lies in the plane

$$z = f(x, y) = 3 - x - y.$$

$$V = \int_0^1 \int_0^x (3 - x - y)\, dy\, dx = \int_0^1 \left[3y - xy - \frac{y^2}{2}\right]_{y=0}^{y=x} dx$$

$$= \int_0^1 \left(3x - \frac{3x^2}{2}\right) dx = \left[\frac{3x^2}{2} - \frac{x^3}{2}\right]_{x=0}^{x=1} = 1.$$

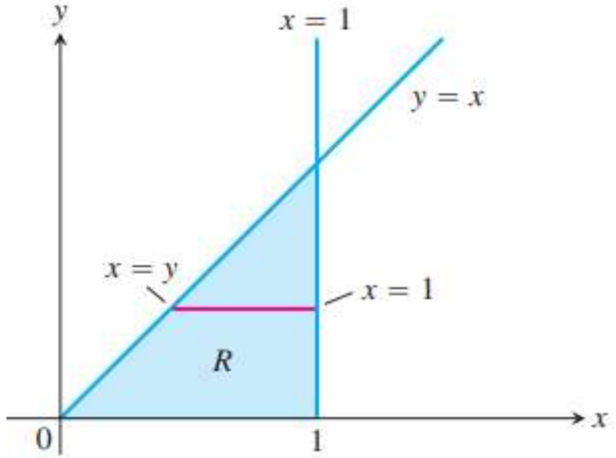


When the order of integration is reversed , the integral for the volume is

$$V = \int_0^1 \int_y^1 (3 - x - y)\, dx\, dy = \int_0^1 \left[3x - \frac{x^2}{2} - xy\right]_{x=y}^{x=1} dy$$

$$= \int_0^1 \left(3 - \frac{1}{2} - y - 3y + \frac{y^2}{2} + y^2\right) dy$$

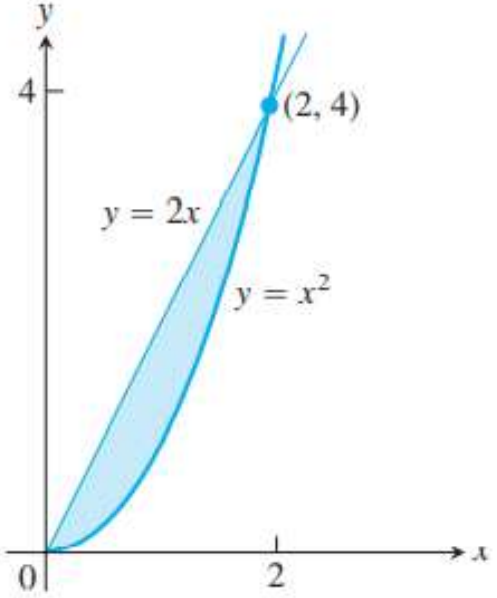
$$= \int_0^1 \left(\frac{5}{2} - 4y + \frac{3}{2}y^2\right) dy = \left[\frac{5}{2}y - 2y^2 + \frac{y^3}{2}\right]_{y=0}^{y=1} = 1.$$



Sketch the region of integration for the integral

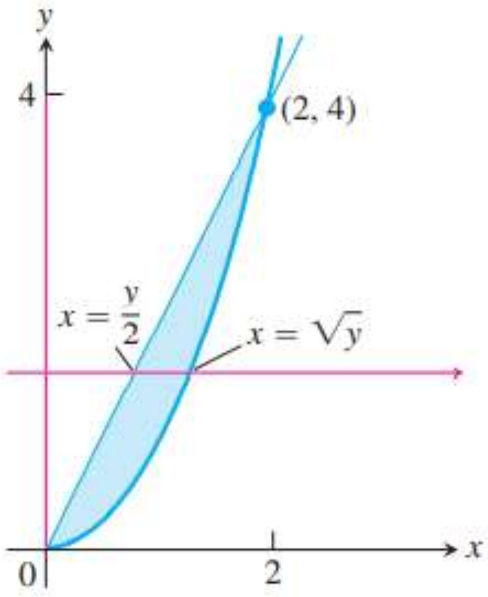
$$\int_0^2 \int_{x^2}^{2x} (4x + 2)\, dy\, dx$$

and write an equivalent integral with the order of integration reversed.



$$\int_0^4 \int_{y/2}^{\sqrt{y}} (4x + 2)\, dx\, dy.$$

integrals is 8.



Solution The region of integration is given by the inequalities $x^2 \le y \le 2x$ and $0 \le x \le 2$. It is therefore the region bounded by the curves $y = x^2$ and $y = 2x$ between $x = 0$ and $x = 2$

To find limits for integrating in the reverse order, we imagine a horizontal line passing from left to right through the region. It enters at $x = y/2$ and leaves at $x = \sqrt{y}$. To include all such lines, we let y run from $y = 0$ to $y = 4$

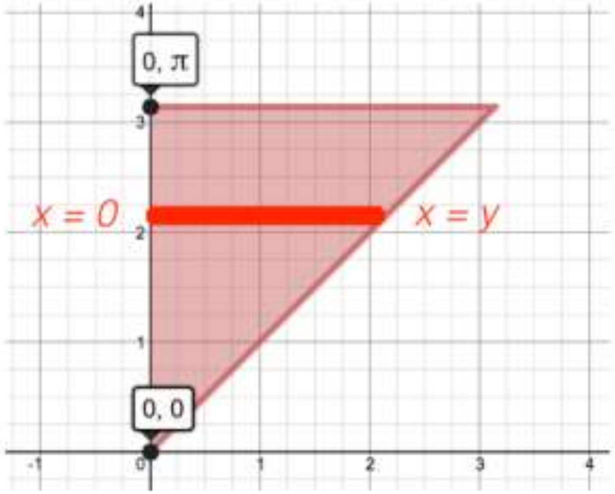
The integral is

$$\int_0^4 \int_{y/2}^{\sqrt{y}} (4x + 2)\, dx\, dy = \int_0^4 (2x^2 + 2x)\Big]_{y/2}^{\sqrt{y}}\, dy$$

$$= \int_0^4 (y + 2\sqrt{y} - y^2/2)\, dy$$

$$= 8$$

2. Sketch the region of integration, reverse the order of integration, and evaluate the integral $\int_0^{\pi}\int_x^{\pi} \frac{\sin y}{y}\, dy\, dx$



The region of Integration can be defined as

$R = \{(x, y) \mid x \le y \le \pi,\ 0 \le x \le \pi\}$

Sketch of the region is shown :

Now the region by using horizontal cross-sections

After reversing the order of integration we get

$$\int_0^{\pi}\int_0^{y} \frac{\sin y}{y}\, dx\, dy = \int_0^{\pi} \left[\frac{x \sin y}{y}\right]_0^{y} dy$$

$$= \int_0^{\pi} \sin y\, dy$$

$$= \Big[-\cos y\Big]_0^{\pi}$$

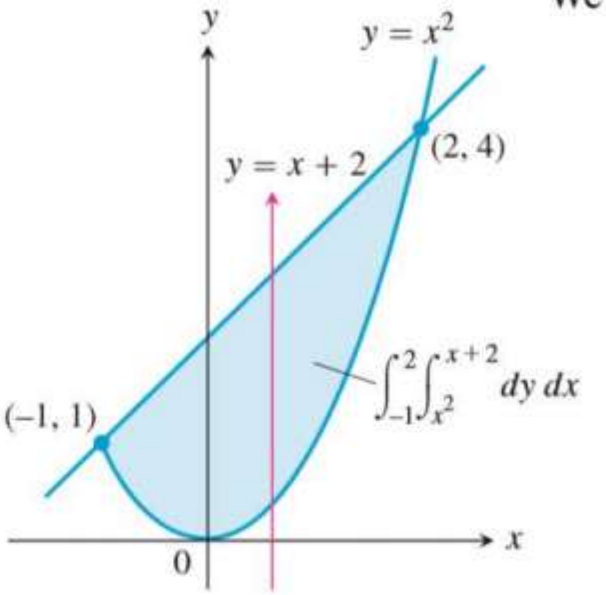
$$= -\cos(\pi) + \cos(0)$$

$$= 1 + 1 = 2$$

DEFINITION The **area** of a closed, bounded plane region R is

$$A = \iint_R dA.$$

Find the area of the region R enclosed by the parabola $y = x^2$ and the line $y = x + 2$?



We sketch the region and calculate the area as

$$A = \iint_R dA$$

$$= \int_{-1}^{2} \Big[y \Big]_{x^2}^{x+2} dx$$

$$= \int_{-1}^{2} (x + 2 - x^2)\, dx$$

$$= \left[\frac{x^2}{2} + 2x - \frac{x^3}{3} \right]_{-1}^{2}$$

$$= \frac{9}{2}$$