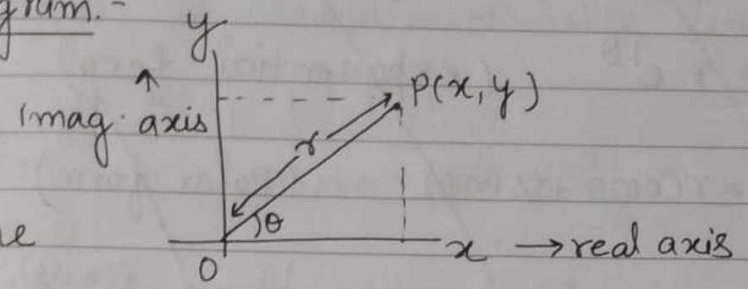
Module - I

Complex number:-

$$z = x + iy$$

Argand diagram:-



$r = \sqrt{x^2+y^2}$ is the distance OP.

Angle that OP makes with x axis is argument of z.

Let $z = x+iy$ be complex number

put $x = r\cos\theta$ $y = r\sin\theta$, $r = \sqrt{x^2+y^2}$

& $\theta = \tan^{-1}\left(\frac{y}{x}\right)$, $\cos\theta = \frac{x}{\sqrt{x^2+y^2}}$, $\sin\theta = \frac{y}{\sqrt{x^2+y^2}}$

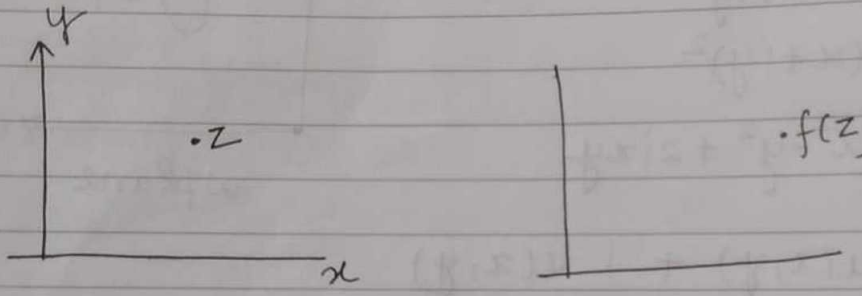
θ is argument or amplitude of z.

* θ has infinite number of values differening by multiples of 2π. The values of θ lying in $-\pi \le \theta \le \pi$ is called principal value of θ. ie Principal values of θ is written either between 0 to π or between 0 and $-\pi$.

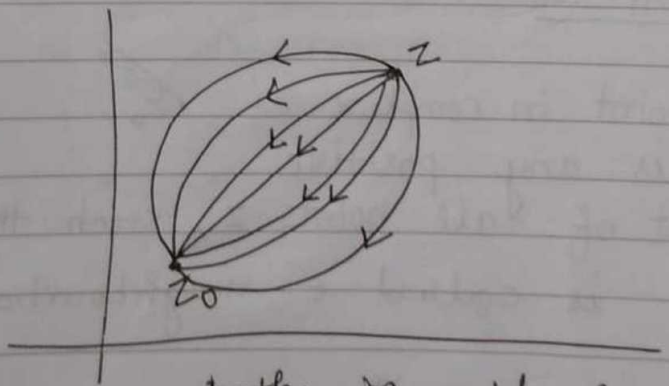
Limit of function of complex variable:-

Let $f(z)$ be a single valued function, define at all point in some neighbourhood of z_0, Then $\lim_{z \to z_0} f(z)$ is exist if limit is independent of path by which z approaches to z_0.

If we get two different limits as $z \to z_0$, along two different paths then limit does not exist.



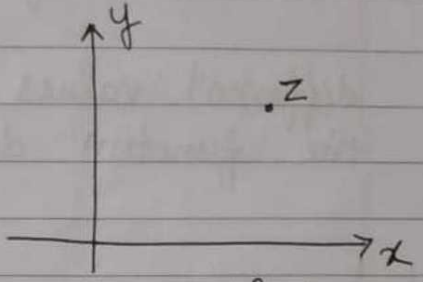
z plane w plane



paths in z plane

(1) Show that $\lim_{z\to 0} \frac{z}{|z|}$ does not exist.

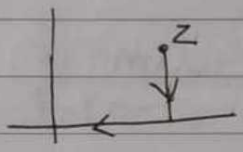
Soln $\quad f(z) = \frac{z}{|z|}$



z plane

$$\lim_{z\to 0} \frac{z}{|z|} = \lim_{x\to 0, y\to 0} \frac{x+iy}{\sqrt{x^2+y^2}}$$

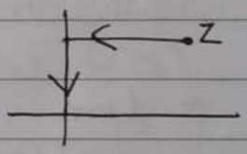
Case I:- Let $z \to 0$ along the path $y \to 0$ then $x \to 0$



$$= \lim_{x\to 0} \left(\frac{x}{\sqrt{x^2}}\right) = 1$$

i.e $\lim_{z\to 0} \frac{z}{|z|} = 1$

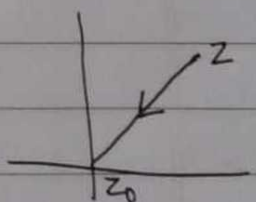
Case II:- Let $z \to 0$ along the path $x \to 0$ then $y \to 0$



$$\lim_{z\to 0} \frac{z}{|z|} = \lim_{y\to 0} \frac{iy}{y}$$

$$\lim_{z\to 0} \frac{z}{|z|} = i$$

Since along two different paths along which $z \to 0$ limit is not unique. therefor $\lim_{z\to 0} \frac{z}{|z|}$ does not exist. or we can take general path $y = mx$.



$$\lim_{z\to 0} \frac{z}{|z|} = \lim_{x\to 0} \frac{x+imx}{\sqrt{x^2+m^2x^2}}$$

$$= \lim_{x \to 0} \frac{1+im}{\sqrt{1+m^2}} = \frac{1+im}{\sqrt{1+m^2}}$$

The value of $\frac{1+im}{\sqrt{1+m^2}}$ is different for different values of m. Therefor limit of the function does not exist.

Q:- find the limit of function

$$\lim_{z \to 1+i} \frac{z^2 - z + 1 - i}{z^2 - 2z + 2}$$

$$= \lim_{z \to 1+i} \frac{(z^2+1) - (z+i)}{(z-1)^2 - i^2}$$

$$= \lim_{z \to 1+i} \frac{(z-i)(z+i) - (z+i)}{(z-1+i)(z-1-i)}$$

$$= \lim_{z \to 1+i} \frac{(z+i)(z-i-1)}{(z-1+i)(z-1-i)}$$

$$= \frac{1+2i}{2i} \quad \text{or} \quad -\frac{(1+2i)}{2} i = \frac{2-i}{2}$$

$$= 1 - \frac{i}{2} \quad \text{Ans}$$

Q:- show that following limits do not exist:-

(a) $\lim_{z \to 0} \frac{Re(z^2)}{Im\, z}$ (b) $\lim_{z \to 0} \frac{z}{(\bar{z})^2}$

(c) $\lim_{z \to 0-i} \frac{z^2}{z+i}$

Q find the following limits:-

(a) $\lim_{z \to 0} \frac{Re(z^2)}{|z|}$ (b) $\lim_{z \to 0} \frac{z^2+6z+3}{z^2+2z+2}$

Ans - (0) Ans - (3/2)

Continuity of f(z):-

f(z) is said to be continous at point $z=z_0$, if $\lim_{z \to z_0} f(z)$ exist and equal to $f(z_0)$.

or interms of real and imaginary parts of f(
if $w = u + iv$ is continous function at $z=$
then $u(x,y)$ and $v(x,y)$ are separately continous. and vice verca.

Q1 $\lim_{z \to 0}$

Q-1 Examine the continuity of following:-

$$f(z) = \begin{cases} \dfrac{z^3 - iz^2 + z - i}{z - i}, & z \neq i \\ 0, & z = i \end{cases}$$

at $z = i$.

Soln:- $\lim_{z \to i} f(z)$

$$= \lim_{z \to i} \frac{z^3 - iz^2 + z - i}{z - i}$$

$$= \lim_{z \to i} z^2 + 1 = -1 + 1 = 0$$

OR

Now

$$= \lim_{(x,y) \to (0,1)} (x^2 + iy)^2 + 1$$

$$= \lim_{x \to 0, y \to 1} x^2 - y^2 + 1 + (2xy)i$$

Case I:- Let first $x \to 0$, then $y \to 1$

$$= \lim_{y \to 1} -y^2 + 1$$

$$= 0$$

Case IInd:- Let first $y \to 1$ then $x \to 0$

$$= \lim_{x \to 0} x^2 + 2ix = 0$$

along two different path limit is unique ie limit exist and equal to $f(0)$. therefor function is continous at $z=0$.

Q2 Show that the function $f(z)$ defined by

$$f(z) = \begin{cases} \frac{Re(z)}{z}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$ is not cont. at $z=0$.

Soln $$\lim_{z\to 0} f(z) = \lim_{\substack{x\to 0 \\ y\to 0}} \frac{x}{x+iy} \quad \text{— (1)}$$

case I:- $y \to 0$ then $x \to 0$.

$$\lim_{z\to 0} f(z) = \lim_{x\to 0} \frac{x}{x} = 1$$

case II:- first $x \to 0$, then $y \to 0$

$$\lim_{z\to 0} f(z) = \lim_{y\to 0} \frac{1}{iy} \rightarrow \text{does not exis}$$

or we can take general case

If $z \to 0$ along $y = mx$

$$\lim_{z\to 0} f(z) = \lim_{z\to 0} \frac{x}{x+imx} = \frac{1}{1+im}$$ which depends on m, i.e. is not unique therefor $\lim_{z\to 0} f(z)$ does not exist & $f(z)$ is not cont. at $z=0$.

Q1- Examine the continuity of the following functions:-

(1) $$f(z) = \begin{cases} \frac{Im(z)}{|z|}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$ at $z = 0$.

Ans (not continous)

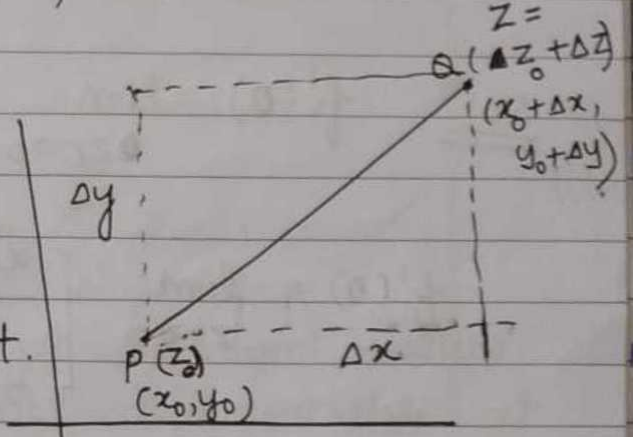
Q:- Show that following functions are continous for all z.

(i) $\cos z$ (ii) e^{2z}

Differentiability of complex function $f(z)$:-

Let $f(z)$ be a single valued function, then derivative of $f(z)$ at a point z_0 is given by

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$, Provided the limit exist and independent of path along which $z \to z_0$.



* Let P be a fixed point and Q be neighbouring point. then.

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z},$$

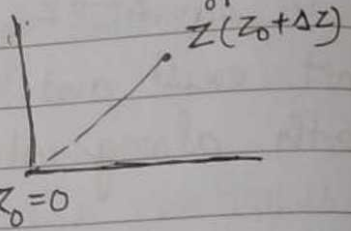
$z = z_0 + \Delta z$ is neighbouring Point of z_0.

Provided the limit exist and independent of path by which $Q(z)$ approaches to $P(z_0)$ [it may be staright line Path or curved path]

Q1 If $f(z) = \begin{cases} \frac{x^3 y(y-ix)}{x^6+y^2}, & z \neq 0 \\ 0, & z = 0 \end{cases}$

check whether $f(z)$ is differentiable at $z=0$ or not.

Soln $f'(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z-0}$



$$f'(0) = \lim_{z \to 0} \frac{\left[\frac{x^3 y(y-ix)}{x^6+y^2} - 0\right]}{x+iy}$$

$$f'(0) = \lim_{\substack{x \to 0 \\ y \to 0}} \left[\frac{x^3 y(y-ix)}{(x+iy)(x^6+y^2)}\right] = \lim_{\substack{x \to 0 \\ y \to 0}} \left[\frac{-ix^3 y}{x^6+y^2}\right]$$

case I :- Let $z \to 0$ along the path $x \to 0$ first then $y \to 0$

$$f'(0) = \lim_{y \to 0} \frac{0}{\cdot} = 0$$

case II :- along $y \to 0$ first then $x \to 0$

$$f'(0) = 0$$

case III → ~~take~~ Let $z \to 0$ along $y = mx$

$$f'(0) = \lim_{x \to 0} \frac{m x^4 (mx - ix)}{x}$$

$$f'(0) = \lim_{x\to 0} \frac{-i m x^4}{x^6 + m^2 x^2}$$

$$f'(0) = \lim_{x\to 0} \frac{-i m x^2}{x^4 + m^2} = 0$$

Case IV
But along $y = x^3$

$$f'(0) = \lim_{x\to 0} \frac{-i x^6}{x^6 + x^6} = \frac{-i}{2}$$

i.e. In different paths, we get different values of $f'(z)$, i.e $f(z)$ is not differentiable at $z=0$.

Q:- check whether $f(z) = |z|^2$ is differentiable at $z=0$ or not.

Soln $f'(z_0) = \lim_{z\to 0} \frac{f(z) - f(0)}{z}$

$$= \lim_{z\to 0} \frac{|z|^2 - 0}{z}$$

$$f'(0) = \lim_{\substack{x\to 0 \\ y\to 0}} \frac{x^2+y^2}{x+iy}$$

f'(0) - Case I:- let first $x\to 0$ then $y\to 0$

$$f'(0) = \lim_{y\to 0} \frac{y^2}{iy} = 0$$

case II^nd consider $y \to 0$ then $x \to 0$

$$f'(0) = \lim_{x \to 0} \frac{x^2}{x} = 0$$

case III^rd let $z \to 0$ along $y = mx$

$$f'(z) = \lim_{x \to 0} \frac{x^2 + m^2x^2}{x + imx}$$

$$= \lim_{x \to 0} \frac{x(1+m^2)}{1+i}$$

$$f'(0) = 0$$

as along all paths through which $z \to 0$, $f'(z)$ is unique therefor. $f(z)$ is differentiable at $z = 0$. and

$$\boxed{f'(0) = 0}$$