Directional Derivatives

Properties:

$$\left(\frac{df}{ds}\right)_{u, P_0} = (\nabla f)_{P_0} \cdot u$$

$$= \left|(\nabla f)\right|_{P_0} \cos\theta$$

1. The function f increases most rapidly when

$\cos\theta = 1 \Rightarrow \theta = 0^\circ$

$\Rightarrow (\nabla f)$ and u are in the same direction.

$\Rightarrow f$ increases most rapidly in the direction of (∇f)

& the derivative in this direction is $|\nabla f|$

2. Similarly, f decreases most rapidly in the direction of

$-(\nabla f)$

The derivative in this direction is given by $-|\nabla f|$

3. Any direction u which is orthogonal to $(\nabla f) \neq 0$ is a direction of zero change in f as here $\theta = \frac{\pi}{2}$.

Prob: Find the directions in which $f(x,y) = \frac{x^2}{2} + \frac{y^2}{2}$

i. increases most rapidly at $(1,1)$?

ii. decreases most rapidly at $(1,1)$?

iii. what are the directions of zero change in f at $(1,1)$?

Soln: i.

$$(\nabla f)_{(1,1)} = \left.\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}\right|_{(1,1)}$$

$$= \left. x\hat{i} + y\hat{j}\right|_{(1,1)}$$

$$= \hat{i} + \hat{j}.$$

The direction of rapid increase is $\hat{u} = \frac{(\hat{i}+\hat{j})}{\sqrt{2}}$ & the derivative is given by

$$|\nabla f| = \sqrt{2}$$

ii. The direction of rapid decrease is $\hat{u} = -\frac{\hat{i}}{\sqrt{2}} - \frac{\hat{j}}{\sqrt{2}}$ and the derivative is given by, $-|\nabla f| = -\sqrt{2}$

iii. The directions of zero change at $(1,1)$ are the directions orthogonal to (∇f) and these are,

$$\frac{\hat{i}-\hat{j}}{\sqrt{2}} \quad \text{and} \quad \frac{-\hat{i}+\hat{j}}{\sqrt{2}}$$

Tangent Plane & Normal Lines

Defn: Tangent plane at $P_0(x_0, y_0, z_0)$ on the level surface $f(x,y,z) = c$ of a differentiable function f is the plane through P_0 and normal to

$$\nabla f \big|_{P_0}$$

The normal line of the surface at P_0 is the line through P_0 and parallel to $\nabla f \big|_{P_0}$.

Eqn: The tangent plane to $f(x,y,z) = c$ at $P_0(x_0, y_0, z_0)$ is given by:

$$f_x(P_0)(x - x_0) + f_y(P_0)(y - y_0) + f_z(P_0)(z - z_0) = 0$$

$\boxed{}$ The normal line to $f(x,y,z) = c$ at $P_0(x_0, y_0, z_0)$ is

$$\frac{x - x_0}{f_x(P_0)} = \frac{y - y_0}{f_y(P_0)} = \frac{z - z_0}{f_z(P_0)}$$

$$x = x_0 + f_x(P_0)t \qquad y = y_0 + f_y(P_0)t$$

$$z = z_0 + f_z(P_0)t$$

Prob: Find the tangent plane and normal line to the surface

$$f(x,y,z) = x^2 + y^2 + z - 9 = 0$$

at $(1,2,4)$

Sol^n: The gradient of f at $(1,2,4)$ is given by

$$\nabla f = 2x\hat{i} + 2y\hat{j} + \hat{k} \Big|_{(1,2,4)}$$

$$= 2\hat{i} + 4\hat{j} + \hat{k}$$

The tangent plane is the plane through $(1,2,4)$ and orthogonal to ∇f at $(1,2,4)$. Hence it is given by

$$2(x-1) + 4(y-2) + (z-4) = 0$$

$$\Rightarrow \quad 2x + 4y + z = 14$$

The line normal to f at $(1,2,4)$ is:

$$\frac{x-1}{2} = \frac{y-2}{4} = \frac{z-4}{1} \quad \Big| \quad x = 1+2t, \quad y = 2+4t, \quad z = 4+t$$

□ The tangent plane for $z = f(x,y)$ at $(x_0, y_0, f(x_0,y_0))$ is given by:

$$f_x(P_0)(x-x_0) + f_y(P_0)(y-y_0) - (z-z_0) = 0$$

Vector Identities

i. $\text{div grad } f = \nabla^2 f = \dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} + \dfrac{\partial^2 f}{\partial z^2}$

ii. $\text{curl grad} f = \nabla \times (\nabla f) = 0$

iii. $\text{div curl } f = \nabla \cdot (\nabla \times f) = 0$

iv. $\text{curl curl } f = \text{grad div } f - \nabla^2 f$

$$\Rightarrow \quad \nabla \times (\nabla \times f) = \nabla(\nabla \cdot f) - \nabla^2 f$$

v. $\text{grad div} f = \text{curl curl } f + \nabla^2 f$

$$\Rightarrow \quad \nabla(\nabla . f) = \nabla \times (\nabla \times f) + \nabla^2 f$$

Proof: i. div grad f = $\nabla.(\nabla f)$

$$= \nabla . \left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k} \right)$$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k} \right) . \left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k} \right)$$

$$= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) f$$

$$= \nabla^2 f$$

∇^2 = Laplacian operator and $\nabla^2 f = 0$ is known as Laplace equation.

Prob: Show that the vector $V = (4xy - z^3)\hat{i} + 2x^2\hat{j} - 3xz^2\hat{k}$ is irrotational. Also show that V can be expressed as the gradient of some scalar function ϕ.

Soln: V is irrotational if curl $V = 0$.

Consider, $V = \nabla \phi$

$$= \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}$$

Equating co-efficients of like vectors:

$$\frac{\partial \phi}{\partial x} = 4xy - z^3 \quad \Rightarrow \quad \phi = 2x^2y - xz^3 + \boxed{\lambda(y,z)} \quad \rightarrow (1)$$

$$\frac{\partial \phi}{\partial y} = 2x^2 \quad \Rightarrow \quad \phi = 2x^2y + \mu(x,z) \quad \rightarrow (2)$$

$$\frac{\partial \phi}{\partial y} = 2z^2 \quad \Rightarrow \quad \phi = 2x^2 y + \mu(x, z) \rightarrow (2)$$

$$\frac{\partial \phi}{\partial z} = -3xz^2 \quad \Rightarrow \quad \phi = -xz^3 + \psi(x, y) \rightarrow (3)$$

(1), (2) and (3) will all be agreed if these are zero or at the most constant.

$$\Rightarrow \quad \phi = 2x^2 y - xz^3 + c$$