Exponential Distribution

A random variable X is said to have an exponential distribution with parameter $\theta > 0$ if its p.d.f is given by

$$f(x) = \begin{cases} \theta e^{-\theta x}, & x \geq 0 \\ 0, & \text{o.w.} \end{cases} \quad \text{or} \quad \begin{cases} \lambda e^{-\lambda x}, & x > 0 \\ 0, & \text{o.w.} \end{cases}$$

θ is called rate parameter.

Mean of exponential distribution:-

$$\boxed{\text{Mean} = \frac{1}{\theta}}$$

Variance of exponential distribution

$$\boxed{\text{Variance} = \frac{1}{\theta^2}} \qquad \boxed{\text{Coeff. of variation} = \frac{\text{Mean}}{\text{S.D}}}$$

Moment Generating function of exponential distn:-

$$\text{M.G.F} = E(e^{tx}) = \int_{x=0}^{\infty} e^{tx} \cdot \theta e^{-\theta x}\, dx$$

$$= \theta \int_{x=0}^{\infty} e^{(t-\theta)x}\, dx$$

$$= \theta \left[\frac{e^{(t-\theta)x}}{t-\theta}\right]_0^{\infty} = \frac{\theta}{t-\theta}\left[e^{-(\theta-t)x}\right]_0^{\infty}$$

M.G.F

$$= \frac{\theta}{t-\theta}(0-1) = \frac{\theta}{\theta - t}$$

$$M.G.F = \frac{\theta}{\theta - t}$$

$$\mu_1' = \text{Mean} = \left[\frac{d}{dt} M_x(t)\right]_{t=0}$$

$$= \left[\frac{\theta}{(\theta-t)^2}(-1)\right]_{t=0}$$

$$= \left[\theta/\theta^2\right] = 1/\theta$$

$$\boxed{\mu_1' = 1/\theta}$$

$$\mu_2' = \left[\frac{d^2}{dt^2} M_x(t)\right]_{t=0}$$

$$= \left[\frac{2\theta}{(\theta-t)^3}\right]_{t=0} = \frac{2\theta}{\theta^3} = \frac{2}{\theta^2}$$

$$\text{variance} = \mu_2' - \mu_1^2$$

$$= \frac{2}{\theta^2} - \frac{1}{\theta^2}$$

$$\boxed{\text{Variance} = \frac{1}{\theta^2}}$$

Commulative dist^n function :-

$$F(x) = P(X \le x)$$

$$= \int_{-\infty}^{x} f(x)\,dx = \int_{0}^{x} \theta e^{-\theta x}\,dx$$

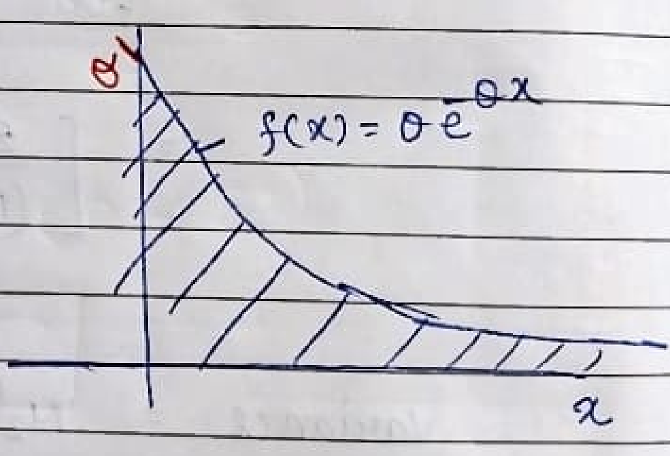
$$= \theta \left(\frac{e^{-\theta x}}{-\theta}\right)_0^x$$

$$= -\left[e^{-\theta x} - 1\right]$$

$$\boxed{F(x) = 1 - e^{-\theta x} = P(X \le x)}$$

Graph of exponential distribution :-

$$\left[\int_{0}^{\infty} \theta e^{-\theta x}\,dx = 1\right]$$



Area under the given curve = 1

= total probability

Note:-

A typical application of exponential distribution is to model waiting times or life times. for example each of the following gives an application of an exponential distribution:-

x = life time of a radioactive particle.
x = how long you have to wait for an accident to occur at a given intersection.
x = length of interval between occurences of poisson distributed events.

The parameter θ is referred as the rate parameter, it represent how quickly event occur.

for example when x denotes the lifetime of radioactive particle, θ would give the rate at which such particle decay.

* To get some intuition for interpretation of an exponential distribution, suppose you are waiting for an event to happen.

* For example you are at store and are waiting for the next customer.

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eg:- The length of telephone conversation is an exponential variate with mean 3 minutes Find the probability that call

(1) end in less than 3 minutes
(2) takes between 3 to 5 minutes

Solution:- mean = 3   X is a random variable denotes length of telephone call.

ie $\frac{1}{\theta} = 3$

$$\left[\theta = \frac{1}{3}\right]$$

P.d.f of exponential distribution is given by

$$f(x) = \theta e^{-\theta x}, \quad x \geq 0$$

$$= \frac{1}{3} e^{-\frac{x}{3}}, \quad x \geq 0$$

(1) to find $P(X < 3) = \int_0^3 f(x)\,dx$

$$= \int_0^3 \frac{1}{3} e^{-x/3}\,dx$$

$$= \frac{1}{3} \left(\frac{e^{-x/3}}{-1/3}\right)_0^3$$

$$= -\left(e^{-x/3}\right)_0^3 = -\left(e^{-1} - e^{0}\right)$$

$$P(X<3) = 1 - 1/e$$

(ii) $P(3 < X < 5) = \int_3^5 \frac{1}{3} e^{-x/3}\,dx$

$$= e^{-1} - e^{-5/3} = \frac{1}{e} - \frac{1}{e^{5/3}}$$

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Q2 A r.v X has exponential distⁿ with p.d.f

$$f(x) = \begin{cases} 2e^{-2x}, & x > 0 \\ 0, & x \le 0 \end{cases}$$

(a) Compute prob. that X is not less than 3.
(b) find mean and S.D.
(c) Prove that the coeff. of variation is unity.

Solⁿ.
(a) $P(X \geq 3) = \int_3^\infty 2e^{-2x}\,dx$

$$= 2\left(\frac{e^{-2x}}{-2}\right)_3^\infty$$

$$= -1(0 - e^{-6})$$

$$\boxed{P(X \geq 3) = e^{-6}}$$

(b) $f(x) = 2e^{-2x}$, comparing with $f(x) = \theta e^{-\theta x}$

$$\boxed{\theta = 2}$$

Mean $= 1/\theta = 1/2$

variance $= 1/\theta^2 = \frac{1}{4}$

S.D $= \sqrt{\text{variance}} = \frac{1}{2}$

(c) coefficient of variation = $\frac{\text{Mean}}{\text{S.D}}$

$$= \frac{1/2}{1/2} = 1$$

[coeff of variation = 1]

Q:- The life of electric bulb is exponentially distributed with failure rate $= \theta = 1/3$ (one failure every 3000 hours). Find

(a) $P(X > 3)$ (b) $P(1.5 \le X \le 3)$ (c) $P(X > 3.5 / X > 2.5)$

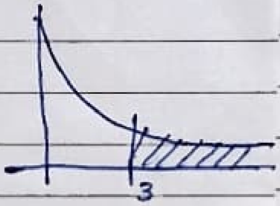
Soln X is a random variable denote the life of electric bulb.

given $\theta = 1/3$

P.d.f is given by

$$f(x) = \theta e^{-\theta x}, \quad x \ge 0$$

$$f(x) = \frac{1}{3} e^{-\frac{x}{3}}, \quad x \ge 0$$



(a) $$P(X > 3) = \int_3^{\infty} \frac{1}{3} e^{-\frac{x}{3}} \, dx$$

$$P(X > 3) = \frac{1}{3} \left(\frac{e^{-x/3}}{-1/3} \right)_3^{\infty} \quad -(0 - e^{-1})$$

= [illegible]

$\boxed{P(X>3) = +e^{-1}}$ $\boxed{P(X > 3) = .3679}$

(b) $P(1.5 \le X \le 3) = \int_{1.5}^{3} \frac{1}{3} e^{-\frac{x}{3}} dx$

$$P(1.5 \le X \le 3) = \frac{1}{3} \left[\frac{e^{-x/3}}{-1/3} \right]_{1.5}^{3}$$

$$= -\left[e^{-1} - e^{-1/2} \right]$$

$$\boxed{P(1.5 \le X \le 3) = e^{-1/2} - e^{-1}} = 0.2386$$

(c) $P(X > 3.5 / X > 2.5)$

$$= \frac{P(X > 3.5 \cap X > 2.5)}{P(X > 2.5)}$$

$$= \frac{P(X > 3.5)}{P(X > 2.5)} = \frac{\cancel{1/3} \int_{3.5}^{\infty} e^{-\frac{x}{3}} dx}{\cancel{1/3} \int_{2.5}^{\infty} e^{-\frac{x}{3}} dx}$$

$$= \frac{\frac{\left[e^{-\frac{x}{3}} \right]_{3.5}^{\infty}}{\cancel{-1/3}}}{\frac{\left[e^{-\frac{x}{3}} \right]_{2.5}^{\infty}}{\cancel{-1/3}}} = \frac{0 - e^{-\frac{3.5}{3}}}{0 - e^{-\frac{2.5}{3}}}$$

$$= e^{-\frac{3.5}{3} + \frac{2.5}{3}} = e^{-\frac{1}{3}} = 0.717$$

$$\boxed{P(X > 3.5 / X > 2.5) = 0.717}$$

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Q The time in hours require to repair a machine is exponentially distributed with parameter $\lambda = \frac{1}{2}$.

(a) what is the probability that the repair time exceeds 2 hour?

(b) what is conditional probability that a repair takes at least 10 hours given that its duration exceeds 9 hours.

Solⁿ: $f(x) = \lambda e^{-\lambda x}, \quad x \geq 0, \qquad \lambda = \frac{1}{2}$

X denotes the repair time of machine in hours

(a) $P(X>2) = \int_2^\infty \frac{1}{2} e^{-\frac{x}{2}} dx$

$$= \frac{1}{2} \left(\frac{e^{-x/2}}{-\frac{1}{2}} \right)_2^\infty$$

$$= -[0 - e^{-1}] = e^{-1} = 0.3679$$

(b) $P(X \geq 10 / X > 9) = \dfrac{P(X \geq 10 \cap X > 9)}{P(X>9)}$

$$= \frac{P(X \geq 10)}{P(X>9)}$$

$$= \frac{\int_{10}^\infty \cancel{\frac{1}{2}} e^{-x/2} dx}{\int_9^\infty \cancel{\frac{1}{2}} e^{-x/2} dx} = \frac{\left(e^{-\frac{x}{2}}\right)_{10}^\infty}{\left(e^{-x/2}\right)_9^\infty}$$

$$= \frac{0 - e^{-5}}{0 - e^{-9/2}} = e^{9/2 - 5} = e^{-1/2}$$

$$= e^{-0.5} = .6065$$

ie $\left[P(X \geq 10 / X > 9) = .6065 \right]$

Ans.

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Ex The mileage which car owners get with a certain kind of radial tire is a random variable having an exponential distribution with mean 40,000 km. Find the probability that one of these tires will last
(a) at least 20,000 km
(b) at most 30,000 km

Sol^n X denotes the mileage obtained with the tire

$$\text{mean} = 40{,}000$$

$$\text{ie } \frac{1}{\theta} = 40{,}000$$

$$\theta = \frac{1}{40{,}000}.$$

P.d.f $\quad f(x) = \theta e^{-\theta x}, \quad x \geq 0$

$$\therefore f(x) = \frac{1}{40{,}000} e^{-\frac{x}{40{,}000}}, \quad x \geq 0.$$

(a) $P(X \geq 20{,}000) = \int_{20{,}000}^{\infty} \frac{1}{40{,}000} e^{-\frac{x}{40{,}000}} dx.$

$$P(X \geq 20{,}000) = -1\left[0 - e^{-1/2}\right] = e^{-1/2} = .6065$$

(b) $P(X \leq 30{,}000) = \int_{0}^{30{,}000} \frac{1}{40{,}000} e^{-\frac{x}{40{,}000}} dx$

$$= 1 - e^{-3/4} = 1 - e^{-0.75}$$

$$\boxed{P(X \leq 30{,}000) = .5270}$$