

Problems on Waves: Module 1

(Assume SI units if units are not given for displacement)

1. A wave displacement is given by $y = 0.1 \sin(0.1x - 0.1t)$. Find (a) the amplitude of the wave, (b) the magnitude of the propagation vector, (c) the wavelength, (d) the time period, and (e) the wave velocity.

Sol:

The various parameters of the given harmonic wave can be found by comparing it with the standard form $y = A \sin(kx - \omega t)$ for a wave propagating in the positive x -direction.

- (a) The amplitude $A = 0.1$ m
- (b) The propagation vector $k = 0.1 \text{ m}^{-1}$
- (c) The wavelength $\lambda = 2\pi/k = 20\pi$ m
- (d) The angular frequency $\omega = 0.1 \text{ s}^{-1}$, time period $T = 2\pi/\omega = 20\pi$ s
- (e) The wave velocity $v = \omega/k = 1$ m/s.

2. Verify that the wave function $y(x, t) = Ae^{-B(x-vt)^2}$ satisfies the one-dimensional wave equation.

Sol: Differentiate twice w.r.t x and t respectively and then equate.

3. A wire of mass m and length l is fixed at the top end, and hangs freely under its own weight. Deduce the time for a transverse wave to travel the length of the wire.

Sol:

$$\text{Mass per unit length of the wire} = \mu = \frac{m}{l}.$$

Consider a small portion of the string of length dx at a distance x from the point of suspension O (Fig. 5.4). The tension at this point is $(l-x)\mu g$ and the velocity of propagation of transverse wave at this point is

$$\sqrt{\frac{(l-x)\mu g}{\mu}} = \sqrt{(l-x)g}.$$

Time taken by the transverse wave to traverse a distance dx at this point is

$$\frac{dx}{\sqrt{(l-x)g}}.$$

Total time to travel the length of the wire is

$$\int_0^l \frac{dx}{\sqrt{(l-x)g}} = 2\sqrt{\frac{l}{g}}.$$

4. A transverse harmonic disturbance is produced in a string. The maximum transverse velocity is 3 m s^{-1} and maximum transverse acceleration is 90 m s^{-2} . If the wave velocity is 20 m s^{-1} then find the waveform.

Sol:

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5. A steel cable 3.0 cm in diameter is kept under a tension of 10 kN. The density of steel is 7.8 g cm^{-3} . Find the speed of transverse waves along the cable.

Sol: Mass per unit length of the steel cable, $\mu = \text{density} \times \text{cross-sectional area}$
 $= 7.8 \text{ g cm}^{-3} \times 3.14 \times 1.5 \times 1.5 \text{ cm}^2$
 $= 55.10 \text{ g cm}^{-1} = 5.51 \text{ kg m}^{-1}$

Speed of transverse waves is given by, $v = \sqrt{\frac{T}{\mu}}$

Therefore, $v = \sqrt{\frac{10000}{5.51}} = 42.6 \text{ ms}^{-1}$

6. A uniform flexible cable is 20 m long. It hangs vertically under its own weight and is vibrated from its upper end. Find the speed of transverse wave on the cable at its midpoint.

Sol: Tension in the cable at its midpoint, $T = mg/2$

Mass per unit length of the cable, $\mu = m/20 \text{ kg m}^{-1}$

Speed of transverse wave, $v = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{mg/2}{m/20}} = \sqrt{10g} = 9.9 \text{ ms}^{-1}$

7. The speed of a wave on a string is 120 ms^{-1} when the tension is 120 N. To what value must the tension be increased in order to raise the wave speed to 150 m s^{-1} ?

Sol: Let $v = 120 \text{ ms}^{-1}$, $T = 120 \text{ N}$

Case I

and $v' = 150 \text{ ms}^{-1}$, $T = ?$

Case II

Mass per unit length μ is same for both the cases.

$$\frac{v}{v'} = \sqrt{\frac{T/\mu}{T'/\mu}}$$

$$\frac{120}{150} = \sqrt{\frac{120}{T'}}$$

$$T' = 187.5 \text{ N}$$

8. The linear density of a vibrating string is $1.6 \times 10^{-2} \text{ kg m}^{-1}$. A transverse wave is propagating on the string and is described by the equation $y = 0.02 \sin (2x + 30t)$ (a) what is the wave speed? (b) What is the tension in the string?

Sol: From $y = 0.02 \sin (2x + 30t)$, we get $k = 2 \text{ m}^{-1}$
and $\omega = 30 \text{ rad s}^{-1}$

Wave speed is given by, $= \frac{\omega}{k}$, Therefore, $v = 15 \text{ ms}^{-1}$

Tension in the string is given by, $T = \mu v^2 = 1.6 \times 10^{-2} \text{ kg m}^{-1} \times 15 \text{ ms}^{-1} \times 15 \text{ ms}^{-1} = 3.6 \text{ N}$

9. A stretched string has a mass per unit length of 2.5 g cm^{-1} and a tension 9 N. A wave on this string has an amplitude of 0.12 mm and a frequency of 100 Hz and is travelling in the negative x-direction. Write an equation for this wave.

Sol: Mass per unit length, $\mu = 2.5 \text{ g cm}^{-1}$

Tension in the string, $T = 9 \text{ N}$

Amplitude of the wave, $A = 0.12 \text{ mm}$, Frequency of the wave, $\nu = 100 \text{ Hz}$

To write equation for this wave we need to find its speed and then wave number k .

$$v = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{9 \text{ N}}{0.25 \text{ kg m}^{-1}}} = 6 \text{ ms}^{-1}$$

$$\text{Now, Wave number, } k = \frac{\omega}{v} = \frac{2 \times 3.14 \times 100}{6} = 104.67 \text{ m}^{-1}$$

Thus, wave function, $y = 0.00012 \sin (104.67x + 200\pi t) \text{ m}$

10. A continuous sinusoidal wave is travelling on a string having linear density 4 g cm^{-1} . The displacement of the particle of the string at $x = 10 \text{ cm}$ is found to vary with time according to the equation $y = 0.05 \sin (1 - 4t)$ (a) What is the frequency of the wave? (b) What is the wavelength of the wave? (c) What is the velocity of the wave? (d) Calculate the tension in the string. (e) Write the general equation giving the transverse displacement of the particles of the string as a function of position and time.

Sol: From $y = 0.05 \sin (1 - 4t)$

we get, $\omega = 4 \text{ rad s}^{-1}$, Thus Frequency of the wave, $\nu = 0.64 \text{ s}^{-1}$

Wavelength of the wave can be found from $kx = 1$ and $x = 10 \text{ cm}$ or 0.1 m

which gives, $k = 10 \text{ m}^{-1}$. Therefore wavelength, $\lambda = 2\pi/k = 0.628 \text{ m}$

Velocity of the wave, $v = \text{frequency} \times \text{wavelength} = 0.64 \times 0.628 = 0.4 \text{ ms}^{-1}$

Tension in the string, $T = \mu v^2 = 0.4 \times 0.4 \times 0.4 = 0.064 \text{ N}$

The general equation for this wave, $y = 0.05 \sin (10x - 4t) \text{ m}$

11. The vibration of a string fixed at both ends is represented by the equation:
 $y = 2 \sin (\pi x/3) \cos 50\pi t$. If the above stationary wave is produced due to superposition of two waves of same frequency, velocity and amplitude travelling in opposite directions,
 $y_1 = A \sin (2\pi/\lambda) (x - vt)$ and $y_2 = A \sin (2\pi/\lambda) (x + vt)$;
(i) Find the equations of the component waves, and
(ii) What is the distance between two consecutive nodes of the stationary wave?

Sol:

(i) We have

$$y = y_1 + y_2 = 2A \sin \frac{2\pi x}{\lambda} \cos \frac{2\pi}{\lambda} vt$$

Thus, $A = 1 \text{ m}$, $\lambda = 6 \text{ m}$, $\frac{2\pi v}{\lambda} = 50\pi$ or $v = 150 \text{ m/s}$.

Hence $y_1 = \sin \frac{\pi}{3}(x - 150t) \text{ m}$

and $y_2 = \sin \frac{\pi}{3}(x + 150t) \text{ m}$

(ii) Distance between two consecutive nodes $= \lambda/2 = 3 \text{ m}$.

12. The phase velocity of a surface wave on a liquid of density ρ and surface tension T is given by $v_p = \left(\frac{g\lambda}{2\pi} + \frac{2\pi T}{\lambda\rho} \right)^{1/2}$ where λ is the wavelength of the wave and g is the acceleration due to gravity. Find the group velocity of the surface wave. Find the wavelength λ for which v_p is minimum. Evaluate the minimum value of v_p and the corresponding value of v_g .

Sol:

Since $v_p = \frac{\omega}{k}$ and $\lambda = \frac{2\pi}{k}$, we have

$$\frac{\omega}{k} = \left(\frac{g}{k} + \frac{kT}{\rho} \right)^{1/2}$$

or $\omega = \left(gk + \frac{k^3 T}{\rho} \right)^{1/2}$

which is the dispersion relation of the surface wave on a liquid.

Thus,
$$v_g = \frac{d\omega}{dk} = \frac{g + 3k^2 T / \rho}{2(gk + k^3 T / \rho)^{1/2}}$$

$$= \frac{g + 12\pi^2 T / (\rho\lambda^2)}{2[2\pi g / \lambda + 8\pi^3 T / (\rho\lambda^3)]^{1/2}}$$

When v_p is minimum, so is v_p^2 , and the conditions for this is

$$\frac{d}{d\lambda}(v_p^2) = 0$$

or $\frac{d}{d\lambda} \left(\frac{g}{k} + \frac{kT}{\rho} \right) = 0$

or $\frac{d}{d\lambda} \left(\frac{\lambda g}{2\pi} + \frac{2\pi T}{\lambda\rho} \right) = 0$

or $\frac{8}{2\pi} - \frac{2\pi T}{\lambda^2 \rho} = 0$

giving $\lambda = 2\pi \left(\frac{T}{\rho g} \right)^{1/2}$

This gives the minimum value of v_p since $\frac{d^2}{d\lambda^2}(v_p^2) > 0$. Thus the minimum value of v_p is

$$(v_p)_{\min} = \sqrt{2} \left(\frac{Tg}{\rho} \right)^{1/4}$$

and the corresponding value of v_g is $\sqrt{2} \left(\frac{Tg}{\rho} \right)^{1/4}$.