CHAPTER

7

Multiple Integrals and Beta, Gamma Functions

1. Double integrals. **2.** Change of order of integration. **3.** Double integrals in Polar coordinates. **4.** Areas enclosed by plane curves. **5.** Triple integrals. **6.** Volume of solids. **7.** Change of variables. **8.** Area of a curved surface. **9.** Calculation of mass. **10.** Centre of gravity. **11.** Centre of pressure. **12.** Moment of inertia. **13.** Product of inertia ; Principal axes. **14.** Beta function. **15.** Gamma function. **16.** Relation between beta and gamma functions. **17.** Elliptic integrals. **18.** Error function or Probability integral. **19.** Objective Type of Questions.

7.1 DOUBLE INTEGRALS

The definite integral $\int_a^b f(x)\,dx$ is defined as the limit of the sum

$$f(x_1)\,\delta x_1 + f(x_2)\,\delta x_2 + \dots + f(x_n)\,\delta x_n,$$

where $n \to \infty$ and each of the lengths $\delta x_1, \delta x_2, \dots$ tends to zero. A double integral is its counterpart in two dimensions.

Consider a function $f(x, y)$ of the independent variables x, y defined at each point in the finite region R of the xy-plane. Divide R into n elementary areas $\delta A_1, \delta A_2, \dots, \delta A_n$. Let (x_r, y_r) be any point within the rth elementary area δA_r. Consider the sum

$$f(x_1, y_1)\,\delta A_1 + f(x_2, y_2)\,\delta A_2 + \dots + f(x_n, y_n)\,\delta A_n, \text{ i.e., } \sum_{r=1}^{n} f(x_r, y_r)\,\delta A_r$$

The limit of this sum, if it exists, as the number of sub-divisions increases indefinitely and area of each sub-division decreases to zero, is defined as the *double integral of f(x, y) over the region R* and is written as $\iint_R f(x, y)\,dA$.

Thus
$$\iint_R f(x, y)\,dA = \underset{\substack{n \to \infty \\ \delta A \to 0}}{\text{Lt}} \sum_{r=1}^{n} f(x_r, y_r)\,\delta A_r \qquad \dots(1)$$

The utility of double integrals would be limited if it were required to take limit of sums to evaluate them. However, there is another method of evaluating double integrals by successive single integrations.

For purpose of evaluation, (1) is expressed as the repeated integral $\int_{x_1}^{x_2} \int_{y_1}^{y_2} f(x, y)\,dxdy$.

Its value is found as follows :

(*i*) *When y_1, y_2 are functions of x and x_1, x_2 are constants*, $f(x, y)$ is first integrated w.r.t. y keeping x fixed between limits y_1, y_2 and then resulting expression is integrated w.r.t. x within the limits x_1, x_2 *i.e.*,

$$\boxed{I_1 = \int_{x_1}^{x_2} \boxed{\int_{y_1}^{y_2} f(x, y)\,dy}\,dx}$$

where integration is carried from the inner to the outer rectangle.

Figure 7.1 illustrates this process. Here AB and CD are the two curves whose equations are $y_1 = f_1(x)$ and $y_2 = f_2(x)$. PQ is a vertical strip of width dx.

Then the inner rectangle integral means that the integration is along one edge of the strip PQ from P to Q (x remaining constant), while the outer rectangle integral corresponds to the sliding of the edge from AC to BD.

Thus the whole region of integration is the area $ABDC$.

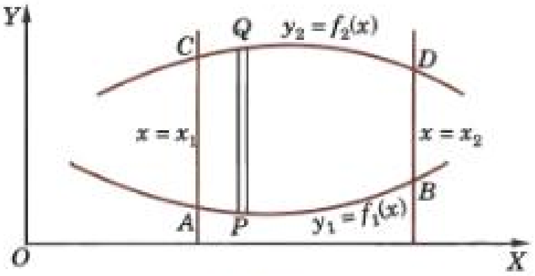

Fig. 7.1

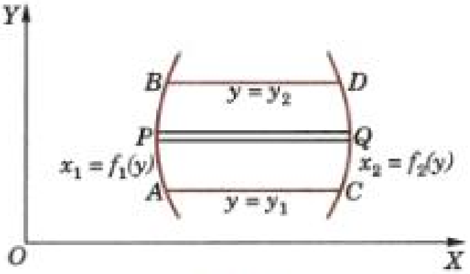

Fig. 7.2

(*ii*) *When x_1, x_2 are functions of y and y_1, y_2 are constants*, $f(x, y)$ is first integrated w.r.t. x keeping y fixed, within the limits x_1, x_2 and the resulting expression is integrated w.r.t. y between the limits y_1, y_2, *i.e.*,

$$I_2 = \boxed{\int_{y_1}^{y_2} \boxed{\int_{x_1}^{x_2} f(x, y)\, dx}\, dy}$$ which is geometrically illustrated by Fig. 7.2.

Here AB and CD are the curves $x_1 = f_1(y)$ and $x_2 = f_2(y)$. PQ is a horizontal strip of width dy.

Then inner rectangle indicates that the integration is along one edge of this strip from P to Q while the outer rectangle corresponds to the sliding of this edge from AC to BD.

Thus the whole region of integration is the area $ABDC$.

(*iii*) *When both pairs of limits are constants*, the region of integration is the rectangle $ABDC$ (Fig. 7.3).

In I_1, we integrate along the vertical strip PQ and then slide it from AC to BD.

In I_2, we integrate along the horizontal strip $P'Q'$ and then slide it from AB to CD.

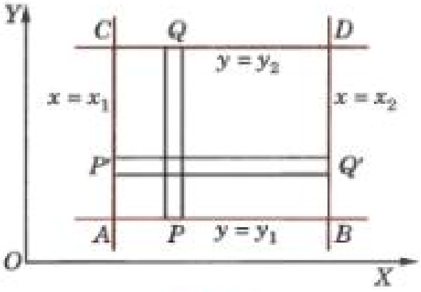

Fig. 7.3

Here obviously $I_1 = I_2$.

Thus *for constant limits, it hardly matters whether we first integrate w.r.t. x and then w.r.t. y or vice versa.*

Example 7.1. *Evaluate* $\int_0^5 \int_0^{x^2} x(x^2 + y^2)\, dxdy$.

Solution. $$I = \int_0^5 dx \int_0^{x^2} (x^3 + xy^3)\, dy = \int_0^5 \left[x^3 y + x \,.\, \frac{y^3}{3}\right]_0^{x^2} dx = \int_0^5 \left[x^3 \,.\, x^2 + x \,.\, \frac{y^6}{3}\right] dx$$

$$= \int_0^5 \left(x^5 + \frac{x^7}{3}\right) dx = \left| \frac{x^6}{6} + \frac{x^8}{24} \right|_0^5 = 5^6 \left[\frac{1}{6} + \frac{5^2}{24}\right] = 18880.2 \text{ nearly.}$$

Example 7.2. *Evaluate* $\iint_A xy\, dx\, dy$, *where A is the domain bounded by x-axis, ordinate $x = 2a$ and the curve $x^2 = 4ay$.*

Solution. The line $x = 2a$ and the parabola $x^2 = 4ay$ intersect at $L(2a, a)$. Figure 7.4 shows the domain A which is the area OML.

Integrating first over a vertical strip PQ, *i.e.*, w.r.t. y from $P(y = 0)$ to $Q(y = x^2/4a)$ on the parabola and then w.r.t. x from $x = 0$ to $x = 2a$, we have

$$\iint_A xy\, dx\, dy = \int_0^{2a} dx \int_4^{x^2/4a} xy\, dy = \int_0^{2a} x \left[\frac{y^2}{2}\right]_0^{x^2/4a} dx$$

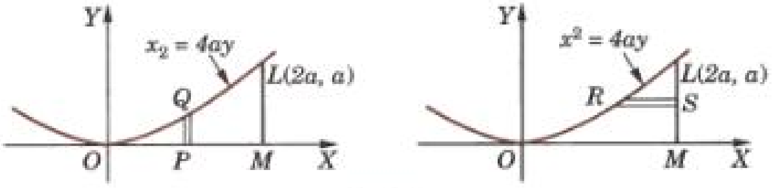

Fig. 7.4

$$= \frac{1}{32a^2}\int_0^{2a} x^5\,dx = \frac{1}{32a^2}\left|\frac{x^6}{6}\right|_0^{2a} = \frac{a^4}{3}.$$

Otherwise integrating first over a horizontal strip RS, *i.e.*, w.r.t. x from, R ($x = 2\sqrt{ay}$) on the parabola to $S(x = 2a)$ and then w.r.t. y from $y = 0$ to $y = a$, we get

$$\iint_A xy\,dx\,dy = \int_0^a dx\int_{2\sqrt{(ay)}}^{2a} xy\,dx = \int_0^a y\left[\frac{x^2}{2}\right]_{2\sqrt{(ay)}}^{2a} dy$$

$$= 2a\int_0^a (ay - y^2)\,dy = 2a\left[\frac{ay^2}{2} - \frac{y^3}{3}\right]_0^a = \frac{a^4}{3}.$$

Example 7.3. *Evaluate* $\iint_R x^2\,dx\,dy$ *where R is the region in the first quadrant bounded by the lines* $x = y$, $y = 0$, $x = 8$ *and the curve* $xy = 16$.

Solution. The line AL ($x = 8$) intersects the hyperbola $xy = 16$ at A (8, 2) while the line $y = x$ intersects this hyperbola at B (4, 4). Figure 7.5 shows the region R of integration which is the area $OLAB$. To evaluate the given integral, we divide this area into two parts OMB and $MLAB$.

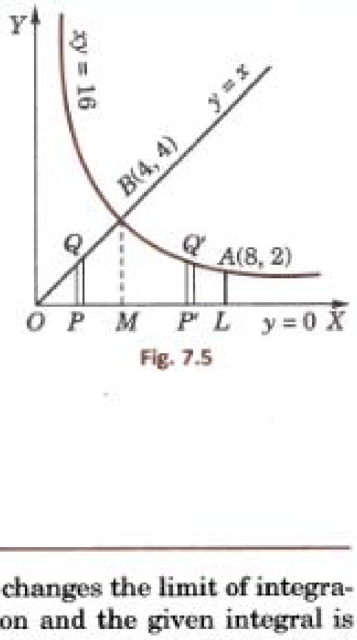

Fig. 7.5

$$\therefore \quad \iint_R x^2\,dxdy = \int_{x\text{ at }0}^{x\text{ at }M}\int_{y\text{ at }P}^{y\text{ at }Q} x^2\,dxdy + \int_{x\text{ at }M}^{x\text{ at }L}\int_{y\text{ at }P'}^{y\text{ at }Q'} x^2\,dxdy$$

$$= \int_0^4\int_0^x x^2\,dxdy + \int_4^8\int_0^{16/x} x^2\,dxdy$$

$$= \int_0^4 x^2\,dx\left|\,y\,\right|_0^x + \int_4^8 x^2\,dx\left|\,y\,\right|_0^{16/x}$$

$$= \int_0^4 x^3\,dx + \int_4^8 16x\,dx = \left|\frac{x^4}{4}\right|_0^4 + 16\left|\frac{x^2}{2}\right|_4^8 = 448$$

7.2 CHANGE OF ORDER OF INTEGRATION

In a double integral with variable limits, the change of order of integration changes the limit of integration. While doing so, sometimes it is required to split up the region of integration and the given integral is expressed as the sum of a number of double integrals with changed limits. To fix up the new limits, it is always advisable to draw a rough sketch of the region of integration.

The change of order of integration quite often facilitates the evaluation of a double integral. The following examples will make these ideas clear.

Example 7.4. *By changing the order of integration of* $\int_0^\infty\int_0^\infty e^{-xy}\sin px\,dxdy$, *show that*

$$\int_0^\infty \frac{\sin px}{x}\,dx = \frac{\pi}{2}.$$

(U.P.T.U., 2004)

Solution. $\int_0^\infty\int_0^\infty e^{-xy}\sin px\,dxdy = \int_0^\infty\left(\int_0^\infty e^{-xy}\sin px\,dx\right)dy$

$$= \int_0^\infty \left| -\frac{e^{-xy}}{p^2+y^2}(p\cos px + y\sin px) \right|_0^\infty dy$$

$$= \int_0^\infty \frac{p}{p^2+y^2}\, dy = \left| \tan^{-1}\left(\frac{y}{p}\right) \right|_0^\infty = \frac{\pi}{2} \qquad ...(i)$$

On changing the order of integration, we have

$$\int_0^\infty \int_0^\infty e^{-xy} \sin px \, dx\, dy = \int_0^\infty \sin px \left\{ \int_0^\infty e^{-xy}\, dy \right\} dx$$

$$= \int_0^\infty \sin px \left| \frac{e^{-xy}}{-x} \right|_0^\infty dx = \int_0^\infty \frac{\sin px}{x}\, dx \qquad ...(ii)$$

Thus from (*i*) and (*ii*), we have $\int_0^\infty \frac{\sin px}{x}\, dx = \frac{\pi}{2}$.

Example 7.5. *Change the order of integration in the integral*

$$I = \int_{-a}^{a} \int_0^{\sqrt{(a^2-y^2)}} f(x,y)\, dx\, dy.$$

Solution. Here the elementary strip is parallel to x-axis (such as PQ) and extends from $x = 0$ to $x = \sqrt{(a^2-y^2)}$ (*i.e.*, to the circle $x^2 + y^2 = a^2$) and this strip slides from $y = -a$ to $y = a$. This shaded semi-circular area is, therefore, the region of integration (Fig. 7.6).

On changing the order of integration, we first integrate w.r.t. y along a vertical strip RS which extends from R [$y = -\sqrt{(a^2-y^2)}$] to S [$y = \sqrt{(a^2-y^2)}$]. To cover the given region, we then integrate w.r.t. x from $x = 0$ to $x = a$.

Thus $$I = \int_0^a dx \int_{-\sqrt{(a^2-x^2)}}^{\sqrt{(a^2-x^2)}} f(x,y)\, dy$$

or $$= \int_0^a \int_{-\sqrt{(a^2-x^2)}}^{\sqrt{(a^2-x^2)}} f(x,y)\, dydx.$$

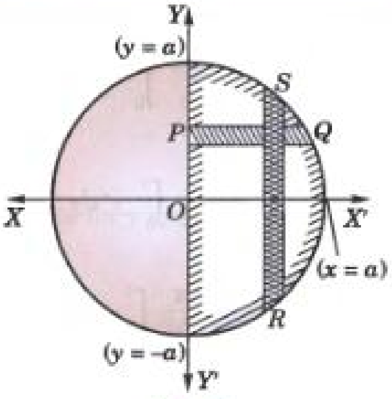

Fig. 7.6

Example 7.6. *Evaluate* $\int_0^1 \int_{e^x}^{e} dydx/\log y$ *by changing the order of integration.*

Solution. Here the integration is first w.r.t. y from P on $y = e^x$ to Q on the line $y = e$. Then the integration is w.r.t. x from $x = 0$ to $x = 1$, giving the shaded region ABC (Fig. 7.7).

On changing the order of integration, we first integrate w.r.t. x from R on $x = 0$ to S on $x = \log y$ and then w.r.t. y from $y = 1$ to $y = e$.

Thus $$\int_0^1 \int_{e^x}^{e} \frac{dydx}{\log y} = \int_1^e \int_0^{\log y} \frac{dxdy}{\log y}$$

$$= \int_1^e \frac{dy}{\log y} \left| x \right|_0^{\log y} = \int_1^e dy = \left| y \right|_1^e = e - 1.$$

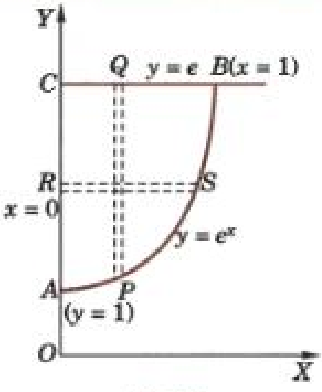

Fig. 7.7

Example 7.7. *Change the order of integration in* $I = \int_0^{4a} \int_{x^2/4a}^{2\sqrt{ax}} dydx$ *and hence evaluate.*

(Nagpur, 2009 ; P.T.U., 2009 S)

Solution. Here integration is first w.r.t. y and P on the parabola $x^2 = 4ay$ to Q on the parabola $y^2 = 4ax$ and then w.r.t. x from $x = 0$ to $x = 4a$ giving the shaded region of integration (Fig. 7.8).

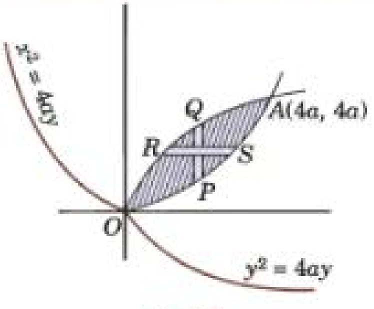

Fig. 7.8

On changing the order of integration, we first integrate w.r.t. x from R to S, then w.r.t. y from $y = 0$ to $y = 4a$

$$\therefore \quad I = \int_0^{4a}\int_{y^2/4a}^{2\sqrt{ay}} dxdy = \int_0^{4a} dy \left| x \right|_{y^2/4a}^{2\sqrt{ay}} = \int_0^{4a} (2\sqrt{ay} - y^2/4a)\, dy$$

$$= \left| 2\sqrt{a}\,\frac{y^{3/2}}{3/2} - \frac{y^3}{12a} \right|_0^{4a} = \frac{32a^2}{3} - \frac{16a^2}{3} = \frac{16a^2}{3}.$$

Example 7.8. *Change the order of integration and hence evaluate*

$$I = \int_0^a \int_{\sqrt{ax}}^a \frac{y^2\, dxdy}{\sqrt{(y^4 - a^2x^2)}}$$ (S.V.T.U., 2006 S)

Solution. Here integration is first w.r.t. y from P on the parabola $y^2 = ax$ to Q on the line $y = a$, then w.r.t. x from $x = 0$ to $x = a$, giving the shaded region OAB of integration (Fig. 7.9).

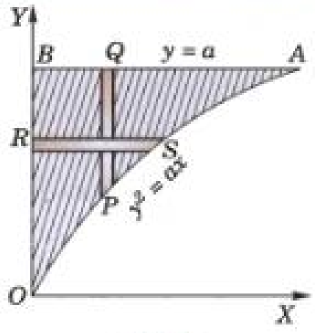

Fig. 7.9

On changing the order of integration, we first integrate w.r.t. x from R to S, then w.r.t. y from $y = 0$ to $y = a$.

$$\therefore \quad I = \int_0^a \int_0^{y^2/a} \frac{y^2\, dy}{\sqrt{(y^4 - a^2x^2)}}\, dx = \frac{1}{a}\int_0^a \int_0^{y^2/a} y^2\, dy \frac{dx}{\sqrt{[(y^2/a)^2 - x^2]}}\, dx$$

$$= \frac{1}{a}\int_0^a y^2 dy \left| \sin^{-1}\left(\frac{xa}{y^2}\right) \right|_0^{y^2/a} = \frac{1}{a}\int_0^a y^2 dy\ [\sin^{-1}(1) - \sin^{-1}(0)]$$

$$= \frac{\pi}{2a}\int_0^a y^2 dy = \frac{\pi}{2a}\left| \frac{y^3}{3} \right|_0^a = \frac{\pi a^2}{6}.$$

Example 7.9. *Change the order of integration in* $I = \int_0^1 \int_{x^2}^{2-x} xy\, dxdy$ *and hence evaluate the same.*

(Bhopal, 2008 ; V.T.U., 2008 ; S.V.T.U., 2007 ; P.T.U., 2005 ; U.P.T.U., 2005)

Solution. Here the integration is first w.r.t. y along a vertical strip PQ which extends from P on the parabola $y = x^2$ to Q on the line $y = 2 - x$. Such a strip slides from $x = 0$ to $x = 1$, giving the region of integration as the curvilinear triangle OAB (shaded) in Fig. 7.10.

On changing the order of integration, we first integrate w.r.t. x along a horizontal strip $P'Q'$ and that requires the splitting up of the region OAB into two parts by the line AC ($y = 1$), *i.e.*, the curvilinear triangle OAC and the triangle ABC.

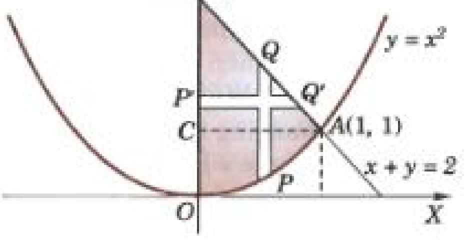

Fig. 7.10

For the region OAC, the limits of integration for x are from $x = 0$ to $x = \sqrt{y}$ and those for y are from $y = 0$ to $y = 1$. So the contribution to I from the region OAC is

$$I_1 = \int_0^1 dy \int_0^{\sqrt{y}} xy\, dx$$

For the region ABC, the limits of integration for x are from $x = 0$ to $x = 2 - y$ and those for y are from $y = 1$ to $y = 2$. So the contribution to I from the region ABC is

$$I_2 = \int_1^2 dy \int_0^{2-y} xy\, dx.$$

Hence, on reversing the order of integration,

$$I = \int_0^1 dy \int_0^{\sqrt{y}} xy\, dx + \int_1^2 dy \int_0^{2-y} xy\, dx$$

$$= \int_0^1 dy \left| \frac{x^2}{2} \cdot y \right|_0^{\sqrt{y}} + \int_1^2 dy \left| \frac{x^2}{2} \cdot y \right|_0^{2-y} = \frac{1}{2}\int_0^1 y^2 dy + \frac{1}{2}\int_1^2 y(2-y)^2\, dy = \frac{1}{6} + \frac{5}{24} = \frac{3}{8}.$$

Example 7.10. *Change the order of integration in* $I = \int_0^1 \int_x^{\sqrt{(2-x^2)}} \frac{x}{\sqrt{(x^2+y^2)}}\, dx$ *and hence evaluate it.*

(*J.N.T.U., 2005 ; Rohtak, 2003*)

Solution. Here the integration is first w.r.t. y along PQ which extends from P on the line $y = x$ to Q on the circle $y = \sqrt{(2 . x^2)}$. Then PQ slides from $y = 0$ to $y = 1$, giving the region of integration OAB as in Fig. 7.11.

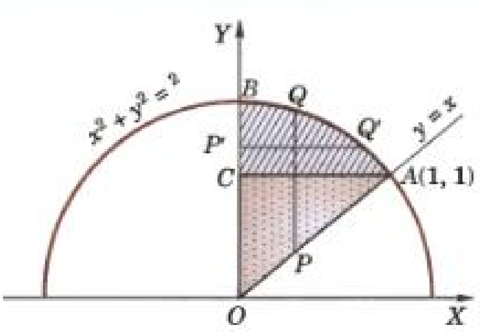

Fig. 7.11

On changing the order of integration, we first integrate w.r.t. x from P' to Q' and that requires splitting the region OAB into two parts OAC and ABC.

For the region OAC, the limits of integration for x are from $x = 0$ to $x = 1$ and those for y are from $y = 0$ to $y = 1$. So the contribution to I from the region OAC is

$$I_1 = \int_0^1 dy \int_0^y \frac{x}{\sqrt{(x^2+y^2)}}\, dx.$$

For the region ABC, the limits of integration for x are 0 to $\sqrt{(2-y^2)}$ and these for y are from 1 to $\sqrt{2}$. So the contribution to I from the region ABC is

$$I_2 = \int_1^{\sqrt{2}} dy \int_0^{\sqrt{(2-y^2)}} \frac{x}{\sqrt{(x^2+y^2)}}\, dx$$

Hence $$I = \int_0^1 \left| (x^2+y^2)^{1/2} \right|_0^y dy + \int_1^{\sqrt{2}} \left| (x^2+y^2)^{1/2} \right|_0^{\sqrt{(2-y^2)}} dy$$

$$= \int_0^1 (\sqrt{2}-1)\, y\, dy + \int_1^{\sqrt{2}} \sqrt{(2-y)}\, dy = \frac{1}{2}(\sqrt{2}-1) + \sqrt{2}\sqrt{(2-1)} - \frac{1}{2} = 1 - 1/\sqrt{2}.$$

7.3 DOUBLE INTEGRALS IN POLAR COORDINATES

To evaluate $\int_{\theta_1}^{\theta_2} \int_{r_1}^{r_2} f(r, \theta)\, dr\, d\theta$, we first integrate w.r.t. r between limits $r = r_1$ and $r = r_2$ keeping θ fixed and the resulting expression is integrated w.r.t. θ from θ_1 to θ_2. In this integral, r_1, r_2 are functions of θ and θ_1, θ_2 are constants.

Figure 7.12 illustrates the process geometrically.

Here AB and CD are the curves $r_1 = f_1(\theta)$ and $r_2 = f_2(\theta)$ bounded by the lines $\theta = \theta_1$ and $\theta = \theta_2$. PQ is a wedge of angular thickness $\delta\theta$.

Then $\int_{r_1}^{r_2} f(r, \theta)\, dr$ indicates that the integration is along PQ from P to Q while the integration w.r.t. θ corresponds to the turning of PQ from AC to BD.

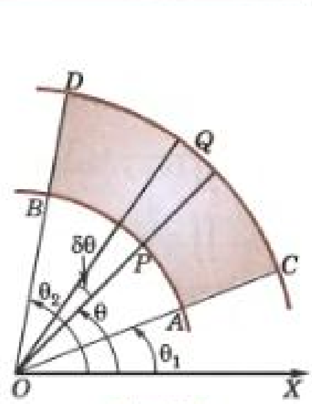

Fig. 7.12

Thus the whole region of integration is the area $ACDB$. The order of integration may be changed with appropriate changes in the limits.

Example 7.11. *Evaluate* $\iint r \sin\theta \, dr \, d\theta$ *over the cardioid* $r = a(1 - \cos\theta)$ *above the initial line.*

(Kerala, 2005)

Solution. To integrate first w.r.t. r, the limits are from 0 ($r = 0$) to P [$r = a(1 - \cos\theta)$] and to cover the region of integration R, θ varies from 0 to π (Fig. 7.13).

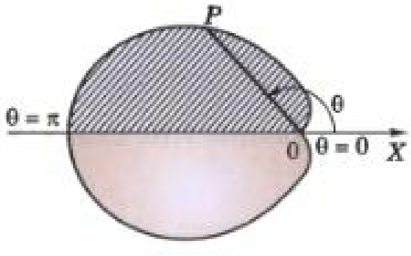

Fig. 7.13

$$\therefore \quad \iint_R r \sin\theta \, dr d\theta = \int_0^\pi \sin\theta \left[\int_0^{r = a(1-\cos\theta)} r dr \right] d\theta$$

$$= \int_0^\pi \sin\theta \, d\theta \left| \frac{r^2}{2} \right|_0^{a(1-\cos\theta)} = \frac{a^2}{2} \int_0^\pi (1 - \cos\theta)^2 \, . \sin\theta \, d\theta$$

$$= \frac{a^2}{2} \left| \frac{(1-\cos\theta)^3}{3} \right|_0^\pi = \frac{a^2}{2} \cdot \frac{8}{3} = \frac{4a^2}{3}.$$

Example 7.12. *Calculate* $\iint r^3 \, dr \, d\theta$ *over the area included between the circles* $r = 2\sin\theta$ *and* $r = 4\sin\theta$.

Solution. Given circles $r = 2\sin\theta$ ...(*i*)

and $r = 4\sin\theta$ ...(*ii*)

are shown in Fig. 7.14. The shaded area between these circles is the region of integration.

If we integrate first w.r.t. r, then its limits are from $P(r = 2\sin\theta)$ to $Q(r = 4\sin\theta)$ and to cover the whole region θ varies from 0 to π. Thus the required integral is

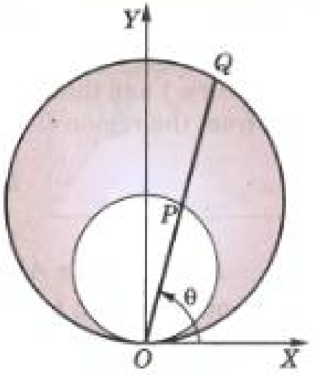

Fig. 7.14

$$I = \int_0^\pi d\theta \int_{2\sin\theta}^{4\sin\theta} r^3 \, dr = \int_0^\pi d\theta \left[\frac{r^4}{4} \right]_{2\sin\theta}^{4\sin\theta}$$

$$= 60 \int_0^\pi \sin^4\theta \, d\theta = 60 \times 2 \int_0^{\pi/2} \sin^4\theta \, d\theta = 120 \times \frac{3 \, . \, 1}{4 \, . \, 2} \cdot \frac{\pi}{2} = 22.5\,\pi.$$

PROBLEMS 7.1

Evaluate the following integrals (1–7) :

1. $\int_1^2 \int_1^3 xy^2 \, dxdy$.
2. $\int_0^1 \int_x^{\sqrt{x}} (x^2 + y^2) \, dx \, dy$. *(V.T.U., 2000)*
3. $\int_0^1 \int_0^x e^{x/y} \, dxdy$. *(P.T.U., 2005)*
4. $\int_0^1 \int_0^{\sqrt{(1+x^2)}} \frac{dydx}{1 + x^2 + y^2}$ *(Rajasthan, 2005)*
5. $\iint xy \, dxdy$ over the positive quadrant of the circle $x^2 + y^2 = a^2$. *(Rajasthan, 2006)*
6. $\iint (x + y)^2 dxdy$ over the area bounded by the ellipse $x^2/a^2 + y^2/b^2 = 1$. *(Kurukshetra, 2009 S ; U.P.T.U., 2004 S)*
7. $\iint xy(x + y) \, dxdy$ over the area between $y = x^2$ and $y = x$. *(V.T.U., 2010)*

Evaluate the following integrals by changing the order of integration (8–15) :

8. $\int_0^a \int_y^a \frac{xdxdy}{x^2 + y^2}$. *(Bhopal, 2008)*
9. $\int_0^3 \int_1^{\sqrt{(4-y)}} (x + y) \, dxdy$. *(V.T.U., 2005 ; Anna, 2003 S ; Delhi, 2002)*

10. $\int_0^1 \int_x^{\sqrt{(2-x^2)}} \frac{x\,dydx}{\sqrt{(x^2+y^2)}}$. (*P.T.U., 2010 ; Marathwada, 2008 ; U.P.T.U., 2006*)

11. $\int_0^{a/\sqrt{2}} \int_y^{\sqrt{(a^2-y^2)}} \log(x^2+y^2)\,dxdy\ (a>0)$.

12. $\int_0^1 \int_x^{\sqrt{x}} xy\,dydx$. (*V.T.U., 2010*)

13. $\int_0^a \int_{a-\sqrt{(a^2-y^2)}}^{a+\sqrt{(a^2-y^2)}} xy\,dx\,dy$. (*Anna, 2009*)

14. $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\,dydx$. (*Bhopal, 2009 ; S.V.T.U., 2009 ; V.T.U., 2007*)

15. $\int_0^\infty \int_0^x xe^{-x^2/y}\,dydx$. (*S.V.T.U., 2006 ; U.P.T.U., 2005 ; V.T.U., 2004*)

16. Sketch the region of integration of the following integrals and change the order of integrations,

(*i*) $\int_0^{2a} \int_{\sqrt{(2ax-x^2)}}^{\sqrt{(2ax)}} f(x)\,dxdy$ (*Rajasthan, 2006*) (*ii*) $\int_0^{ae^{\pi/4}} \int_{2\log(r/a)}^{\pi/2} f(r,\theta)\,r\,drd\theta$.

17. Show that $\iint_R r^2 \sin\theta\,drd\theta = 2a^3/3$, where R is the semi-circle $r = 2a\cos\theta$ above the initial line.

18. Evaluate $\iint \frac{r\,drd\theta}{\sqrt{a^2+r^2}}$ over one loop of the lemniscate $r^2 = a^2\cos 2\theta$. (*Rohtak, 2006 S ; P.T.U., 2005*)

19. Evaluate $\iint r^3\,drd\theta$ over the area bounded between the circles $r = 2\cos\theta$ and $r = 4\cos\theta$.

(*Anna, 2009 ; Madras, 2006*)

7.4 AREA ENCLOSED BY PLANE CURVES

(**1**) *Cartesian coordinates.*

Consider the area enclosed by the curves $y = f_1(x)$ and $y = f_2(x)$ and the ordinates $x = x_1$, $x = x_2$ [Fig. 7.15 (*a*)].

Divide this area into vertical strips of width δx. If $P(x, y)$, $Q(x + \delta x, y + \delta y)$ be two neighbouring points, then the area of the small rectangle $PQ = \delta x \delta y$.

$\therefore$ area of strip $KL = \underset{\delta y \to 0}{\text{Lt}} \Sigma\, \delta x\, \delta y$.

Since for all rectangles in this strip δx is the same and y varies from $y = f_1(x)$ to $y = f_2(x)$.

$$\therefore \quad \text{area of the strip } KL = \delta x \underset{\delta y \to 0}{\text{Lt}} \sum_{f_1(x)}^{f_2(x)} dy = \delta x \int_{f_1(x)}^{f_2(x)} dy.$$

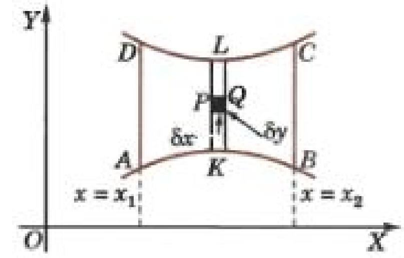

Fig. 7.15(*a*)

Now adding up all such strips from $x = x_1$ to $x = x_2$, we get the area $ABCD$

$$= \underset{\delta x \to 0}{\text{Lt}} \sum_{x_1}^{x_2} \delta x \cdot \int_{f_1(x)}^{f_2(x)} dy = \int_{x_1}^{x_2} dx \int_{f_1(x)}^{f_2(x)} dy = \int_{x_1}^{x_2} \int_{f_1(x)}^{f_2(x)} dxdy$$

Similarly, dividing the area $A'B'C'D$ [Fig. 7.15(*b*)] into horizontal strips of width δy, we get the area $A'B'C'D'$.

$$= \int_{y_1}^{y_2} \int_{f_1(y)}^{f_2(y)} dxdy$$

(**2**) *Polar coordinates.*

Consider an area A enclosed by a curve whose equation is in polar coordinates.

Let $P(r, \theta)$, $Q(r + \delta r, \theta + \delta\theta)$ be two neighbouring points. Mark circular areas of radii r and $r + \delta r$ meeting OQ in R and OP (produced) in S (Fig. 7.16).

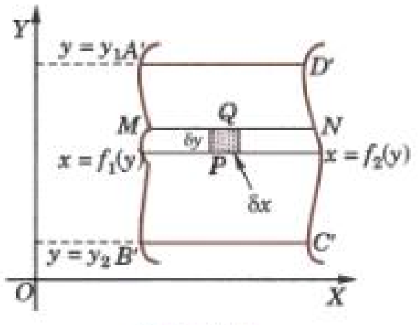

Fig. 7.15 (*b*)

Since arc $PR = r\delta\theta$ and $PS = \delta r$.

$\therefore$ area of the curvilinear rectangle $PRQS$ is approximately $= PR \,.\, PS = r\delta\theta \,.\, \delta r$.

If the whole area is divided into such curvilinear rectangles, the sum $\Sigma\Sigma r\delta\theta\delta r$ taken for all these rectangles, gives in the limit the area A.

Hence $$A = \underset{\substack{\delta r \to 0 \\ \delta\theta \to 0}}{\text{Lt}} \Sigma\Sigma r\delta\theta\delta r = \iint r\,d\theta dr$$

where the limits are to be so chosen as to cover the entire area.

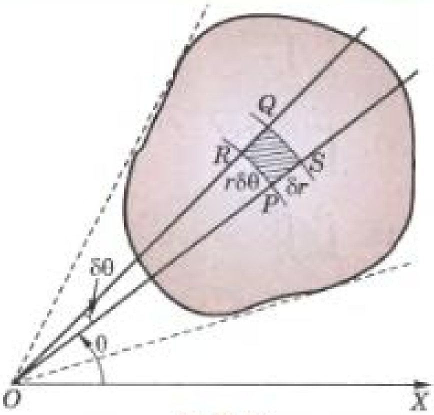

Fig. 7.16

Example 7.13. *Find the area of a plate in the form of a quadrant of the ellipse*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$ *(V.T.U., 2001 ; Osmania, 2000 S)*

Solution. Dividing the area into vertical strips of width δx, y varies from $K(y = 0)$ to $L[y = b\sqrt{(1 - x^2/b^2)}]$ and then x varies from 0 to a (Fig. 7.17).

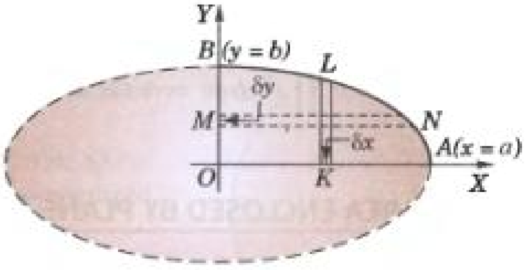

Fig. 7.17

$\therefore$ required area

$$= \int_0^a dx \int_0^{b\sqrt{(1-x^2/a^2)}} dy = \int_0^a dx\, [y]_0^{b\sqrt{(1-x^2/a^2)}}$$

$$= \frac{b}{a}\int_0^a \sqrt{(a^2 - x^2)}\, dx = \pi ab/4.$$

Otherwise, dividing this area into horizontal strips of width δy, x varies from $M(x = 0)$ to $N[x = a\sqrt{(1 - y^2/b^2)}]$ and then y varies from 0 to b.

$\therefore$ required area $$= \int_0^b dy \int_0^{a\sqrt{(1-y^2/b^2)}} dx = \int_0^b dy\, [x]_0^{a\sqrt{(1-y^2/b^2)}}$$

$$= \frac{a}{b}\int_0^b \sqrt{(b^2 - y^2)}\, dy = \pi ab/4.$$

Obs. The change of the order of integration does not in any way affect the value of the area.

Example 7.14. *Show that the area between the parabolas $y^2 = 4ax$ and $x^2 = 4ay$ is $\frac{16}{3}a^2$.*

(Kerala, 2005 ; Rohtak, 2003)

Solution. Solving the equations $y^2 = 4ax$ and $x^2 = 4ay$, it is seen that the parabolas intersect at O (0, 0) and A ($4a$, $4a$). As such for the shaded area between these parabolas (Fig. 7.18) x varies from 0 to $4a$ and y varies from P to Q *i.e.*, from $y = x^2/4a$ to $y = 2\sqrt{(ax)}$. Hence the *required area*

$$= \int_0^{4a}\int_{x^2/4a}^{2\sqrt{(ax)}} dydx = \int_0^{4a} (2\sqrt{(ax)} - x^2/4a)\, dx$$

$$= \left| 2\sqrt{a}\,.\,\frac{2}{3}x^{3/2} - \frac{1}{4a}\,.\,\frac{x^3}{3} \right|_0^{4a} = \frac{32}{3}a^2 - \frac{16}{3}a^2 = \frac{16}{3}a^2.$$

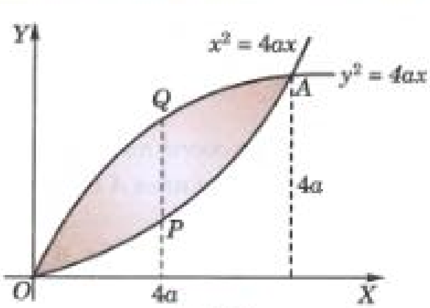

Fig. 7.18

Example 7.15. *Calculate the area included between the curve $r = a\,(\sec\theta + \cos\theta)$ and its asymptote.*

Solution. The curve is symmetrical about the initial line and has an asymptote $r = a \sec\theta$ (Fig. 7.19).

Draw any line OP cutting the curve at P and its asymptote at P'. Along this line, θ is constant and r varies from $a\sec\theta$ at P' to $a\,(\sec\theta + \cos\theta)$ at P. Then to get the upper half of the area, θ varies from 0 to $\pi/2$.

$$\therefore \quad \text{required area} = 2\int_0^{\pi/2}\int_{a\sec\theta}^{a(\sec\theta+\cos\theta)} r\,dr d\theta$$

$$= 2\int_0^{\pi/2}\left[\frac{r^2}{2}\right]_{a\sec\theta}^{a(\sec\theta+\cos\theta)} d\theta$$

$$= a^2\int_0^{\pi/2}(2+\cos^2\theta)\,d\theta = 5\pi a^2/4.$$

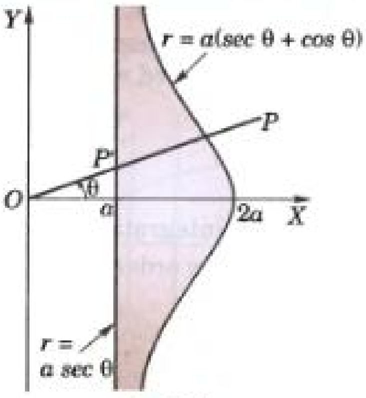

Fig. 7.19

Example 7.16. *Find the area lying inside the cardioid $r = a(1 + \cos\theta)$ and outside the circle $r = a$.*

Solution. In Fig. 7.20, $ABODA$ represents the cardioid $r = a(1 + \cos\theta)$ and $CBA'DC$ is the circle $r = a$.

Required area (shaded) = 2 (area $ABCA$)

$$= 2\int_0^{\pi/2}\int_{r=OP'}^{r=OP} r\,d\theta\,dr = 2\int_0^{\pi/2}\int_a^{a(1+\cos\theta)}(rdr)\,d\theta$$

$$= 2\int_0^{\pi/2}\left|\frac{r^2}{2}\right|_a^{a(1+\cos\theta)} d\theta = a^2\int_0^{\pi/2}[(1+\cos\theta)^2 - 1]\,d\theta$$

$$= a^2\int_0^{\pi/2}(\cos^2\theta + 2\cos\theta)\,d\theta = a^2\left(\frac{1}{2}\cdot\frac{\pi}{2}+2\right) = \frac{a^2}{4}(\pi + 8).$$

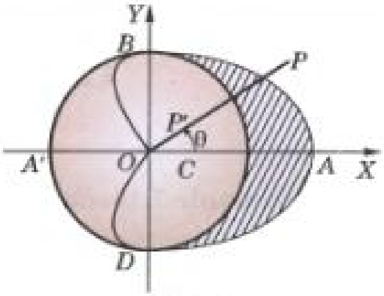

Fig. 7.20

PROBLEMS 7.2

1. Find, by double integration, the area lying between the parabola $y = 4x - x^2$ and the line $y = x$.
2. Find the area lying between the parabola $y = x^2$ and the line $x + y - 2 = 0$. (*Anna, 2009*)
3. By double integration, find the whole area of the curve $a^2x^2 = y^3(2a - y)$. (*U.P.T.U., 2001*)
4. Find, by double integration, the area enclosed by the curves $y = 3x/(x^2 + 2)$ and $4y = x^2$. (*J.N.T.U., 2005*)
5. Find, by double integration, the area of the lemniscate $r^2 = a^2 \cos 2\theta$. (*Madras, 2000 S*)
6. Find, by double integration, the area lying inside the circle $r = a\sin\theta$ and outside the cardioid $r = a(1 - \cos\theta)$. (*Anna 2009 ; Mumbai, 2006*)
7. Find the area lying inside the cardioid $r = 1 + \cos\theta$ and outside the parabola $r\,(1 + \cos\theta) = 1$.
8. Find the area common to the circles $r = a\cos\theta$, $r = a\sin\theta$ by double integration. (*Mumbai, 2007*)

7.5 TRIPLE INTEGRALS

Consider a function $f(x, y, z)$ defined at every point of the 3-dimensional finite region V. Divide V into n elementary volumes $\delta V_1, \delta V_2, \ldots, \delta V_n$. Let (x_r, y_r, z_r) be any point within the rth sub-division δV_r. Consider the sum

$$\sum_{r=1}^{\infty} f(x_r, y_r, z_r)\delta V_r\,.$$

The limit of this sum, if it exists, as $n \to \infty$ and $\delta V_r \to 0$ is called the *triple integral of $f(x, y, z)$ over the region V* and is denoted by

$$\iiint f(x, y, z)\,dV.$$

For purposes of evaluation, it can also be expressed as the repeated integral

$$\int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} f(x, y, z)\,dxdydz.$$

If x_1, x_2 are constants ; y_1, y_2 are either constants or functions of x and z_1, z_2 are either constants or functions of x and y, then this integral is evaluated as follows :

First $f(x, y, z)$ is integrated w.r.t. z between the limits z_1 and z_2 keeping x and y fixed. The resulting expression is integrated w.r.t. y between the limits y_1 and y_2 keeping x constant. The result just obtained is finally integrated w.r.t. x from x_1 to x_2.

Thus
$$I = \boxed{\int_{x_1}^{x_2} \boxed{\int_{y_1(x)}^{y_2(x)} \boxed{\int_{z_1(x,y)}^{z_2(x,y)} f(x, y, z)dz}\, dy}\, dx}$$

where the integration is carried out from the innermost rectangle to the outermost rectangle.

The order of integration may be different for different types of limits.

Example 7.17. *Evaluate* $\int_{-1}^{1} \int_{0}^{z} \int_{x-z}^{x+z} (x + y + z)\, dxdydz.$ *(J.N.T.U., 2006 ; Cochin, 2005)*

Solution. Integrating first w.r.t. y keeping x and z constant, we have

$$I = \int_{-1}^{1} \int_{0}^{z} \left| xy + \frac{y^2}{2} + yz \right|_{x-z}^{x+z} dx\, dz = \int_{-1}^{1} \int_{0}^{z} \left[(x + z)(2z) + \frac{1}{2} 4xz\right] dx\, dz$$

$$= 2 \int_{-1}^{1} \left| \frac{x^2 z}{2} + z^2 x + \frac{x^2}{2} z \right|_{0}^{z} dz = 2 \int_{-1}^{1} \left(\frac{z^3}{2} + z^3 + \frac{z^3}{2}\right) dz = 4 \left| \frac{z^4}{4} \right|_{-1}^{1} = 0.$$

Example 7.18. *Evaluate* $\int_{0}^{1} \int_{0}^{\sqrt{(1-x^2)}} \int_{0}^{\sqrt{(1-x^2-y^2)}} xyz\, dxdydz.$ *(V.T.U., 2003 S)*

Solution. We have

$$I = \int_0^1 x \left[\int_0^{\sqrt{(1-x^2)}} y \left\{ \int_0^{\sqrt{(1-x^2-y^2)}} z\, dz \right\} dy \right] dx = \int_0^1 x \left\{ \int_0^{\sqrt{(1-x^2)}} y \,.\, \left| \frac{z^2}{2} \right|_0^{\sqrt{(1-x^2-y^2)}} dy \right\} dx$$

$$= \int_0^1 x \left\{ \int_0^{\sqrt{(1-x^2)}} y \,.\, \frac{1}{2}(1 - x^2 - y^2)\, dy \right\} dx = \frac{1}{2} \int_0^1 x \left| (1 - x^2)\frac{y^2}{2} - \frac{y^4}{4} \right|_0^{\sqrt{(1-x^2)}} dx$$

$$= \frac{1}{8} \int_0^1 [(1 - x^2)^2 \,.\, 2x - (1 - x^2)^4 \,.\, x]\, dx = \frac{1}{8} \int_0^1 (x - 2x^3 + x^5)\, dx$$

$$= \frac{1}{8} \left| \frac{x^2}{2} - \frac{2x^4}{4} + \frac{x^6}{6} \right|_0^1 = \frac{1}{8}\left(\frac{1}{2} - \frac{1}{2} + \frac{1}{6}\right) = \frac{1}{48}.$$

PROBLEMS 7.3

Evaluate the following integrals :

1. $\int_0^a \int_0^b \int_0^c (x^2 + y^2 + z^2)\, dx\, dy\, dz.$ *(Anna, 2009)*
2. $\int_{-c}^{c} \int_{-b}^{b} \int_{-a}^{a} (x^2 + y^2 + z^2)\, dx\, dy\, dz$ *(S.V.T.U., 2009 ; V.T.U. 2000)*
3. $\int_0^1 \int_{y^2}^1 \int_0^{1-x} x\, dz\, dx\, dy$ *(Nagpur, 2009)*
4. $\int_0^a \int_0^x \int_0^{x+y} e^{x+y+z}\, dz\, dy\, dx.$ *(V.T.U., 2010 ; Kurukshetra, 2009 S ; J.N.T.U., 2005)*
5. $\int_0^{\log 2} \int_0^x \int_0^{x+\log y} e^{x+y+z}\, dx\, dy\, dz.$ *(Bhopal, 2008)*
6. $\int_1^e \int_1^{\log y} \int_1^{e^x} \log z\, dz\, dx\, dy.$ *(S.V.T.U., 2008 ; Rohtak, 2005)*
7. $\int_0^{\pi/2} \int_0^{a \sin\theta} \int_0^{\frac{a^2-r^2}{a}} r dz\, dr\, d\theta.$ *(V.T.U., 2009)*

7.6 VOLUMES OF SOLIDS

(1) Volumes as double integrals. Consider a surface $z = f(x, y)$. Let the orthogonal projection on XY-plane of its portion S' be the area S (Fig. 7.21).

Divide S into elementary rectangles of area $\delta x\, \delta y$ by drawing lines parallel to X and Y-axes. With each of these rectangles as base, erect a prism having its length parallel to OZ.

$\therefore$ volume of this prism between S and the given surface $z = f(x, y)$ is $z\delta x\delta y$.

Hence the volume of the solid cylinder on S as base, bounded by the given surface with generators parallel to the Z-axis.

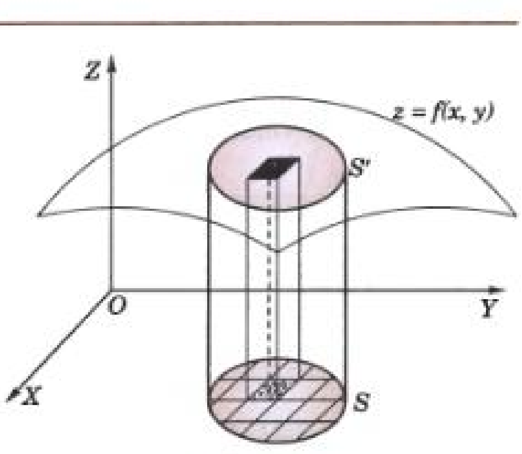

Fig. 7.21

$$= \underset{\substack{\delta x \to 0 \\ \delta y \to 0}}{\text{Lt}} \Sigma\Sigma\, z\delta x\delta y$$

$$= \iint z\, dx\, dy \quad \text{or} \quad \iint f(x, y)\, dx\, dy$$

where the integration is carried over the area S.

Obs. While using polar coordinates, divide S into elements of area $r\delta\theta\, \delta r$.

$\therefore$ replacing $dxdy$ by $r\delta\theta\delta r$, we get the required volume $= \iint zr\, d\theta\, dr$.

Example 7.19. *Find the volume bounded by the cylinder $x^2 + y^2 = 4$ and the planes $y + z = 4$ and $z = 0$.*

(S.V.T.U., 2007 ; Cochin, 2005 ; Madras, 2000 S)

Solution. From Fig. 7.22, it is self-evident that $z = 4 - y$ is to be integrated over the circle $x^2 + y^2 = 4$ in the XY-plane. To cover the shaded half of this circle, x varies from 0 to $\sqrt{(4 - y^2)}$ and y varies from -2 to 2.

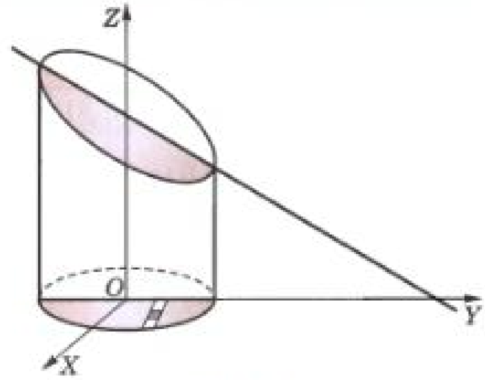

Fig. 7.22

$\therefore$ Required volume

$$= 2\int_{-2}^{2}\int_{0}^{\sqrt{(4-y^2)}} z\, dx dy = 2\int_{-2}^{2}\int_{0}^{\sqrt{(4-y^2)}} (4 - y)\, dx dy$$

$$= 2\int_{-2}^{2} (4 - y)\, [x]_0^{\sqrt{(4-y^2)}}\, dy = 2\int_{-2}^{2} (4 - y)\sqrt{(4 - y^2)}\, dy$$

$$= 2\int_{-2}^{2} 4\sqrt{(4 - y^2)}\, dy - 2\int_{-2}^{2} y\sqrt{(4 - y^2)}\, dy$$

$$= 8\int_{-2}^{2} \sqrt{(4 - y^2)}\, dy$$ [The second term vanishes as the integrand is an odd function.]

$$= 8\left|\frac{y\sqrt{(4 - y)^2}}{2} + \frac{4}{2}\sin^{-1}\frac{y}{2}\right|_{-2}^{2} = 16\pi.$$

(2) Volume as triple integral

Divide the given solid by planes parallel to the coordinate planes into rectangular parallelopipeds of volume $\delta x\, \delta y\, \delta z$ (Fig. 7.23).

$$\therefore \text{ the total volume} = \underset{\substack{\delta x \to 0 \\ \delta y \to 0 \\ \delta z \to 0}}{\text{Lt}} \Sigma\Sigma\Sigma\, \delta x\, \delta y\, \delta z$$

$$= \int\int\int dx\, dy\, dz$$

with appropriate limits of integration.

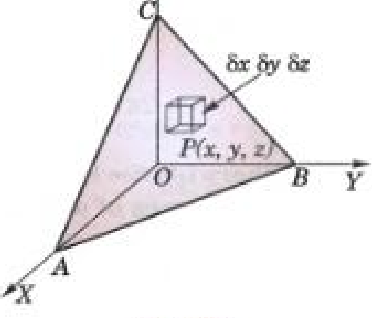

Fig. 7.23

Example 7.20. *Calculate the volume of the solid bounded by the planes $x = 0$, $y = 0$, $x + y + z = a$ and $z = 0$.* (*P.T.U., 2009*)

Solution. Volume required $= \int_0^a \int_0^{a-x} \int_0^{a-x-y} dz\, dy\, dx$

$$= \int_0^a \int_0^{a-x} (a - x - y)\, dy\, dx = \int_0^a \left| (a-x)y - \frac{y^2}{2} \right|_0^{a-x} dx$$

$$= \int_0^a \left\{ (a-x)^2 - \frac{(a-x)^2}{2} \right\} dx = \frac{1}{2}\int_0^a (a-x)^2\, dx = \frac{1}{2}\left| -\frac{(a-x)^3}{3} \right|_0^a = \frac{a^3}{6}.$$

Example 7.21. *Find the volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.* (*Anna, 2009 ; P.T.U., 2006 ; Kottayam, 2005*)

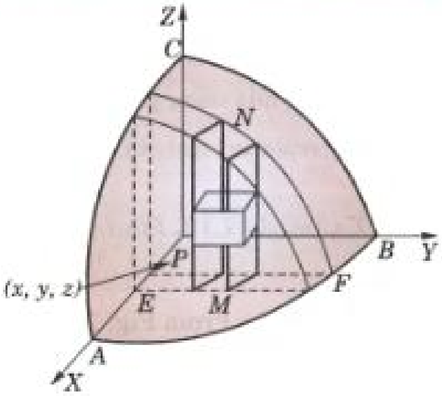

Fig. 7.24

Solution. Let $OABC$ be the positive octant of the given ellipsoid which is bounded by the planes OAB ($z = 0$), OBC ($x = 0$), OCA ($y = 0$) and the surface ABC, *i.e.*,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Divide this region R into rectangular parallellopipeds of volume $\delta x \delta y \delta z$. Consider such an element at $P(x, y, z)$. (Fig. 7.24)

$\therefore$ the required volume $= 8 \iiint_R dx\, dy\, dz$.

In this region R,

(*i*) z varies from 0 to MN where

$$MN = c\sqrt{(1 - x^2/a^2 - y^2/b^2)}.$$

(*ii*) y varies from 0 to EF, where $EF = b\sqrt{(1 - x^2/a^2)}$ from the equation of the ellipse OAB, *i.e.*,

$$x^2/a^2 + y^2/b^2 = 1.$$

(*iii*) x varies from 0 to $OA = a$.

Hence the volume of the whole ellipsoid

$$= 8 \int_0^a \int_0^{b\sqrt{(1-x^2/a^2)}} \int_0^{c\sqrt{(1-x^2/a^2-y^2/b^2)}} dx\, dy\, dz = 8 \int_0^a dx \int_0^{b\sqrt{(1-x^2/a^2)}} dy \,\big|\, z\, \big|_0^{c\sqrt{(1-x^2/a^2-y^2/b^2)}}$$

$$= 8c \int_0^a dx \int_0^{b\sqrt{(1-x^2/a^2)}} \sqrt{(1 - x^2/a^2 - y^2/b^2)}\, dy$$

$$= \frac{8c}{b} \int_0^a dx \int_0^{\rho} \sqrt{(\rho^2 - y^2)}\, dy \quad \text{when } \rho = b\sqrt{1 - x^2/a^2}.$$

$$= \frac{8c}{b} \int_0^a dx \left[\frac{y\sqrt{(\rho^2 - y^2)}}{2} + \frac{\rho^2}{2}\sin^{-1}\frac{y}{\rho} \right]_0^{\rho} = \frac{8c}{b}\int_0^a \frac{b^2}{2}\left(1 - \frac{x^2}{a^2}\right)\frac{\pi}{2}\, dx$$

$$= 2\pi bc \int_0^a \left(1 - \frac{x^2}{a^2}\right) dx = 2\pi bc \left| x - \frac{x^3}{3a^2} \right|_0^a = \frac{4\pi abc}{3}.$$

Otherwise. See Problem 27 page 292.

(3) Volumes of solids of revolution

Consider an elementary area $\delta x \delta y$ at the point $P(x, y)$ of a plane area A. (Fig. 7.25)

As this elementary area revolves about x-axis, we get a ring of volume

$$= \pi[(y + \delta y)^2 - y^2]\, \delta x = 2\pi y\, \delta x\, \delta y,$$

nearly to the first powers of δy.

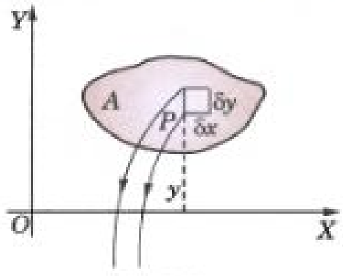

Fig. 7.25

Hence the total volume of the solid formed by the revolution of the area A about x-axis.

$$= \iint_A 2\pi y \, dxdy.$$

In polar coordinates, the above formula for the volume becomes

$$\iint_A 2\pi r \sin\theta \, . \, r d\theta dr, \text{ i.e. } \iint_A 2\pi r^2 \sin\theta \, d\theta dr$$

Similarly, the volume of the solid formed by the revolution of the area A about y-axis $= \iint_A 2\pi x \, dx \, dy$.

Example 7.22. *Calculate by double integration, the volume generated by the revolution of the cardioid $r = a\,(1 - \cos\theta)$ about its axis.*

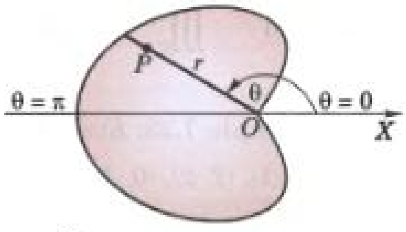

Fig. 7.26

Solution. Required volume

$$= \int_0^{\pi} \int_0^{a(1-\cos\theta)} 2\pi r^2 \sin\theta \, dr \, d\theta$$

$$= 2\pi \int_0^{\pi} \left| \frac{r^3}{3} \right|_0^{a(1-\cos\theta)} \sin\theta \, d\theta$$

$$= \frac{2\pi a^3}{3} \int_0^{\pi} (1-\cos\theta)^3 \, . \sin\theta \, d\theta = \frac{2\pi a^3}{3} \left| \frac{(1-\cos\theta)^4}{4} \right|_0^{\pi} = \frac{8\pi a^3}{3}.$$

7.7 CHANGE OF VARIABLES

An appropriate choice of co-ordinates quite often facilitates the evaluation of a double or a triple integral. By changing the variables, a given integral can be transformed into a simpler integral involving the new variables.

(1) **In a double integral,** let the variables x, y be changed to the new variables u, v by the transformation.

$$x = \phi\,(u, v),\ y = \psi\,(u, v)$$

where $\phi(u, v)$ and $\psi(u, v)$ are continuous and have continuous first order derivatives in some region R'_{uv} in the uv-plane which corresponds to the region R_{xy} in the xy-plane. Then

$$\iint_{R_{xy}} f(x, y)\, dxdy = \iint_{R'_{uv}} f[\phi(u, v), \psi(u, v)] \,|\, J \,|\, dudv \qquad ...(1)$$

where $$J = \frac{\partial(x, y)}{\partial(u, v)} \ (\neq 0)$$

is the *Jacobian of transformation** from (x, y) to (u, v) coordinates.

(2) **For triple integrals,** the formula corresponding to (1) is

$$\iiint_{R_{xyz}} f(x, y, z)\, dxdydz = \iiint_{R'_{uvw}} f[x(u, v, w), y(u, v, w), z(u, v, w)] \,|\, J \,|\, dudvdw$$

where $$J = \frac{\partial(x, y, z)}{\partial(u, v, w)} \ (\neq 0)$$

is the *Jacobian of transformation* from (x, y, z) to (u, v, w) coordinates.

Particular cases :

(*i*) ***To change cartesian coordinates (x, y) to polar coordinates (r, θ),*** *we have* $x = r\cos\theta$, $y = r\sin\theta$ *and*

$$J = \frac{\partial(x, y)}{\partial(r, \theta)} = r \qquad \text{[Ex. 5.25, p. 216]}$$

$$\therefore \quad \iint_{R_{xy}} f(x, y)\, dx\, dy = \iint_{R'_{r\theta}} f(r\cos\theta, r\sin\theta)\, . \, r\, dr\, d\theta.$$

---

* See footnote page 215.

(*ii*) ***To change rectangular coordinates (x, y, z) to cylindrical coordinates*** (ρ, ϕ, z) — *Fig. 8.27, we have*

$$x = \rho \cos \phi, y = \rho \sin \phi, z = z$$

and
$$J = \frac{\partial(x, y, z)}{\partial(\rho, \phi, z)} = \rho \qquad \text{[Ex. 5.25]}$$

Then $\iiint_{R_{xyz}} f(x, y, z)\, dxdydz = \iiint_{R'_{\rho\phi z}} f(\rho \cos \phi, \rho \sin \phi, z) \,.\, \rho d\rho d\phi dz.$

(*iii*) ***To change rectangular coordinates (x, y, z) to spherical polar coordinates (r,*** θ, ϕ***)***—*Fig. 8.28, we have*

$$x = r \sin \theta \cos \phi, y = r \sin \theta \sin \phi, z = r \cos \theta$$

and
$$J = \frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = r^2 \sin \theta \qquad \text{[Ex. 5.25]}$$

Then $\iiint_{R_{xyz}} f(x, y, z)\, dxdydz = \iiint_{R'_{r\theta\phi}} f(r \sin \theta \cos \phi, r \sin \theta, \sin \phi, r \cos \theta) \,.\, r^2 \sin \theta \, dr d\theta d\phi$

Example 7.23. *Evaluate* $\iint_R (x + y)^2 \, dxdy$, *where R is the parallelogram in the xy-plane with vertices (1, 0), (3, 1), (2, 2), (0, 1) using the transformation u = x + y and v = x – 2y.* (*U.P.T.U., 2004*)

Solution. The region R, *i.e.*, parallelogram $ABCD$ in the xy-plane becomes the region R', *i.e.*, rectangle $A'B'C'D'$ in the uv-plane as shown in Fig. 7.27, by taking

$$u = x + y \quad \text{and} \quad v = x - 2y \qquad \ldots(i)$$

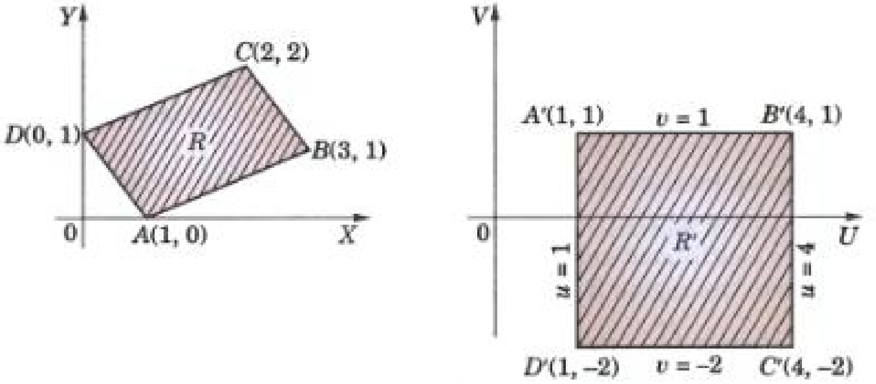

Fig. 7.27

From (*i*), we have
$$x = \frac{1}{3}(2u + v), y = \frac{1}{3}(u - v)$$

$$\therefore \qquad \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} \end{vmatrix} = -\frac{1}{3}$$

Hence, the given integral

$$= \iint_{R'} u^2 \,|\, J \,|\, dudv = \int_1^4 \int_{-2}^1 u^2 \,.\, \frac{1}{3} \,.\, dudv = \frac{1}{3} \left| \frac{u^3}{3} \right|_1^4 . \,| \, v \,|_{-2}^1 = 21.$$

Example 7.24. *Evaluate* $\iint_D xy\sqrt{(1 - x - y)}\, dx\, dy$ *where D is the region bounded by x = 0, y = 0 and x + y = 1 using the transformation x + y = u, y = uv.* (*Marathwada, 2008*)

Solution. We have $x = u - uv,\ y = uv$

$$\therefore \qquad J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \partial x/\partial u & \partial y/\partial u \\ \partial x/\partial v & \partial y/\partial v \end{vmatrix} = \begin{vmatrix} 1-v & v \\ -u & u \end{vmatrix} = u.$$

Also when $x = 0,\ u = 0,\ v = 1$; when $y = 0,\ u = 0,\ v = 0$ and when $x + y = 1,\ u = 1$

$\therefore$ the limits of u are from 0 to 1 and limits of v are from 0 to 1.

Thus
$$\iint_D xy\sqrt{(1-x-y)}\,dxdy = \int_0^1\int_0^1 u(1-v)\,uv\,(1-u)^{1/2}\,|J|\,dudv$$

$$= \int_0^1\int_0^1 u^3(1-u)^{1/2}\,v(1-v)\,du\,dv$$

where $u = \sin^2\theta$
$du = 2\sin\theta\cos\theta\,d\theta$
$u = 0, \theta = 0$
$u = 1, \theta = \pi/2$

$$= \int_0^1 u^3(1-u)^{1/2}\,du \times \int_0^1 v(1-v)\,dv$$

$$= \int_0^{\pi/2} \sin^6\theta\cos\theta\,.\,2\sin\theta\cos d\theta \times \left|\frac{v^2}{2} - \frac{v^3}{3}\right|_0^1$$

$$= 2\int_0^{\pi/2}\sin^7\theta\cos^2\theta\,d\theta\left(\frac{1}{6}\right) = \frac{1}{3}\cdot\frac{6\,.\,1}{9\,.\,7\,.\,5\,.\,3} = \frac{2}{945}.$$

Example 7.25. *Evaluate* $\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dxdy$ *by changing to polar coordinates.* (*Anna, 2003*)

Hence show that $\int_0^\infty e^{-x^2}\,dx = \sqrt{\pi/2}$. (*Madras, 2003 ; U.P.T.U., 2003 ; J.N.T.U., 2000*)

Solution. The region of integration being the first quadrant of the xy-plane, r varies from 0 to ∞ and θ varies from 0 to $\pi/2$. Hence,

$$I = \int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dxdy = \int_{\theta=0}^{\pi/2}\int_{r=0}^\infty e^{-r^2}\,rdrd\theta$$

$$= -\frac{1}{2}\int_0^{\pi/2}\left\{\int_0^\infty e^{-r^2}(-2r)\,dr\right\}d\theta = -\frac{1}{2}\int_0^{\pi/2}\left|e^{-r^2}\right|_0^\infty d\theta = \frac{1}{2}\int_0^{\pi/2}d\theta = \frac{\pi}{4}. \qquad \text{...(i)}$$

Also
$$I = \int_0^\infty e^{-x^2}\,dx \times \int_0^\infty e^{-y^2}\,dy = \left\{\int_0^\infty e^{-x^2}\,dx\right\}^2 \qquad \text{...(ii)}$$

Thus, from (*i*) and (*ii*), we have $\int_0^\infty e^{-x^2} = \sqrt{\pi/2}$. ...(*iii*)

Example 7.26. *Find the volume bounded by the paraboloid* $x^2 + y^2 = az$, *the cylinder* $x^2 + y^2 = 2ay$ *and the plane* $z = 0$.

Solution. The required volume is found by integrating $z = (x^2 + y^2)/a$ over the circle $x^2 + y^2 = 2ay$.

Changing to polar coordinates in the xy-plane, we have $x = r\cos\theta$, $y = r\sin\theta$ so that $z = r^2/a$ and the polar equation of the circle is $r = 2a\sin\theta$.

To cover this circle, r varies from 0 to $2a\sin\theta$ and θ varies from 0 to π. (Fig. 7.28)

Hence the required volume

$$= \int_0^\pi\int_0^{2a\sin\theta} z\,.\,r\,d\theta\,dr = \frac{1}{a}\int_0^\pi d\theta\int_0^{2a\sin\theta} r^3\,dr$$

$$= \frac{1}{a}\int_0^\pi d\theta\left|\frac{r^4}{4}\right|_0^{2a\sin\theta} = 4a^3\int_0^\pi \sin^4\theta\,d\theta = \frac{3\pi a^3}{2}.$$

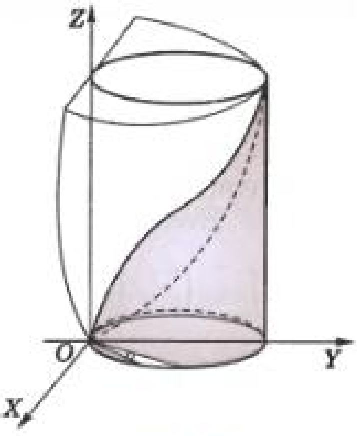

Fig. 7.28

Example 7.27. *Find, by triple integration, the volume of the sphere* $x^2 + y^2 + z^2 = a^2$.

(Bhopal, 2009 ; Madras, 2006 ; V.T.U., 2003 S)

Solution. Changing to polar spherical coordinates by putting

$$x = r \sin\theta \cos\phi,\ y = r \sin\theta \sin\phi,\ z = r\cos\theta$$

we have $dx\, dy\, dz = r^2 \sin\theta\, dr\, d\theta\, d\phi$.

Also the volume of the sphere is 8 times the volume of its portion in the positive octant for which r varies from 0 to a, θ varies from 0 to $\pi/2$ and ϕ varies from 0 to $\pi/2$.

$\therefore$ volume of the sphere

$$= 8 \int_0^a \int_0^{\pi/2} \int_0^{\pi/2} r^2 \sin\theta\, dr\, d\theta\, d\phi = 8 \int_0^a r^2\, dr \,.\int_0^{\pi/2} .\sin\theta\, d\theta \,.\int_0^{\pi/2} d\phi$$

$$= 8 \,.\left| \frac{r^3}{3} \right|_0^a .\left| -\cos\theta \right|_0^{\pi/2} .\frac{\pi}{2} = 4\pi \,.\frac{a^3}{3} \,.(-0+1) = \frac{4}{3}\pi a^3.$$

Example 7.28. *Find the volume of the portion of the sphere* $x^2 + y^2 + z^2 = a^2$ *lying inside the cylinder* $x^2 + y^2 = ay$.

(Rohtak, 2003)

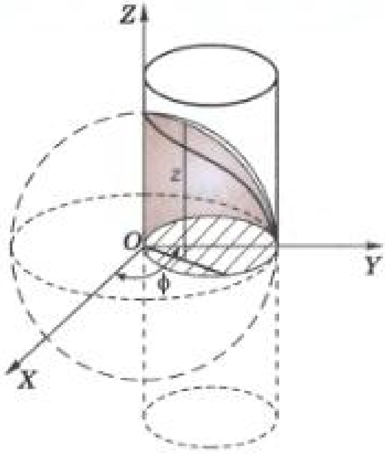

Fig. 7.29

Solution. The required volume is easily found by changing to cylindrical coordinates (ρ, ϕ, z). We therefore, have

$$x = \rho\cos\phi,\ y = \rho\sin\phi,\ z = z$$

and $$J = \frac{\partial(x, y, z)}{\partial(\rho, \phi, z)} = \rho.$$

Then the equation of the sphere becomes $\rho^2 + z^2 = a^2$ and that of cylinder becomes $\rho = a\sin\phi$.

The volume inside the cylinder bounded by the sphere is twice the volume shown shaded in the Fig. 7.29 for which z varies from 0 to $\sqrt{(a^2 - \rho^2)}$, ρ varies from 0 to $a\sin\phi$ and ϕ varies from 0 to π.

Hence the required volume $= 2 \int_0^{\pi} \int_0^{a\sin\phi} \int_0^{\sqrt{(a^2-\rho^2)}} \rho\, dz\, d\rho\, d\phi$

$$= 2 \int_0^{\pi} \int_0^{a\sin\phi} \rho\sqrt{(a^2 - \rho^2)}\, d\rho\, d\phi = 2\int_0^{\pi} \left| -\frac{1}{3}(a^2 - \rho^2)^{3/2} \right|_0^{a\sin\phi} d\phi$$

$$= \frac{2a^3}{3} \int_0^{\pi} (1 - \cos^3\phi)\, d\phi = \frac{2a^3}{9}(3\pi - 4).$$

Example 7.29. *Evaluate* $\int_0^1 \int_0^{\sqrt{(1-x^2)}} \int_{\sqrt{(x^2+y^2)}}^1 \frac{dz\, dy\, dx}{\sqrt{(x^2+y^2+z^2)}}$.

(V.T.U., 2008)

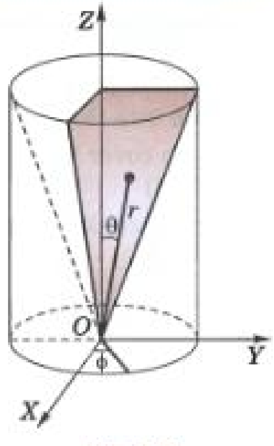

Fig. 7.30

Solution. We change to spherical polar coordinates (r, θ, ϕ), so that

$$x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi,\ z = r\cos\theta$$

and $$J = r^2 \sin\theta,\ x^2 + y^2 + z^2 = r^2.$$

The region of integration is common to the cone $z^2 = x^2 + y^2$ and the cylinder $x^2 + y^2 = 1$ bounded by the plane $z = 1$ in the positive octant (Fig. 7.30). Hence θ varies from 0 to $\pi/4$, r varies from 0 to $\sec\theta$ and ϕ varies from 0 to $\pi/2$.

$\therefore$ given integral becomes

$$\int_0^{\pi/2} \int_0^{\pi/4} \int_0^{\sec\theta} \frac{1}{r} .r^2 \sin\theta\, dr\, d\theta\, d\phi = \int_0^{\pi/2} d\phi \int_0^{\pi/4} \left| \frac{r^2}{2} \right|_0^{\sec\theta} \sin\theta\, d\theta$$

$$= \frac{\pi}{2} \int_0^{\pi/4} \frac{\sec^2\theta}{2} \sin\theta\, d\theta = \frac{\pi}{4} \int_0^{\pi/4} \sec\theta\tan\theta\, d\theta = \frac{\pi}{4} \left| \sec\theta \right|_0^{\pi/4} = \frac{(\sqrt{2}-1)\pi}{4}.$$

Example 7.30. *Find the volume of the solid surrounded by the surface*

$$(x/a)^{2/3} + (y/b)^{2/3} + (z/c)^{2/3} = 1.$$ *(Hissar, 2005 S)*

Solution. Changing the variables, x, y, z to X, Y, Z where, $(x/a)^{1/3} = X$, $(y/b)^{1/3} = Y$, $(z/c)^{1/3} = Z$

i.e., $x = aX^3, y = bY^3, z = cZ^3$ so that $J = \partial(x, y, z)/\partial(X, Y, Z) = 27\, abc\, X^2Y^2Z^2$.

$$\therefore \text{ required volume} = \iiint dx\, dy\, dz = 27\, abc \iiint X^2Y^2Z^2\, dX\, dY dZ$$

taken throughout the sphere $X^2 + Y^2 + Z^2 = 1$. ...(*i*)

Now change X, Y, Z to spherical polar coordinates r, θ, ϕ so that $X = r \sin\theta \cos\phi$, $Y = r \sin\theta \sin\phi$, $Z = r\cos\theta$, and $\partial(X, Y, Z)/\partial(r, \theta, \phi) = r^2 \sin\theta$. To describe the positive octant of the sphere (*i*), r varies from 0 to 1, θ from 0 to $\pi/2$ and ϕ from 0 to $\pi/2$.

$$\therefore \text{ required volume} = 27abc \times 8 \int_0^1 \int_0^{\pi/2} \int_0^{\pi/2} r^2 \sin^2\theta \cos^2\phi \times r^2 \sin^2\theta \sin^2\phi \,.\, r^2\cos^2\theta \,.\, r^2 \sin\theta\, dr d\theta d\phi$$

$$= 216\, abc \int_0^1 r^8\, dr \int_0^{\pi/2} \sin^4\theta \cos^2\theta\, d\theta \int_0^{\pi/2} \sin^2\phi \cos^2\phi\, d\phi = 4\pi\, abc/35.$$

PROBLEMS 7.4

Evaluate the following integrals by changing to polar co-ordinates :

1. $\int_0^1 \int_0^{\sqrt{(1-y^2)}} (x^2 + y^2)\, dy\, dx$. *(P.T.U., 2010)*

2. $\int_0^2 \int_0^{\sqrt{(2x-x^2)}} \frac{x\, dx\, dy}{x^2 + y^2}$ *(Anna, 2009)*

3. $\int_0^{4a} \int_{y^2/4a}^{y} \frac{x^2 - y^2}{x^2 + y^2} dxdy$ *(Mumbai, 2006)*

4. $\iint xy(x^2 + y^2)^{n/2}\, dxdy$ over the positive quadrant of $x^2 + y^2 = 4$, supposing $n + 3 > 0$. *(S.V.T.U., 2007)*

5. $\iint \frac{dxdy}{(1 + x^2 + y^2)^2}$ over one loop of the lemniscate $(x^2 + y^2) = x^2 - y^2$. *(Mumbai, 2007)*

6. Transform the following to cartesian form and hence evaluate $\int_0^{\pi} \int_0^{a} r^3 \sin\theta \cos\theta\, dr d\theta$. *(P.T.U., 2005)*

7. $\iint y^2\, dx\, dy$ over the area outside $x^2 + y^2 - ax = 0$ and inside $x^2 + y^2 - 2ax = 0$. *(Mumbai, 2006)*

8. By using the transformation $x + y = u$, $y = uv$, show that $\int_0^1 \int_0^{1-x} e^{y/(x+y)}\, dydx = \frac{1}{2}(e - 1)$. *(P.T.U., 2003)*

9. Transform $\int_0^{\pi/2} \int_0^{\pi/2} \sqrt{\frac{\sin\phi}{\sin\theta}}\, d\phi\, d\theta$ by the substitution $x = \sin\phi \cos\theta$, $y = \sin\phi \sin\theta$ and show that its value is π. *(U.P.T.U., 2001)*

Evaluate the following integrals by changing to spherical coordinates :

10. $\int_0^1 \int_0^{\sqrt{(1-x^2)}} \int_0^{\sqrt{(1-x^2-y^2)}} \frac{dx\, dy\, dz}{\sqrt{(1 - x^2 - y^2 - z^2)}}$. *(V.T.U., 2006 ; Kottayam, 2005)*

11. $\iiint_V \frac{dx\, dy\, dz}{x^2 + y^2 + z^2}$ where V is the volume of the sphere $x^2 + y^2 + z^2 = a^2$. *(Anna, 2009)*

12. Evaluate $\iiint \frac{dx\, dy\, dz}{(1 + x + y + z)^3}$ over the volume of the tetrahedron $x = 0, y = 0, z = 0, x + y + z = 1$. *(Mumbai, 2007)*

13. Show that $\iiint \frac{dx\, dy\, dz}{\sqrt{(a^2 - x^2 - y^2 - z^2)}} = \frac{\pi^2 a^3}{8}$, the integral being extended for all the values of the variables for which the expression is real. *(U.T.U., 2010)*

14. $\iiint z^2\, dxdydz$, taken over the volume bounded by the surfaces $x^2 + y^2 = a^2$, $x^2 + y^2 = z$ and $z = 0$.