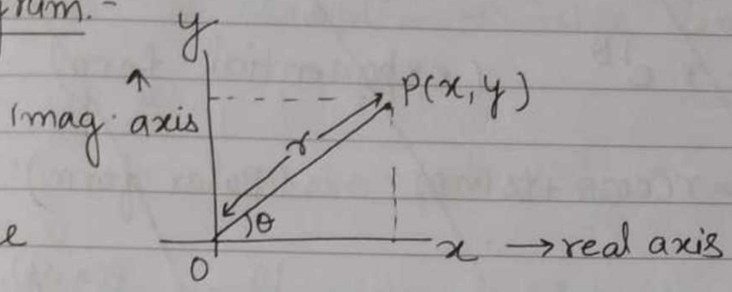
Module - I

complex number:-

$$z = x + iy$$

Argand diagram:-



$r = \sqrt{x^2+y^2}$ is the distance OP.

Angle that OP makes with x axis is argument of z.

Let $z = x + iy$ be complex number

put $x = r\cos\theta$ $y = r\sin\theta$, $r = \sqrt{x^2+y^2}$

& $\theta = \tan^{-1}\left(\frac{y}{x}\right)$, $\cos\theta = \frac{x}{\sqrt{x^2+y^2}}$, $\sin\theta = \frac{y}{\sqrt{x^2+y^2}}$

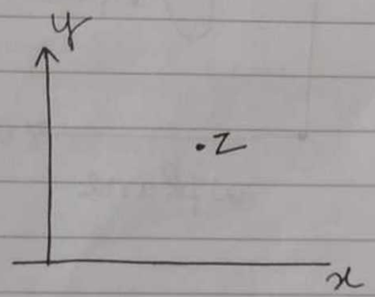
θ is argument or amplitude of z.

* θ has infinite number of values differening by multiples of 2π. The values of θ lying in $-\pi \le \theta \le \pi$ is called principal value of θ.
ie Principal values of θ is written either between 0 to π or between 0 and $-\pi$.

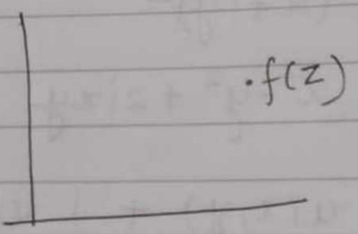
Limit of function of complex variable :-

Let $f(z)$ be a single valued function, define at all point in some neighbourhood of z_0, Then $\lim_{z \to z_0} f(z)$ is exist if limit is independent of path by which z approaches to z_0.

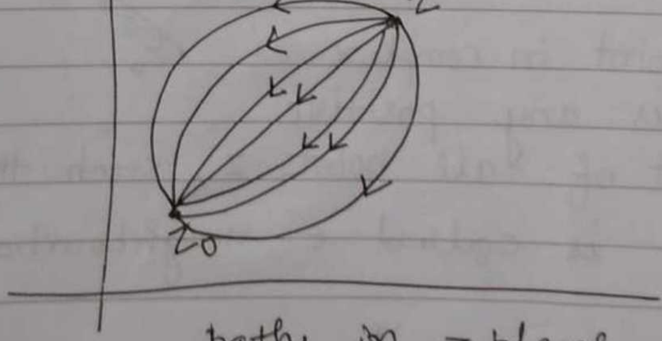
If we get two different limits as $z \to z_0$, along two different paths then limit does not exist.



z plane



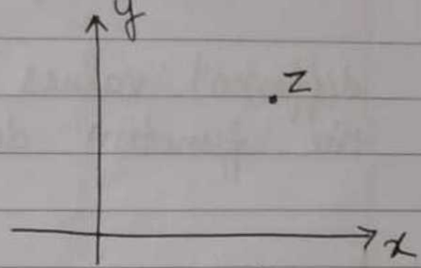
w plane



paths in z plane

(1) Show that $\lim\limits_{z\to 0} \frac{z}{|z|}$ does not exist.

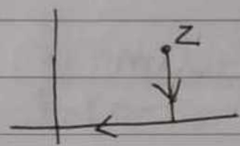
<u>sol^n</u> $\quad f(z) = \frac{z}{|z|}$



<u>z plane</u>

$$\lim_{z\to 0} \frac{z}{|z|} = \lim_{x\to 0, y\to 0} \frac{\cancel{x}+iy}{\sqrt{\cancel{x}^2+y^2}}$$

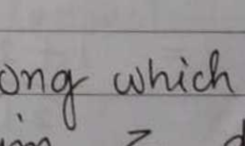
<u>case I</u> :- Let $z\to 0$ along the path $y \to 0$ then $x\to 0$



$$= \lim_{x\to 0} \left(\frac{x}{\sqrt{x^2}}\right) = 1$$

i.e $\quad \lim\limits_{z\to 0} \frac{z}{|z|} = 1$

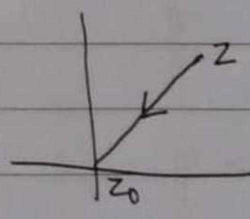
<u>Case II</u> :- Let $z\to 0$ along the path $x\to 0$ then $y\to 0$



$$\lim_{z\to 0} \frac{z}{|z|} = \lim_{y\to 0} \frac{iy}{y}$$

$$\lim_{z\to 0} \frac{z}{|z|} = i$$

since along two different paths along which $z\to 0$ limit is not unique. therefor $\lim\limits_{z\to 0} \frac{z}{|z|}$ does not exist. or we can take general path $y = mx$.



$$\lim_{z\to 0} \frac{z}{|z|} = \lim_{x\to 0} \frac{x+imx}{\sqrt{x^2+m^2x^2}}$$

$$= \lim_{x \to 0} \frac{1+im}{\sqrt{1+m^2}} = \frac{1+im}{\sqrt{1+m^2}}$$

The value of $\frac{1+im}{\sqrt{1+m^2}}$ is different for different values of m. Therefor limit of the function does not exist.

Q:- find the limit of function

$$\lim_{z \to 1+i} \frac{z^2 - z + 1 - i}{z^2 - 2z + 2}$$

$$= \lim_{z \to 1+i} \frac{(z^2+1) - (z+i)}{(z-1)^2 - i^2}$$

$$= \lim_{z \to 1+i} \frac{(z-i)(z+i) - (z+i)}{(z-1+i)(z-1-i)}$$

$$= \lim_{z \to 1+i} \frac{(z+i)\cancel{(z-i-1)}}{(z-1+i)\cancel{(z-1-i)}}$$

$$= \frac{1+2i}{2i} \text{ or } -\frac{(1+2i)\,i}{2} = \frac{2-i}{2}$$

$$= 1 - \frac{i}{2} \quad \underline{\text{Ans}}$$

Q:- show that following limits do not exist :-

(a) $\lim_{z \to 0} \frac{Re(z^2)}{Im\, z}$ (b) $\lim_{z \to 0} \frac{z}{(\bar{z})^2}$

(c) $\lim_{z \to -i} \frac{z^2}{z+i}$

find the following limits :-

(a) $\lim_{z \to 0} \frac{Re(z^2)}{|z|}$ (b) $\lim_{z \to 0} \frac{z^2+6z+3}{z^2+2z+2}$

Ans - (0) Ans - (3/2)

Continuity of $f(z)$:-

$f(z)$ is said to be continous at point $z=z_0$, if $\lim_{z \to z_0} f(z)$ exist and equal to $f(z_0)$.

or interms of real and imaginary parts of f(
if $w = u + iv$ is continous function at z=
then $u(x,y)$ and $v(x,y)$ are separately continous. and vice verca.

Q1 $\lim_{z \to 0}$ ...

Q-1 Examine the continuity of following:-

$$f(z) = \begin{cases} \dfrac{z^3 - iz^2 + z - i}{z - i}, & z \neq i \\ 0, & z = i \end{cases}$$

at $z = i$.

Sol^n:- $\lim_{z \to i} f(z)$

$$= \lim_{z \to i} \frac{z^3 - iz^2 + z - i}{z - i}$$

$$= \lim_{z \to i} z^2 + 1 \quad = -1 + 1 = 0$$

or

Now

$$= \lim_{(x,y) \to (0,1)} (x + iy)^2 + 1$$

$$= \lim_{x \to 0,\, y \to 1} x^2 - y^2 + 1 + (2xy)i$$

Case I :- Let first $x \to 0$, then $y \to 1$

$$= \lim_{y \to 1} -y^2 + 1$$

$$= 0$$

Case IInd :- Let first $y \to 1$ then $x \to 0$

$$= \lim_{x \to 0} x^2 + 2ix = 0$$

along two different path limit is unique ie limit exist and equal to $f(0)$.
therefor function is continous at $z=0$.

Q2 Show that the function $f(z)$ defined by

$$f(z) = \begin{cases} \frac{Re(z)}{z}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$ is not cont. at

$z=0$.

Sol^n

$$\lim_{z \to 0} f(z) = \lim_{\substack{x \to 0 \\ y \to 0}} \frac{x}{x+iy} \quad \text{---} (1)$$

case I:- $y \to 0$ then $x \to 0$.

$$\lim_{z \to 0} f(z) = \lim_{x \to 0} \frac{x}{x} = 1$$

case II:- first $x \to 0$, then $y \to 0$

$$\lim_{z \to 0} f(z) = \lim_{y \to 0} \frac{1}{iy} \rightarrow \text{does not exis}$$

or we can take general case
If $z \to 0$ along $y = mx$

$$\lim_{z \to 0} f(z) = \lim_{z \to 0} \frac{x}{x+imx} = \frac{1}{1+im}$$ which

depends on m, i.e. is not unique therefor $\lim_{z \to 0} f(z)$ does not exist & $f(z)$ is not cont. at $z=0$.

Q1- Examine the continuity of the following functions:-

(1) $f(z) = \begin{cases} \frac{Im(z)}{|z|}, & z \neq 0 \\ 0, & z = 0 \end{cases}$ at $z=0$.

Ans (not continous)

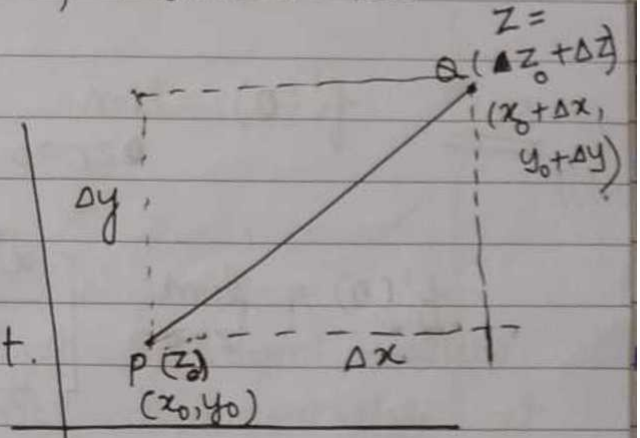
Q:- Show that following functions are continous for all z.

(i) $\cos z$ (ii) e^{2z}

Differentiability of complex function $f(z)$:-

Let $f(z)$ be a single valued function, then derivative of $f(z)$ at a point z_0 is given by

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0},$$ Provided the limit exist and independent of path along which $z \to z_0$.



* Let P be a fixed point and Q be neighbouring point. then

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z},$$

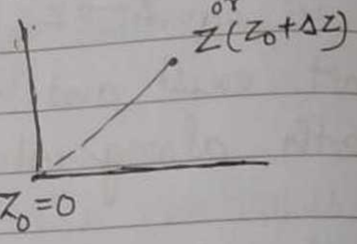
$z = z_0 + \Delta z$ is neighbouring Point of z_0.

Provided the limit exist and independent of path by which $Q(z)$ approaches to $P(z_0)$ [it may be staright line Path or curved path]

Q1 If $f(z) = \begin{cases} \dfrac{x^3 y(y-ix)}{x^6+y^2}, & z \neq 0 \\ 0, & z=0 \end{cases}$

check whether $f(z)$ is differentiable at $z=0$ or not.

Sol^n $f'(0) = \lim\limits_{z \to 0} \dfrac{f(z)-f(0)}{z-0}$



$$f'(0) = \lim_{z\to 0} \left[\frac{\frac{x^3y(y-ix)}{x^6+y^2} - 0}{x+iy} \right]$$

$$f'(0) = \lim_{\substack{x\to 0\\ y \to 0}} \left[\frac{x^3y(y-ix)}{(x+iy)(x^6+y^2)} \right] = \lim_{\substack{x\to 0\\ y\to 0}} \left[\frac{-ix^3y}{x^6+y^2} \right]$$

Case I :- Let $z \to 0$ along the path $x \to 0$ first then $y \to 0$

$$f'(0) = \lim_{y\to 0} 0 = 0$$

Case II :- along $y \to 0$ first then $x \to 0$

$$f'(0) = 0$$

Case III → ~~take~~ Let $z \to 0$ along $y = mx$

$$f'(0) = \lim_{x\to 0} \frac{m x^4 (mx - ix)}{r}$$

$$f'(0) = \lim_{x \to 0} \frac{-i\, m x^4}{x^6 + m^2 x^2}$$

$$f'(0) = \lim_{x \to 0} \frac{-i\, m x^2}{x^4 + m^2} = 0$$

Case IV

But along $y = x^3$

$$f'(0) = \lim_{x \to 0} \frac{-i x^6}{x^6 + x^6} = \frac{-i}{2}$$

i.e. In different paths, we get different values of $f'(z)$, i.e $f(z)$ is not differentiable at $z = 0$.

Q:- check whether $f(z) = |z|^2$ is differentiable at $z = 0$ or not.

Sol

$$f'(z_0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z}$$

$$= \lim_{z \to 0} \frac{|z|^2 - 0}{z}$$

$$f'(0) = \lim_{\substack{x \to 0 \\ y \to 0}} \frac{x^2 + y^2}{x + iy}$$

~~f'(0)~~ Case I:- Let first $x \to 0$ then $y \to 0$

$$f'(0) = \lim_{y \to 0} \frac{y^2}{iy} = 0$$

case II^{nd} conster ~~xy~~ $y \to 0$ then $x \to 0$

$$f'(0) = \lim_{x\to 0} \frac{x^2}{x} = 0$$

case III^{rd} Let $z \to 0$ along $y = mx$

$$f'(z) = \lim_{x\to 0} \frac{x^2 + m^2x^2}{x + imx}$$

$$= \lim_{x\to 0} \frac{x(1+m^2)}{1+i}$$

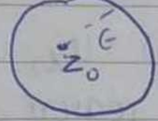
$$f'(0) = 0$$

as along all paths through which $z \to 0$, $f'(z)$ is unique therefor. $f(z)$ is differentiable at $z=0$. and

$$\boxed{f'(0) = 0}$$

Analytic function :-

A function $f(z)$ is said to be analytic at point z_0 if f is differential not only at z_0 but every point of some neighbourhood of z_0.



A function $f(z)$ is analytic in a domain D, if it is analytic at every point of domain D. An analytic function is also known as "holomorphic" or "regular" function.

Singular point :-

The point at which function is not differentiable is called a singular point of the function.

Entire function :-

A function which is analytic everywhere (for all points z in the complex plane) is known as entire function.

Ex :- Polynomial functions are entire.

Note :- Analytic function is always differentiable and continous but converse is not true.

Cauchy Riemann equations:- (C-R equations)

1) Necessary condition for $f(z)$ to be analytic :-

<u>Theorem</u>:- The necessary conditions for a function $f(z) = u + iv$ to be analytic at all points in a region R are

(i) u_x, u_y, v_x, v_y are continous functions in region R.

(ii) $u_x = v_y$, $u_y = -v_x$
called C-R equations.

i.e. $f(z)$ must satisfy C-R equations

or If $f(z)$ is analytic in a domain D $\Rightarrow$ $f(z)$ satisfy C-R equations.

<u>Proof</u>:- Let $f(z)$ be analytic function in R
$w = f(z) = u + iv$
then $f'(z) = \frac{dw}{dz}$ exist at every points of the domam.

i.e. <u>consider</u>

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$f'(z) = \lim_{\Delta z \to 0} \frac{(u+\Delta u) + i(v+\Delta v) - (u+iv)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\Delta u + i\Delta v}{\Delta z}.$$

i.e $f'(z) = \lim_{\Delta z \to 0} \left(\frac{\Delta u}{\Delta z} + i \frac{\Delta v}{\Delta z} \right) = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\Delta u + i\Delta v}{\Delta x + i\Delta y}$

(1) Let $\Delta z \to 0$, as first $\Delta y \to 0$ then $\Delta x \to 0$. i.e (along real axis)

i.e

$$f'(z) = \lim_{\Delta x \to 0} \left(\frac{\Delta u}{\Delta x} + \frac{i\Delta v}{\Delta x} \right)$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \quad \text{—— } (1)$$

(2) Let $\Delta z \to 0$ along y axis (i.e first $\Delta x \to 0$ then $\Delta y \to 0$)

then

$$f'(z) = \lim_{\Delta y \to 0} \frac{\Delta u}{i\Delta y} + \frac{\cancel{i}\Delta v}{\cancel{i}\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{\Delta v}{\Delta y} - i \lim_{\Delta y \to 0} \frac{\Delta u}{\Delta y}$$

$$f'(z) = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} \quad \text{—— } (2)$$

but $f(z)$ is analytic i.e differentiable at all point R i.e $f'(z)$ exist and unique

from (1) and (2)

$$f'(z) = u_x + i v_x = v_y - i u_y$$

comparing $\boxed{\begin{array}{l} u_x = v_y \\ u_y = -v_x \end{array}}$ C-R equations

Sufficient conditions:-

Theorem:- The sufficient condⁿ for $f(z)$ to be analytic in Region $\mathbb{R}$ are

(I) $u_x = v_y$, $u_y = -v_x$

(II) u_x, u_y, v_x, v_y are continous function in region $\mathbb{R}$.

i.e If a function $f(z)$ satisfy C-R eqⁿ in $\mathbb{R}$ and u_x, u_y, v_x, v_y are continous then $f(z)$ is analytic in $\mathbb{R}$.

Note:- 1) If a function $f(z) = u+iv$ is analytic in a domain D, then u and v satisfy C-R equations at all point in D.

2) C-R eqⁿ are necessary but not sufficient for analytic function.

3) C-R eqⁿ are sufficient if u_x, v_x, u_y, v_y are continous.

Q(1) Determine whether $\frac{1}{z}$ is analytic or not.

Solⁿ:- $f(z) = \frac{1}{z} = \frac{1}{x+iy}$

$$f(z) = \frac{x-iy}{(x+iy)(x-iy)}$$

$$f(z) = \frac{x}{x^2+y^2} - \frac{iy}{x^2+y^2} = u+iv$$

$$u = \frac{x}{x^2+y^2}, \quad v = \frac{-y}{x^2+y^2}$$

* function will be analytic if it satisfy C-R eqⁿ and has continous partial derivatives u_x, u_y, v_x, v_y.

$$u_x = \frac{y^2-x^2}{(x^2+y^2)^2} \qquad u_y = \frac{-2xy}{(x^2+y^2)^2}$$

$$v_x = \frac{2xy}{(x^2+y^2)^2} \qquad v_y = \frac{y^2-x^2}{(x^2+y^2)^2}$$

thus C-R eqⁿs are satisfied. we see that partial derivatives are continous everywhere except at (0,0).
therefor f(z) is analytic everywhere except (0,0).

Also from $f'(z) = -\frac{1}{z^2}$ we can see that $f'(z)$ exists everywhere except at $z=0$. Hence f(z) is analytic everywhere except at 0.

Q:- Discuss the analyticity of $z\bar{z}$.

Sol^n $f(z) = z\bar{z} = (x+iy)(x-iy)$

$$f(z) = x^2+y^2 = u+iv$$

$$u = x^2+y^2, \quad v=0$$

$$\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial u}{\partial y} = 2y, \quad \frac{\partial v}{\partial x} = 0, \quad \frac{\partial v}{\partial y} = 0$$

C-R eq^{n}s are

$$u_x = v_y \qquad 2x = 0$$
$$u_y = -v_x \qquad 2y = 0$$

C-R eq^{n}s are satisfied at $(0,0)$ only.

i.e function is not analytic anywhere.

check derivative at $(0,0)$

at $(0,0)$

$$f'(0) = \lim_{z\to 0} \frac{(f(z) - f(0)}{z}$$

$$f'(0) = \lim_{z\to 0} \frac{x^2+y^2}{x+iy} = \lim_{\substack{x\to 0\\ y\to 0}} \frac{x^2+y^2}{x+iy}$$

Now Let $x, y \to 0$ along $y = mx$

i.e.

$$f'(0) = \lim_{x\to 0} \frac{x^2+m^2x^2}{x+imx} = \lim_{x\to 0} \frac{x(1+m^2)}{1+m} = 0$$

which is independent of m.

i.e ~~lim~~ i.e. limit exist and independent of path ~~z→~~ therefor ~~f'(0)~~ function is differentiable at $z=0$ & $f'(0)=0$.

Q:- Show that the function $f(z)=u+iv$ where

$$f(z)=\begin{cases} \dfrac{x^3(1+i)-y^3(1-i)}{x^2+y^2}, & z\neq 0 \\ 0, & z=0 \end{cases}$$

satisfies C R eq^n at $z=0$. Is the function differentiable at $z=0$.

sol^n:- need to check $\left.\begin{aligned} u_x &= v_y \\ u_y &= -v_x \end{aligned}\right\}$ at $z=0$

$u=\dfrac{x^3-y^3}{x^2+y^2}$, $v=\dfrac{x^3+y^3}{x^2+y^2}$

$$\frac{\partial u}{\partial x}=\lim_{x\to 0}\frac{u(x,0)-u(0,0)}{x}=\lim_{x\to 0}\frac{\frac{x^3}{x^2}-0}{x}$$

$$\frac{\partial u}{\partial x}=1$$

$$\frac{\partial u}{\partial y}=\lim_{y\to 0}\frac{u(0,y)-u(0,0)}{y}=\lim_{y\to 0}\frac{\frac{-y^3}{y^2}-0}{y}=-1$$

$$\frac{\partial v}{\partial x}=\lim_{x\to 0}\frac{v(x,0)-v(0,0)}{x}=1$$

$$\frac{\partial v}{\partial y}=\lim_{y\to 0}\frac{v(0,y)-v(0,0)}{y}=1$$

we see that $\left.\begin{aligned} u_x &= v_y \\ u_y &= -v_x \end{aligned}\right\}$ i.e. C-R eq^n are satisfied at $z=0$.

Now

$$f'(0) = \lim_{z\to 0} \frac{f(z) - f(0)}{z - 0}$$

$$= \lim_{z\to 0} \frac{x^3-y^3+i(x^3+y^3)}{(x^2+y^2)(x+iy)}$$

$$f'(0) = \lim_{\substack{x\to 0,\\ y\to 0}} \frac{(x^3-y^3)+i(x^3+y^3)}{(x^2+y^2)(x+iy)}$$

case I st

Let ~~z~~ $z\to 0$ along $y = mx$

$$f'(0) = \lim_{x\to 0} \frac{x^3 - m^3x^3 + i(x^3+m^3x^3)}{(x^2+m^2x^2)(x+imx)}$$

$$f'(0) = \cancel{\tfrac{1}{2}(1+i)} = \frac{x^3(1+i)+m^3x^3(i-1)}{x^3(1+m^2)(1+im)}$$ depends on m

$$= \frac{(1+i)+m^3(i-1)}{(1+m^2)(1+im)} = \frac{i(1+i)m^3}{(1+m^2)(1+im)}$$

case II nd :- Let $z\to 0$ along x axis ie first $y\to 0$ then $x\to 0$

$$f'(0) = \lim_{x\to 0} \frac{x^3+ix^3}{x^3} = (1+i)$$ - optional

by taking two different paths we saw that limit is not unique ie function is not differentiable at $z=0$.

Tutorial - I

Q1- Show that the function $e^x(\cos y + i \sin y)$ is an analytic function, find its derivative

Hint: show $u_x = v_y$ & $u_y = -v_x$ and u_x, v_y, u_y, v_x are continous.
& find $f'(z) = u_x + i v_x$ or $v_y - i u_y$

Q2- Show that $\bar{z}$ is not analytic anywhere.

Q3 Show that $f(z) = xy + iy$ is everywhere continuous but is nowhere analytic.

Q4 find a, b, c if $f(z) = x + ay + i(bx + cy)$ is analytic. Hence find $f'(z)$.

Q5 check the analyticity of function $f(z) = |z|^2$

Q6 Determine which of following functions are analytic

(1) $\dfrac{x - iy}{x^2 + y^2}$ (2) $2xy + i(x^2 - y^2)$ (3) $x^2 + iy^2$

Ans:- (NO) (NO) (diff at all pts except $y = x$)

4) $xy + iy^2$

Q7 Show that $f(z) = \sqrt{|xy|}$ is not differentiable at origin ~~even~~ though C-R eqn are satisfied at origin.

C-R eqⁿs in polar form:- $f(z) = u(r,\theta) + i v(r,\theta)$
$w = u + iv$

$$\left.\begin{aligned} u_r &= \frac{1}{r} v_\theta \\ u_\theta &= -r v_r \end{aligned}\right\}$$ is polar form of C-R equations.

derivative in polar coordinates:-

$$\frac{dw}{dz} = f'(z) = (\cos\theta - i\sin\theta)\frac{\partial w}{\partial r} \quad \text{or} \quad f'(z) = \frac{-i}{r}(\cos\theta - i\sin\theta)\frac{\partial w}{\partial \theta}$$

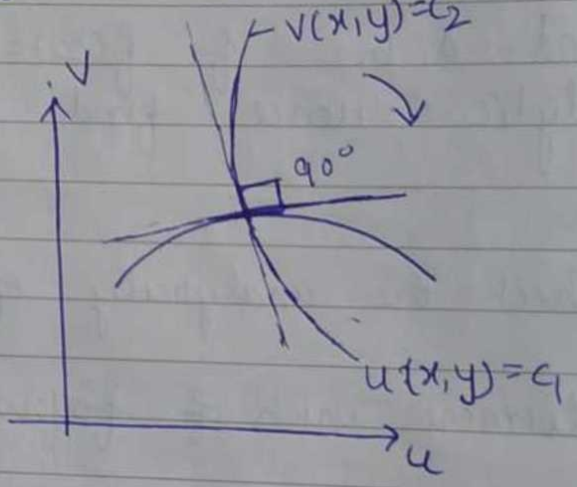
ORTHOGONAL SYSTEM:-

Every analytic function $f(z) = u + iv$ defines two families of curves $u(x,y) = c_1$ and $v(x,y) = c_2$, which form an orthogonal system.

consider

$u(x,y) = c_1$ —— (i)

$v(x,y) = c_2$ —— (ii)



differentiating (1) with r. to x:-

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{\left(\frac{\partial u}{\partial x}\right)}{\left(\frac{\partial u}{\partial y}\right)} = m_1 \text{ (say)}$$

diff. (2) w. r. to x

we get

$$\frac{dy}{dx} = -\frac{\left(\frac{\partial v}{\partial x}\right)}{\left(\frac{\partial v}{\partial y}\right)} = m_2 \text{ (say)}$$

$\frac{dy}{dx}$ $\quad m_1 \cdot m_2 = \dfrac{\left(\frac{\partial u}{\partial x}\right)}{\left(\frac{\partial u}{\partial y}\right)} \cdot \dfrac{\left(\frac{\partial v}{\partial x}\right)}{\left(\frac{\partial v}{\partial y}\right)} = \dfrac{u_x \, v_x}{u_y \, v_y}$

$f(z)$ is analytic therefor $u_x = v_y$, $u_y = -v_x$

$$m_1 \cdot m_2 = \frac{v_y \cdot v_x}{-v_x \, v_y}$$

$$\boxed{m_1 \cdot m_2 = -1}$$

ie curves intersect at right angle i.e. they form orthogonal system.

Examples of analytical function

(1) $f(z) = z^n$ is analytic in $\mathbb{C}$.

(2) $f(z) = e^z$ is " " ".

(3) $f(z) = 1/z$ is analytic everywhere except $z=0$.

(4) Polynomial rational fun. $f(z) = \dfrac{P(z)}{Q(z)}$ where P and Q are ~~analytic~~ polynomial, ~~are~~ is analytic everywhere in $\mathbb{C}$ except the points where $Q(z)=0$.

(5) If $f(z)$ and $g(z)$ are analytic then $f(z) \pm g(z)$, $f(z) \cdot g(z)$, $\dfrac{f(z)}{g(z)}$ (except where $g(z)=0$), all are analytic functions of z.

Tutorial - I (cont.)

Q. 8 find P such that $f(z) = r^2\cos(2\theta) + i r^2 \sin(p\theta)$ is analytic.

Ans ([redacted])

Q. 9 Determine P such that $f(z) = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\left(\frac{px}{y}\right)$ be analytic function.

Ans [redacted]

Q.-10 Show that real and imaginary part of $w = \log(z)$ satisfy C-R equations, when z is not zero. find derivative.

Hint:-

$$f(z) = \log(z)$$

Let $z = r\cos\theta + ir\sin\theta$

$$f(z) = \log(r(\cos\theta + i\sin\theta))$$

$$= \log r + \log e^{i\theta} = \log r + i\theta$$

$$u + iv = \log r + i\theta$$

$$u = \log r = \log\sqrt{x^2+y^2}, \quad v = \theta = \tan^{-1}(y/x)$$

Q. 11 It is given that $f(z)$ and its conjugate $\overline{f(z)}$ are both analytic. Determine the function $f(z)$.

solⁿ $f(z) = u + iv$, $\overline{f(z)} = u - iv$

$f(z)$ is analytic i.e

$$\left.\begin{array}{l} u_x = v_y \\ v_y = -v_x \end{array}\right\} \text{—} (1)$$

$\overline{f(z)}$ is also analytic

$$\left.\begin{array}{l} u_x = -v_y \\ u_y = v_x \end{array}\right\} \text{—} (2)$$

from (1) & (2) $u_x = u_y = v_x = v_y = 0$

ie $\left.\begin{array}{l} u_x = 0 \Rightarrow u = \text{constant} \\ v_x = 0 \Rightarrow v = \text{constant} \end{array}\right\} \Rightarrow f(z) = u + iv$ is constant function.

Harmonic function :-

Any function ~~f(z)~~, which satisfy the Laplace's equation is known as harmonic function.

Note
* Laplace eq^n is second order PDE, given as

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Theorem: If $f(z) = u + iv$ is an analytic function, then u and v both are Harmonic functions.

Proof: $f(z) = u + iv$ is analytic i.e.

$$u_x = v_y \quad \text{--- (1)}$$
$$u_y = -v_x \quad \text{--- (2)}$$

from (1) $\quad u_{xx} = v_{yx}$
" (2) $\quad u_{yy} = -v_{xy}$

we know $v_{yx} = v_{xy}$

i.e. $u_{xx} + u_{yy} = 0$ ie u satisfy Laplace eq^n

Similarly we can show

$v_{xx} + v_{yy} = 0$ ie v also " " ".

ie. u and v are harmonic function.

ie if $f(z)$ is analytic then u & v both are Harmonic functions

but converse may or may not be true.

* such functions u & v are called harmonic conjugate or conjugate harmonic functions if $u+iv$ is also analytic functions.

Q:- Show that the function $u = e^x \sin y$ is harmonic.

Soln $u = e^x \sin y$

To check $u_{xx} + u_{yy} = 0$ —— (1)

$u_x = e^x \sin y$, $u_{xx} = e^x \sin y$
$u_y = e^x \cos y$ $u_{yy} = -e^x \sin y$

adding

$$u_{xx} + u_{yy} = e^x \sin y - e^x \sin y$$
$$= 0$$

ie u satisfy Laplace eqn therefor u is harmonic function.

Q:- Prove that $u = x^2 - y^2$ and $v = \frac{y}{x^2+y^2}$ are harmonic functions of (x, y) but are not harmonic congugates.

Sol^n $u_x = 2x$, $u_{xx} = 2$

$u_y = -2y$ $u_{yy} = -2$

$u_{xx} + u_{yy} = 0$ ie u satisfy Laplace eq^n
ie u is harmonic function.

Slly. $v = \frac{y}{x^2+y^2}$

$V_x = \frac{-y(2x)}{(x^2+y)^2}$ $\quad V_{xx} = \frac{(x^2+y^2)^2(-2y) - (-2xy)2(x^2+y^2)2x}{(x^2+y^2)^4}$

i.e $V_{xx} = \frac{(x^2+y^2)(-2y) + 8x^2y}{(x^2+y^2)^3}$

$V_{xx} = \frac{-2yx^2 - 2y^3 + 8yx^2}{(x^2+y^2)^3}$

i.e $V_{xx} = \frac{6x^2y - 2y^3}{(x^2+y^2)^3}$

Now $V_y = \frac{x^2-y^2}{(x^2+y^2)^2}$

$V_{yy} = \frac{-6x^2y + 2y^3}{(x^2+y^2)^3}$

adding

$$V_{xx} + V_{yy} = 0$$

i.e V is also harmonic function.

Proved

Now u, v will be harmonic conjugate if $u+iv$ is analytic function.
i.e if u & v satisfy C-R eq^n.

we have

$u_x = 2x \quad u_y = -2y$

$$V_x = \frac{-2xy}{(x^2+y^2)^2} \qquad V_y = \frac{x^2-y^2}{(x^2+y^2)^2}$$

$u_x \neq V_y$

$u_y \neq -V_x$

i.e u & v do not satisfy C-R eq^n
i.e $f(z) = u+iv$ is not analytic
i.e u & v are not harmonic condugate.

~~$v = 2yx + x^2 - 3x +$~~

Method to find conjugate function:-

If $f(z) = u + iv$ is analytic function, then u and v are called conjugate or conjugat harmonic of each other.

Method:- Let u is given and v is unknown then

write $dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$ —— (1)

replace $v_x = -u_y$, $v_y = u_x$ } CR eq^ns

$$dv = \left(-\frac{\partial u}{\partial y}\right) dx + \left(\frac{\partial u}{\partial x}\right) dy \quad \text{—— (2)}$$

$$dv = M\,dx + N\,dy \quad \text{—— (3)}$$

say $M = -\frac{\partial u}{\partial y}$, $N = \frac{\partial u}{\partial x}$

above differential eq^n is exact differential eq^n

as $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ (check)

$M = -\frac{\partial u}{\partial y}$, $\frac{\partial M}{\partial y} = -\frac{\partial^2 u}{\partial y^2}$

$N = \frac{\partial u}{\partial x}$, $\frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x^2}$

$-\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial x^2}$ [since u satisfy Laplace eq^n]

Now solution of exact diff. eq^n (3) will be

$$v = \int dv = \int M\,dx + \int N\,dx$$

treat y as constant

integrale only term not containing x.

Date
Page

Method:- similarly if v is given, ~~then~~ to find u, we start with

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$$

$$u = \int \frac{\partial u}{\partial x} dx + \int \frac{\partial u}{\partial y} dy$$

$$u = \int \frac{\partial v}{\partial y} dx + \int \left(-\frac{\partial v}{\partial x}\right) dy$$

$$u = \int M dx + \int N dy \quad \text{—— (1)}$$

diff eq^n (1) is exact differential eq^n.

$$\boxed{\text{check } \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.}$$ $cond^n$ for exact

Sol^n will be

$$u = \int_{y\,\text{const}} M dx + \int_{\substack{\text{not terms} \\ \text{not containing} \\ x}} N dy.$$

once u is determine

$$\boxed{f(z) = u + iv}$$

Q1 → Prove that $u = x^2 - y^2 - 2xy - 2x + 3y$ is harmonic. find function v such that $f(z) = u + iv$ is analytic. ~~Also~~ express $f(z)$ in terms of z.

Sol^n $\quad u = x^2 - y^2 - 2xy - 2x + 3y$.

$u_x = 2x - 2y - 2$, $\quad u_{xx} = 2$

$u_y = -2y - 2x + 3$ $\quad u_{yy} = -2$

$u_{xx} + u_{yy} = +2 + (-2) = 0$.

u satisfy Laplace eq^n $\Rightarrow$ u is harmonic. Proved

To find v :-

$$dv = \left(\frac{\partial v}{\partial x}\right) dx + \left(\frac{\partial v}{\partial y}\right) dy$$

using C-R eq^{n}s $\quad u_x = v_y$, $u_y = -v_x$

$$dv = \left(-\frac{\partial u}{\partial y}\right) dx + \left(\frac{\partial u}{\partial x}\right) dy$$

$$dv = (2y + 2x - 3)\, dx + (2x - 2y - 2)\, dy \quad (1)$$

$$\cancel{\int dv = \int (2y + 2x - 3)\, dx + \int (2x - 2y - 2)\, dy}$$

$dv = M\, dx + N\, dy$.

$\frac{\partial M}{\partial y} = 2$, $\frac{\partial N}{\partial x} = 2$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, i.e (1) is exact diff. eq^n.

Sol^n is $\int dv = \underbrace{\int (2y + 2x - 3)\, dx}_{y\ \text{const}} + \underbrace{\int (-2y - 2)\, dy}_{\text{only term not containing } x}$

$$v = 2yx + x^2 - 3x - y^2 - 2y + c.$$

Now, $f(z) = u + iv$

$$f(z) = (x^2 - y^2 - 2xy - 2x + 3y) + i(2xy + x^2 - 3x - y^2 - 2y + c)$$

$$f(z) = (x^2 - y^2 + 2ixy) + i(x^2 - y^2 + 2ixy) + (-2x - 2yi) + (-3xi + 3y) + ic$$

$$f(z) = (x+iy)^2 + i(x+iy)^2 - 2(x+iy) - 3i(x+iy) + ic$$

$$f(z) = z^2 + iz^2 - 2z - 3iz + ic$$

$$f(z) = (1+i)z^2 - (2+3i)z + ic$$

Q.12 find the imaginary part of analytic function whose real part is $x^3 - 3xy^2 + 3x^2 - 3y^2$.

Ans $v = 3x^2y + 6xy - y^3 + c$

Q13 find a function $w = u + iv$ such that w is analytic if $u = e^x \sin y$

Q14 show that $u = \sin x \cosh y + 2\cos x \sinh y + x^2 - y^2 + 4xy$ satisfies Laplace eqn and find analytic function $u + iv$.

classmate
Date
Page

In polar coordinates

Q: Let $f(z) = u + iv$ be an analytic function & $u = -r^3 \sin 3\theta$ then construct corresponding analytic function $f(z)$ in terms of z.

Solⁿ

$$u = -r^3 \sin 3\theta$$

$$\frac{\partial u}{\partial r} = -3r^2 \sin 3\theta, \quad u_\theta = -3r^3 \cos 3\theta$$

$$dv = \frac{\partial v}{\partial r} dr + \frac{\partial v}{\partial \theta} d\theta$$

C-R eqⁿs

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$$

$$\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$$

$$dv = \left(-\frac{1}{r}\frac{\partial u}{\partial \theta}\right) dr + \left(r\frac{\partial u}{\partial r}\right) d\theta$$

$$dv = -\frac{1}{r}(-3r^3 \cos 3\theta)\, dr + r(-3r^2 \sin 3\theta)\, d\theta$$

$$dv = \underbrace{3r^2 \cos 3\theta}_{M}\, dr + (\underbrace{-3r^3 \sin 3\theta}_{N})\, d\theta$$

above diff-eqⁿ is exact if

$$\cancel{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}} \quad \frac{\partial M}{\partial \theta} = \frac{\partial N}{\partial r}$$

$$\frac{\partial M}{\partial \theta} = -9r^2 \sin 3\theta, \quad \frac{\partial N}{\partial r} = -9r^2 \sin 3\theta$$

ie $\dfrac{\partial M}{\partial \theta} = \dfrac{\partial N}{\partial r}$

Sol^n is

$$\int dv = \underset{\theta\ \text{const}}{\int} (3r^2\cos 3\theta)\,dr + \underset{\substack{\text{ignoring}\\ r\ \text{terms}}}{\int} 0 + c$$

$$v = r^3\cos 3\theta + c$$

i.e

$$f(z) = u + iv$$

$$f(z) = -r^3\sin 3\theta + i\,r^3\cos 3\theta + ic$$

$$= i\,(r^3\cos 3\theta + i\,r^3\sin 3\theta) + ic$$

$$= i\,(z^3) + ic$$

$z = re^{i\theta}$
$= r(\cos\theta + i\sin\theta)$

$$\boxed{f(z) = iz^3 + ic}$$

Q.15 find u such that $f(z) = u + iv$ is analytic given $v(r,\theta) = r^2\cos 2\theta - r\cos\theta + 2$

Ans:- $u(r,\theta) = -r^2\sin 2\theta + r\sin\theta + c$

$$f(z) = i(r^2e^{2i\theta} - re^{i\theta}) + 2ic$$

Finding an analytic function $f(z) = u + iv$ when $u - v$ is given

Q!- If $u - v = (x-y)(x^2+4xy+y^2)$ and $f(z) = u + iv$ is an analytic fun of $z = x+iy$ find $f(z)$ in terms of z.

Sol^n:
$$f(z) = u + iV$$

$$u + iv = f(z) \quad \text{--- (I)}$$
$$iu - v = i f(z) \quad \text{--- (II)} \quad \text{— multiply (I) by } i$$

Adding

$$(u-v) + i(u+v) = (1+i)\, f(z)$$

$$U + iV = F(z) \quad \text{--- (III)}$$

(Say $u - v = U$
$u + v = V$
$if(z) = F(z)$

Now
$$U = u - v = (x-y)(x^2+4xy+y^2) \text{ is given}$$
$$= x^3 - y^3 + 3x^2y - 3xy^2$$

V can be obtained by

$$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy$$

$$dV = -\frac{\partial U}{\partial y}dx + \frac{\partial U}{\partial x}dy \quad \text{(C-R eq}^n\text{)}$$

Date
Page

~~div~~ $\frac{\partial U}{\partial x} = 3x^2+6xy-3y^2$

$\frac{\partial U}{\partial y} = 3x^2-6xy-3y^2$

$dV = (-3x^2+6xy+3y^2)\,dx + (3x^2+6xy-3y^2)\,dy$ — Ⓐ

on integrating

$\int dV = \int_{y\,const} (-3x^2+6xy+3y^2)\,dx + \int_{ignoring\ x} -3y^2\,dy + C$

diff. eqn Ⓐ is exact

$\frac{\partial M}{\partial y} = \cancel{6x} + 6x + 6y$
$= 6y+6y$

$\frac{\partial N}{\partial x} = \cancel{6x} + 6\cancel{y} + 6y$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$V = \frac{-3x^3}{3} + 6\frac{x^2}{2}y + 3y^2x - \frac{3y^3}{3} + C$

$V = -x^3 - y^3 + 3x^2y + 3y^2x + C$

$F(z) = U + iV$
$= (x^3 - y^3 + 3x^2y - 3xy^2) + i(-x^3 - y^3 + 3x^2y + 3xy^2) + ic$

$= (x^3 - iy^3 + 3ix^2y - 3xy^2)$
$+ (-ix^3 - y^3 + 3x^2y + 3ixy^2) + ic$

$(x+iy)^3$
$= x^3 - iy^3 + 3x^2iy$
$-3xy^2$

$= (x+iy)^3 - i(x^3 - iy^3 + 3ix^2y - 3xy^2) + ic$

$= z^3 - iz^3 + ic$

$F(z) = (1-i)z^3 + ic$

NOW $F(z) = (1+i)f(z)$

$$(1+i)\, f(z) = (1-i)\, z^3 + ic$$

$$f(z) = \left(\frac{1-i}{1+i}\right) z^3 + ic$$

$$f(z) = \frac{(1-i)^2}{(1+i)(1-i)}\, z^3 + \frac{i(1-i)}{(1+i)(1-i)}\, c$$

$$\boxed{f(z) = -i z^3 + \frac{(1+i)}{2} c}$$ Ans

Milne Thomson Method for constructing analytic function $f(z) = u + iv$

by this method $f(z)$ is directly constructed without finding v (when u is given) or viceversa.

Case Ist:- when u is given:

we have $f(z) = u + iv$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \quad \text{(using C-R eq}^n\text{s)}$$

Say $\frac{\partial u}{\partial x} = \phi_1(x,y)$, $\frac{\partial u}{\partial y} = \phi_2(x,y)$

i.e $f'(z) = \phi_1(x,y) - i\,\phi_2(x,y)$

replace x by z, and y by 0 in ϕ_1 & ϕ_2

$$f'(z) = \phi_1(z,0) - i\,\phi_2(z,0)$$

On integrating

$$\int f'(z)\,dz = \int \phi_1(z,0)\,dz - i\int \phi_2(z,0)\,dz$$

$$\boxed{f(z) = \int \phi_1(z,0)\,dz - i\int \phi_2(z,0)\,dz + C}$$

case IInd. when v is given:-

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$f'(z) = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x} \quad \text{(C-R eq}^n\text{)}$$

say $v_x = \psi_1(x,y)$, $v_y = \psi_2(x,y)$

$$f'(z) = \psi_1(x,y) + i\,\psi_2(x,y)$$

replace x by z, y by 0

$$f'(z) = \psi_1(z,0) + i\,\psi_2(z,0)$$

$$\int f'(z)\,dz = \int \psi_1(z,0)\,dz + i\int \psi_2(z,0)\,dz$$

$$\boxed{f(z) = \int \psi_1(z,0)\,dz + i\int \psi_2(z,0)\,dz + C}$$

Case III rd :- <u>when $u-v$ is given.</u>

we have $f(z) = u + iv$

$$i f(z) = iu - v$$

adding $(1+i) f(z) = (u-v) + i(u+v)$

$$F(z) = U + iV$$

Now $F'(z) = \dfrac{\partial U}{\partial x} + i\dfrac{\partial V}{\partial x}$

$U = u - v$

$$F'(z) = \phi_1(z,0) + i\,\phi_2(z,0)$$

integrate & solve then $\boxed{f(z) = \dfrac{F(z)}{1+i}}$

Case IV th :- <u>when $u+v$ is given</u>

we have $f(z) = u + iv$,

$$i f(z) = iu - v$$

$(1+i) f(z) = (u-v) + i(u+v)$

$F(z) = U + iV$

$V = u+v$ is given i.e

$F'(z) = U_x + iV_x$

$F'(z) = V_y + iV_x$ (C-R eqⁿ)

$F'(z) = \varphi_1(x,y) + i\,\varphi_2(x,y)$

$F'(z) = \varphi_1(z,0) + i\,\varphi_2(z,0)$

Now integrale.

once $F(z)$ is Known

$$\boxed{f(z) = \frac{F(z)}{1+i}}$$

Date
Page

Q-1. ~~If~~ Show that $e^x(x\cos y - y\sin y)$ is harmonic function. Find analytic function for which $e^x(x\cos y - y\sin y)$ is imaginary part.

Soln:- given $V = e^x(x\cos y - y\sin y)$

$$\frac{\partial v}{\partial x} = e^x(x\cos y - y\sin y) + e^x\cos y$$

$$\frac{\partial^2 v}{\partial x^2} = e^x(x\cos y - y\sin y) + e^x\cos y + e^x\cos y$$
$$= e^x(x\cos y - y\sin y) + 2e^x\cos y \quad \text{--- (1)}$$

$$\frac{\partial v}{\partial y} = e^x(-x\sin y - y\cos y - \sin y)$$

$$\frac{\partial^2 v}{\partial y^2} = e^x(-x\cos y + y\sin y - 2\cos y) \quad \text{--- (2)}$$

adding (1) & (2)

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

$\Rightarrow$ v is harmonic function.

Now

Let $f(z) = u + iv$

$f'(z) = u_x + iv_x$

$f'(z) = \cancel{v}_y + iV_x$ —— (3)

$\varphi_1(x,y) = V_y = e^x(-x\sin y - y\cos y - \sin y)$

$\varphi_1(z,0) = 0$

$\varphi_2(x,y) = V_x = e^x(x\cos y - y\sin y) + e^x\cos y$

$\varphi_2(z,0) = e^z(z) + e^z$

from (iii)

$f'(z) = \varphi_1(z,0) + i\,\varphi_2(z,0)$

$f'(z) = 0 + i(ze^z + e^z)$

$\int f'(z)\,dz = \int i(ze^z + e^z)\,dz + C$

$f(z) = i(ze^z \cancel{-e^z} + \cancel{e^z}) + C$

$$\boxed{f(z) = ize^z + C}$$

required analytic function.

Q:- If $u = \dfrac{\sin 2x}{\cosh 2y + \cos 2x}$, find $f(z)$.

Hint:- $\phi_1(z,0) = \dfrac{2\cos 2z + 2}{(1+\cos 2z)^2}$

$\phi_2(z,0) = 0$

$f(z) = \int \phi_1(z,0)\,dz - i\int \phi_2(z,0)\,dz + C$

$f(z) = 2\int \dfrac{1}{1+\cos 2z}\,dz + C$

$f(z) = \int \sec^2 z\,dz + C$, $f(z) = \tan z + C$

Application to fluid flow problems:-

Consider two dimensional irrotational motion of fluid in plane (x-y)

Velocity v of fluid can be expressed as

$$v = v_{(x)}\hat{i} + v_{(y)}\hat{j} \quad \text{—— (1)}$$

velocity in x direction

velocity in y direction

motion is irrotational therefor $\exists$ a function $\phi(x,y)$ such that

$$V = \text{grad } \phi$$

$$V = \nabla \phi(x,y)$$

$$V = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y}\right)\phi$$

$$v = \hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} \quad \text{—— (2)}$$

i.e comparing (1) and (2)

$$v_{(x)} = \frac{\partial \phi}{\partial x}, \quad v_{(y)} = \frac{\partial \phi}{\partial y}$$

x component of velocity

this Scalar function $\phi(x,y)$ is called the velocity potential.

Note :-

* The velocity potential $\phi(x,y)$ which gives velocity components is always satisfy Laplace eq^n i.e $\phi(x,y)$ is harmonic function

i.e $\boxed{\dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2} = 0}$

as fluid is incompressible
i.e. $\operatorname{div} v = 0$

$$\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y}\right)\left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y}\right) = 0$$

$$\Rightarrow \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$$

complex potential :-

* ϕ is a real part of complex function $f(z)$ given as

$$f(z) = \phi + i\psi$$

where ϕ is velocity potential
ψ is Stream function

$f(z)$ is complex potential, which represent the flow pattern.

$\phi(x,y)=c_1$ & $\psi(x,y)=c_2$ intersects orthogonally :

* Family of curves $\phi(x,y)=c_1$ are called equipotential lines. and family of curves $\psi(x,y)=c_2$ are called stream lines. and these two curves intersects orthogonally

&
$$m_1 = \frac{\left(-\frac{\partial\phi}{\partial x}\right)}{\left(\frac{\partial\phi}{\partial y}\right)} \qquad m_2 = \frac{-\left(\frac{\partial\psi}{\partial x}\right)}{\left(\frac{\partial\psi}{\partial y}\right)}$$

$\left[\begin{array}{c} \text{C-R } eq^n \\ \phi_x = \psi_y \\ \phi_y = -\psi_x \end{array}\right]$

$\boxed{m_1 m_2 = -1}$ i.e $\phi=c_1$ & $\psi=c_2$ intersects orthogonally.

Velocity of fluid :-

$$f(z) = \phi + i\psi$$

$$f'(z) = \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x}$$

CR eqn $\left[\frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}\right]$

$$f'(z) = \frac{\partial \phi}{\partial x} - i\frac{\partial \phi}{\partial y}$$

$$f'(z) = v_x - iv_y$$

$$\boxed{\overline{f'(z)} = v_x + iv_y}$$

is ~~calle~~ the expression for velocity of fluid.

Speed or magnitude of resultant velocity

$$|\overline{f'(z)}| = \sqrt{v_x^2 + v_y^2}$$

(1) In two dimentional fluid flow if velocity potential is $\phi = x^4 - 6x^2y^2 + y^4$ then

(i) find the stream function ψ & corresponding complex potential.

(2) write expression for velocity & hence find speed.

(3) verify that family of curves $\phi(x,y) = c_1$ & $\psi(x,y) = c_2$ intersect orthogonally.

Solⁿ:- $\phi = x^4 - 6x^2y^2 + y^4$

$\phi_x = 4x^3 - 12xy^2 \qquad \phi_y = 4y^3 - 12x^2y$.

$$d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy$$

$$d\psi = -\frac{\partial \phi}{\partial y} dx + \frac{\partial \phi}{\partial x} dy$$

$$d\psi = (-4y^3 + 12x^2y)\,dx + (4x^3 - 12xy^2)\,dy \quad \text{--- (1)}$$

∴ diff eqⁿ (1) is exact

$$\int d\psi = \int_{y\ \text{const}} (-4y^3 + 12y\,x^2)\,dx + \int_{\text{ignoring } x} (4x^3 - 12xy^2)\,dy$$

$$\psi = -4y^3x + 4x^3y + 0 + c$$

$$\boxed{\psi = 4x^3y - 4y^3x + c} \rightarrow \text{stream function.}$$

$$f(z) = \phi + i\psi$$

$$f(z) = (x^4 - 6x^2y^2 + y^4) + i(4x^3y - 4xy^3 + c)$$

$$f(z) = (x+iy)^4 + ic \qquad f(z) = z^4 + ic$$

$$\boxed{f(z) = z^4 + ic}$$ complex potential

(ii) velocity of fluid is

$= \overline{f'(z)}$

$f'(z) = 4z^3$

$$\boxed{\overline{f'(z)} = 4\bar{z}^3}$$ velocity

$\overline{f'(z)} = 4(x-iy)^3$

$= 4(x^3 + iy^3 - 3ix^2y - 3xy^2)$

$\overline{f'(z)} = 4[(x^3 - 3xy^2) - i(-y^3 + 3x^2y)]$

Speed is $|\overline{f'(z)}|$

$|\overline{f'(z)}| = \sqrt{[4(x^3-3xy^2)]^2 + [4(-y^3+3x^2y)]^2}$

$= 2\sqrt{x^6 + y^6 + 9x^2y^4 - 6x^4y^2 + 9x^4y^2 - 6x^2y^4}$

$= 2\sqrt{x^6 + y^6 + 3x^4y^2 + 3x^2y^4}$

$= 2\sqrt{(x^2+y^2)^3} = 2(x^2+y^2)^{3/2}$ Speed

$$\boxed{\text{Speed} = 2(x^2+y^2)^{3/2}}$$

(iii) To show $\phi = c_1$ & $\psi = c_2$ are orthogonal

$$m_1 = \frac{-\frac{\partial \phi}{\partial x}}{\frac{\partial \phi}{\partial y}} = \frac{-(4x^3 - 12xy^2)}{(4y^3 - 12x^2y)}$$

$$m_2 = \frac{-\left(\frac{\partial \psi}{\partial x}\right)}{\left(\frac{\partial \psi}{\partial y}\right)} = \frac{(4y^3 - 12x^2y)}{4x^3 - 12xy^2}$$

$$m_1 . m_2 = \frac{-(4x^3 - 12xy^2)}{(4y^3 - 12x^2y)} \times \frac{(4y^3 - 12x^2y)}{(4x^3 - 12xy^2)}$$

$\boxed{m_1 \; m_2 = -1}$ i.e $\phi = c_1$ & $\psi = c_2$ are orthogonal.

Q:- If $w = \phi + i\psi$ represents the complex potential for electric field and $\psi = x^2 - y^2 + \frac{x}{x^2+y^2}$, determine the function ϕ.

Solⁿ:- $w = f(z) = \phi + i\psi$

$$d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} \cdot dy$$

$$\boxed{d\phi = \left(+\frac{\partial \psi}{\partial y}\right) dx - \left(\frac{\partial \psi}{\partial x}\right) dy} \quad (C-R \text{ eq}^n)$$

$$\frac{\partial \psi}{\partial x} = 2x + \frac{(x^2+y^2) - 2x^2}{(x^2+y^2)^2}$$

$$\frac{\partial \psi}{\partial x} = 2x + \frac{y^2 - x^2}{(x^2+y^2)^2}$$

$$\frac{\partial \psi}{\partial y} = -2y - \frac{2xy}{(x^2+y^2)^2}$$

Page

$$d\phi = \left(-2y - \frac{2xy}{(x^2+y^2)^2}\right)dx + \left(-2x - \frac{y^2-x^2}{(y^2+x^2)^2}\right)dy$$

This is exact diff-eq^n

sol^n is

$$\int d\phi = \underset{y\,const}{\int}\left(-2y - \frac{2xy}{(x^2+y^2)^2}\right)dx + \underset{\substack{ignoring\\ x\ terms}}{\int} 0\, dy + C$$

$$\boxed{\phi = -2yx + \frac{y}{x^2+y^2} + C} \rightarrow \text{velocity potential.}$$

:- Show that $\psi = x^2 - y^2 - 3x - 2y + 2xy$ can represent the stream function of an incompressible fluid flow. Also find corresponding velocity potential ϕ and hence complex potential $f(z) = \phi + i\psi$.

int!- To show ψ can be represent as the stream function, it should satisfy Laplace eq^n.

i.e To show $\dfrac{\partial^2\psi}{\partial x^2} + \dfrac{\partial^2\psi}{\partial y^2} = 0.$

In electric field problems

$w = u + iv$

complex potential

u is called potential function

v is called flux function

In electric field problems

$w = u + iv$

complex potential	u is called potential function	v is called flux function

Q1(b)

Q1(c)

Q 3 (

* Consider the complex potential

$$w = \phi(x, y) + i\,\psi(x, y)$$

(b)

(i) The curves $\phi(x, y) = a$ and $\psi(x, y) = b$ are called equipotential lines and stream lines in the field of fluid flow.

(c

2) The curves $\phi(x, y) = a$ and $\psi(x, y) = b$ are called equipotential lines and lines of force in the field of electrostatic and gravitational field.

(c

3. In heat flow problems $\phi(x, y) = a$ and $\psi(x, y) = b$ are called isothermal and heat flow lines.

Date
Page

Problems based on Properties of analytic function

Theo:- If $f(z)$ is an analytic function with constant modulus show that $f(z)$ is constant.

pf:-

Let $f(z) = u + iv$ be analytic function.

given $|f(z)| = c$ (c is constant)

i.e $\sqrt{u^2 + v^2} = c$

ie $u^2 + v^2 = c^2$

To show $f(z) = \text{const}$.

$u^2 + v^2 = c$

ie diff w.r.t. x

$$2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} = 0 \Rightarrow 2uu_x + 2vv_x = 0 \quad \text{— (1)}$$

diff w.r.t. y

$$2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y} = 0 \Rightarrow 2uu_y + 2vv_y = 0 \quad \text{— (2)}$$

$f(z)$ is analytic $\Rightarrow u_x = v_y$, $u_y = -v_x$.

ie

$2uu_x + 2vv_x = 0$ — (1)

$-2uv_x + 2vu_x = 0$. — (2)

squaring and adding

$$4u^2u_x^2 + 4v^2v_x^2 + \cancel{8uvu_xv_x} + 4u^2v_x^2 + 4v^2u_x^2 - \cancel{8uvu_xv_x} = 0$$

$$4u^2(u_x^2 + v_x^2) + 4v^2(u_x^2 + v_x^2) = 0$$

$$4(u^2+v^2)(u_x^2 + v_x^2) = 0$$

$$4c^2(u_x^2 + v_x^2) = 0$$

ie $u_x^2 + v_x^2 = 0$

ie $|f'(z)|^2 = 0$

ie $|f'(z)| = 0$.

$\Rightarrow f'(z) = 0$

$\Rightarrow f(z) = \text{const}$.

Date
Page

If $f(z)$ is analytic function of z, prove that

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4\,|f'(z)|^2$$

$$\nabla^2\left[Re(f(z))\right]^2 = 2\,|f'(z)|^2$$

$$\nabla^2\left[\log|f(z)|\right]=0$$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^p = p^2\,|f(z)|^{p-2}\,|f'(z)|^2$$

Note $\nabla^2 = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4\,|f'(z)|^2$$

$f(z)=u+iv$, $|f(z)|=\sqrt{u^2+v^2}$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)(u^2+v^2)$$

$$= \frac{\partial^2}{\partial x^2}(u^2+v^2)+\frac{\partial^2}{\partial y^2}(u^2+v^2)$$

$$\frac{\partial}{\partial x}\left(2u\,u_x+2v\,v_x\right)+\frac{\partial}{\partial y}\left(2u\,u_y+2v\,v_y\right)$$

$$= 2\left[u\,u_{xx}+u_x^2+v\,v_{xx}+v_x^2\right]+2\left[u\,u_{yy}+u_y^2+v\,v_{yy}+v_y^2\right]$$

$$= 2u\left[u_{xx}+u_{yy}\right]+2v\left[v_{xx}+v_{yy}\right]+2\left[u_x^2+v_x^2+u_y^2+v_y^2\right]$$

$$= 2u(0)+2v(0)+2\left[u_x^2+v_x^2+u_y^2+v_y^2\right]$$

[$\because f(z)$ is analytic ie u & v are harmonic].

$= 2[u_x^2 + v_x^2 + u_y^2 + v_y^2]$

$u_x = v_y, \quad u_y = -v_x$

$= 2[u_x^2 + v_x^2 + v_x^2 + u_x^2]$

$= 4[u_x^2 + v_x^2]$

$= 4\,|f'(z)|^2$

$f'(z) = u_x + iv_x$
$|f'(z)| = \sqrt{u_x^2 + v_x^2}$

(b) $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)[Re(f(z))]^2$

$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u^2$

$= \frac{\partial^2}{\partial x^2} u^2 + \frac{\partial^2}{\partial y^2} u^2$

$= \frac{\partial}{\partial x}[2uu_x] + \frac{\partial}{\partial y}[2uu_y]$

$= 2[uu_{xx} + u_x^2 + uu_{yy} + u_y^2]$

$= 2u(u_{xx} + u_{yy}) + 2(u_x^2 + u_y^2)$

$= 0 + 2(u_x^2 + u_y^2)$ [as $u_{xx} + u_{yy} = 0$]

To prove
$2|f'(z)|^2$
$= 2(u_x^2 + v_x^2)$

$u_x = v_y, \quad u_y = -v_x$

$= 2(u_x^2 + v_x^2)$

$= 2|f'(z)|^2$

Q Show that

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z \partial \bar{z}}$$

Proof:- $x = \frac{z+\bar{z}}{2}$, $y = \frac{z-\bar{z}}{2i}$.

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial z} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial z}$$

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x}\cdot\frac{1}{2} + \frac{\partial f}{\partial y}\left(\frac{1}{2i}\right)$$

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y}\right)f$$

$$\left[2\frac{\partial}{\partial z} = \frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y}\right] \text{ — (1)}$$

Now $$\frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial \bar{z}} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial \bar{z}}$$

$$\frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial x}\cdot\frac{1}{2} + \frac{\partial}{\partial y}\cdot\left(\frac{-1}{2i}\right)$$

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y}\right)f$$

$$2\frac{\partial f}{\partial \bar{z}} = \left(\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y}\right)f$$

or $$\left[2\frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y}\right] \text{ — (2)}$$

Now $$4\left(\frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial \bar{z}}\right) = \left(\frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y}\right)$$

classmate

Date

Page

$$\left[4\frac{\partial^2}{\partial z\,\partial \bar{z}} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right]$$

$(a+ib)(a-ib)$
$= a^2+b^2$

Q:- If $f(z) = u+iv$ is regular function of z in a domain D, then

$$\nabla^2(u^p) = p(p-1)\,u^{p-2}\,|f'(z)|^2$$

Solⁿ:. $$\nabla^2(u^p) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(u^p)$$

$$= \frac{\partial^2}{\partial x^2}u^p + \frac{\partial}{\partial y^2}u^p$$

$$= \frac{\partial}{\partial x}\left(p\,u^{p-1}u_x\right) + \frac{\partial}{\partial y}\left(p\,u^{p-1}u_y\right)$$

$$= p\left(u^{p-1}u_{xx} + (p-1)\,u^{p-2}u_x^2\right)$$
$$+ p\left(u^{p-1}u_{yy} + (p-1)\,u^{p-2}u_y^2\right)$$

$$= p\,u^{p-1}(u_{xx}+u_{yy}) + p(p-1)\,u^{p-2}(u_x^2+u_y^2)$$

$$= 0 + p(p-1)\,u^{p-2}\,|f'(z)|^2$$

$u_{xx}+u_{yy}=0$

$|f'(z)| = |u_x + iv_x|$
$= |u_x - iu_y|$
$= \sqrt{u_x^2+u_y^2}$

Q If $f(z)$ is a holomorphic function of z show that

$$\left\{\frac{\partial}{\partial x}|f(z)|\right\}^2 + \left\{\frac{\partial}{\partial y}|f(z)|\right\}^2 = |f'(z)|^2$$

Solⁿ $f(z) = u + iv$

$|f(z)| = \sqrt{u^2+v^2}$

L.H.S

$$\left\{\frac{\partial}{\partial x}|f(z)|\right\}^2 + \left\{\frac{\partial}{\partial y}|f(z)|\right\}^2 \quad \text{——} (1)$$

taking

$$\frac{\partial}{\partial x}|f(z)| = \frac{\partial}{\partial x}\sqrt{u^2+v^2}$$

$$= \frac{1}{2}(u^2+v^2)^{-1/2}(2uu_x + 2vv_x) \quad \text{——} (A)$$

similarly,

$$\frac{\partial}{\partial y}|f(z)| = \frac{\partial}{\partial y}\sqrt{u^2+v^2}$$

$$= \frac{1}{2}(u^2+v^2)^{-1/2}(2uu_y + 2vv_y) \quad \text{——} (B)$$

from (A) & (B)

$$\left(\frac{\partial}{\partial x}|f(z)|\right)^2 + \left(\frac{\partial}{\partial y}|f(z)|\right)^2$$

$$= \frac{1}{4}(u^2+v^2)^{-1}\left[(2uu_x+2vv_x)^2 + (2uu_y+2vv_y)^2\right]$$

$$= \frac{1}{4(u^2+v^2)}\left[4u^2u_x^2 + 4v^2v_x^2 + 8uvu_xv_x + 4u^2u_y^2 + 4v^2v_y^2 + 8uvu_yv_y\right]$$

$$u_x = v_y, \quad u_y = -v_x$$

$$= \frac{1}{4(u^2+v^2)} \left[4u^2u_x^2 + 4v^2v_x^2 + 4u^2v_x^2 + 4v^2u_x^2 + \cancel{8uv\,u_x v_x} - \cancel{8uv\,u_x v_x} \right]$$

$$= \frac{1}{\cancel{4(u^2+v^2)}} \left[\cancel{4(u^2+v^2)} (u_x^2 + v_x^2) \right]$$

$$= u_x^2 + v_x^2$$

$$= |f'(z)|^2$$

$f'(z) = u_x + i v_x$

$|f'(z)| = \sqrt{u_x^2 + v_x^2}$

ie $$\left(\frac{\partial}{\partial x} |f(z)|\right)^2 + \left(\frac{\partial}{\partial y} |f(z)|\right)^2 = |f'(z)|^2$$