5.3: Half Range Fourier Series

(i) The half range Sine series for $f(x)$ in the interval $(0, l)$ is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right),$$

where $b_n = \frac{2}{l} \int_0^l f(x) \cdot \sin\left(\frac{n\pi x}{l}\right) dx$

(ii) The half range cosine series for $f(x)$ in the interval $(0, l)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right),$$

where $a_0 = \frac{2}{l} \int_0^l f(x) dx$

and $a_n = \frac{2}{l} \int_0^l f(x) \cdot \sin\left(\frac{n\pi x}{l}\right) dx.$

Problems:

1. Obtain the half range cosine series for the function $f(x) = x$, $0 < x < \pi$ and hence deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$.

Sol:

The half rang cosine series for $f(x)$ in the interval $(0, \pi)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \qquad (\text{here } l = \pi)$$

where $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$

$$= \frac{2}{\pi}\int_0^{\pi} x\,dx = \frac{2}{\pi}\left(\frac{x^2}{2}\right)_0^{\pi} = \pi$$

and $a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$

$$= \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx$$

$$= \frac{2}{\pi}\left\{\left[x \cdot \frac{\sin(nx)}{n}\right]_0^{\pi} - \left[1 \cdot \left(\frac{-\cos nx}{n^2}\right)\right]_0^{\pi}\right\}$$

$$= \frac{2}{\pi}\left\{0 + \frac{1}{n^2}(\cos n\pi - 1)\right\}$$

$$= \frac{2}{\pi n^2}\left((-1)^n - 1\right)$$

Therefore, $f(x) = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{((-1)^n - 1)}{n^2}\cos nx$

i.e., $x = \frac{\pi}{2} + \frac{2}{\pi}\left(\frac{-2}{1^2}\cos x - \frac{2}{3^2}\cos 3x - \frac{2}{5^2}\cos 5x - \cdots\right.$

$$\Rightarrow x = \frac{\pi}{2} \cancel{\frac{4}{\pi}}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right)$$

Taking $x=0$, we get

$$0=\frac{\pi}{2}-\frac{4}{\pi}\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\right)$$

Hence, $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots=\frac{\pi^2}{8}$.

(2) Find the half range sine series for

$f(x)=2x-x^2,\ 0<x<2$.

Sol: The half range sine series for $f(x)=2x-x^2$ in the interval $(0,2)$ is given by

$$f(x)=\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{2}\right)$$

(here $l=2$).

Now,

$$b_n=\frac{2}{2}\int_0^2 f(x)\cdot\sin\left(\frac{n\pi x}{2}\right)dx$$

$$=\int_0^2 (2x-x^2)\sin\left(\frac{n\pi x}{2}\right)dx$$

$$=\left[(2x-x^2)\cdot\frac{\left(-\cos\left(\frac{n\pi x}{2}\right)\right)}{\left(\frac{n\pi}{2}\right)}\right]_0^2-\left[(2-2x)\frac{\left(-\sin\left(\frac{n\pi x}{2}\right)\right)}{\left(\frac{n\pi}{2}\right)^2}\right]_0^2$$

$$+\left[(-2)\frac{\cos\left(\frac{n\pi x}{2}\right)}{\left(\frac{n\pi}{2}\right)^3}\right]_0^2$$

$$= 0 + \frac{4}{n^2\pi^2}(0) - \frac{16}{n^3\pi^3}(\cos n\pi - 1)$$

$$= \frac{-16}{n^3\pi^3}((-1)^n - 1) = \frac{16}{n^3\pi^3}(1-(-1)^n)$$

Therefore, $f(x) = \frac{16}{\pi^3} \sum_{n=1}^{\infty} \frac{(1-(-1)^n)}{n^3} \sin\left(\frac{n\pi x}{2}\right)$

3. Find the halfrange cosine series for

$f(x) = x.\sin x$, $0 < x < \pi$ and hence deduce

~~that~~ $\frac{1}{1(3)} - \frac{1}{3(5)} + \frac{1}{5(7)} - \frac{1}{7(9)} + \cdots = \frac{\pi - 2}{4}$.

Answer: Here $l = \pi$. $\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx}$

$a_0 = 2$, $a_n = \frac{2}{n^2-1}(-1)^{n+1}$ $(n \neq 1)$

$a_1 = -\frac{1}{2}$.

Therefore, $f(x) = \frac{a_0}{2} + a_1 \cos x + \sum_{n=2}^{\infty} a_n \cos nx$

i.e., $x.\sin x = 1 - \frac{1}{2}\cos x + 2\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n^2-1} \cos nx$

Taking $x = \frac{\pi}{2}$, we get

$$\frac{\pi}{2} = 1 - \frac{1}{2}(0) + 2\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n^2-1} \cos\left(\frac{n\pi}{2}\right)$$

$$\frac{\pi}{2} = 1 + 2\left(\frac{-1}{3}\cdot\cos\pi + \frac{1}{8}\cdot\cos\frac{3\pi}{2} - \frac{1}{15}\cos 2\pi + \frac{1}{24}\cos\left(\frac{5\pi}{2}\right) - \frac{1}{35}\cos 3\pi + \cdots\right)$$

$$\Rightarrow \quad \frac{1}{1(3)} - \frac{1}{3(5)} + \frac{1}{5(7)} - \frac{1}{7(5)} + \cdots = \frac{1}{2}\left(\frac{\pi}{2} - 1\right)$$

Hence, $\frac{1}{1(3)} - \frac{1}{3(5)} + \frac{1}{5(7)} - \frac{1}{7(5)} + \cdots = \frac{\pi - 2}{4}$.

4. Find the half range cosine series for

$$f(x) = \begin{cases} x & \text{if } 0 < x < \frac{\pi}{2} \\ \pi - x & \text{if } \frac{\pi}{2} < x < \pi \end{cases}$$

__Answer:__ $f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left(\frac{1}{1^2}\cos 2x + \frac{1}{3^2}\cos 6x + \frac{1}{5^2}\cos 10x + \cdots\right)$

5. Obtain the half range sine series for $f(x) = e^x$ in the interval $(0, \pi)$.

__Answer:__ $b_n = \frac{2n}{\pi(1+n^2)}\left[1 + (-1)^{n+1} e^{\pi}\right]$