

Dynamic Programming

Matrix Chain Multiplication Problem

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SCOPE

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DP - Matrix chain Multiplication

Suppose, I have 2 matrices A and B.

$$A \times B \quad \left\{ \begin{array}{l} 21 + 01, 02 + 0 \end{array} \right\} \text{ etc.} =$$
$$\left[\begin{array}{|c|} \hline \\ \hline \end{array} \right] \times \left[\begin{array}{|c|} \hline \\ \hline \end{array} \right] = \left[\begin{array}{|c|} \hline \\ \hline \end{array} \right]$$

$$5 \times 4 \quad 4 \times 3$$

Condition for

matrix multiplication

No. of cols of
1st matrix.

No. of rows

of 2nd matrix.

no. of
rows of

1st matrix

no. of cols.
of 2nd matrix.

$$B \times A$$

$$\left[\begin{array}{|c|} \hline \\ \hline \end{array} \right] \times \left[\begin{array}{|c|} \hline \\ \hline \end{array} \right]$$

$$4 \times 3$$

$$5 \times 4$$

we have 15 elements
in the resultant matrix

To get one single element
we have to do 4
multiplications.

$$\text{So, } 15 \times 4 = 60 \text{ multiplication}$$

Cannot be multiplied, as cond. is not satisfied.

So, If $A \times B$ is possible, then $B \times A$ is possible is
not necessary.

To Find No. of multiplications

$$\begin{matrix} A & B & C \\ 5 \times 4 & 4 \times 3 & = 5 \times 3 \end{matrix}$$

$5 \times 4 \times 3 = 60 \rightarrow$ No. of multiplications
 Cost of multiplying 2 matrices.

60 multiplications needed to get result matrix.
 depends on no. of multiplications.

Matrix chain multiplication

$\{ A_1, A_2, A_3, A_4 \}$
 $5 \times 4 \quad 4 \times 6 \quad 6 \times 2 \quad 2 \times 7$

Four matrices with dimensions are given. They are arranged in such a way that they can be multiplied.

Question is not multiplying and getting final answer.

At a time, we cannot multiply all matrices. we can multiply only a pair at a time.

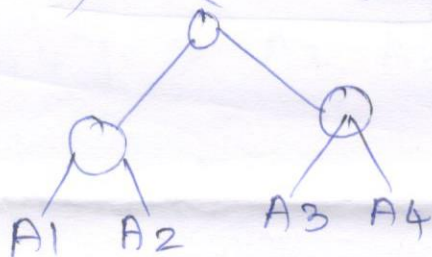
Question is, which pair I should select such that the total cost of multiplying all of them is minimum.

So, problem is not to multiply and get the answer. Problem is ^{to} know, how to multiply. This is matrix chain multiplication problem.

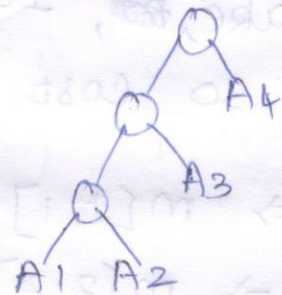
It means, how should I parenthesis them.

$$((A_1 \cdot A_2) \cdot A_3) \cdot A_4 \Rightarrow$$

$$(A_1 \cdot A_2) \cdot (A_3 \cdot A_4)$$



Tree with 3 nodes.



Tree with 3 nodes.

m	1	2	3	4
1	0	120	88	158
2		0	48	104
3			0	84
4				0

S	1	2	3	4
1		1	1	3
2			2	3
3				3
4				

To fill the table, use bottom-up approach.

That is, take the smaller values and fill the table.

only one matrix.

Just take ~~A~~, It is not multiplied with any thing. So cost is 0.

$$A_1 \Rightarrow m[1,1] = 0$$

Similarly, $A_2 \Rightarrow m[2,2] = 0$

$$A_3 \Rightarrow m[3,3] = 0$$

$$A_4 \Rightarrow m[4,4] = 0$$

$$m[i,i] = 0$$

Now take 2 matrices.

$$A_1 \cdot A_2 \Rightarrow m[1,2]$$

$$5 \times 4 \quad 4 \times 6$$

Total cost = $5 \times 4 \times 6 = 120$

we have 1 in left side.

1 is multiplied with 2, so let us write as

$$1 \text{ in } S\text{-table. (i.e.) } S[1,2] = 1$$

$$m[2,3] \Rightarrow A_2 \cdot A_3$$

$$4 \times 6 \quad 6 \times 2$$

$$\text{Total cost} = 4 \times 6 \times 2 = 48$$

$$\therefore s[2,3] = 2$$

$$m[3,4] \Rightarrow A_3 \cdot A_4$$

$$6 \times 2 \quad 2 \times 7$$

$$\text{Total cost} = 6 \times 2 \times 7 = 84$$

$$s[3,4] = 3$$

Consider 3 matrices.

$$m[1,3] \Rightarrow A_1, A_2, A_3$$

cost of this multiplication

Two possibilities $\Rightarrow$



$$(A_1 \cdot A_2) \cdot A_3$$

$5 \times 4 \quad 4 \times 6 \quad 6 \times 2$

$$m[1,2] + m[3,3] + 5 \times 6 \times 2$$

$$= 120 + 0 + 60$$

$$= 180$$

$$A_1 \cdot (A_2 \cdot A_3)$$

$$5 \times 4 \quad 4 \times 6 \quad 6 \times 2$$

$$\text{cost}(A_1) + \text{cost}(A_2 \cdot A_3) + 5 \times 4 \times 2$$

$m[1,1] \quad m[2,3]$

$$= 0 + 48 + 40$$

$$= 88$$

Pick minimum. minimum is 88.

$$\text{So, } m[1,3] = 88. \quad S[1,3] = 1$$

$$m[2,4] \Rightarrow A_2, A_3, A_4$$

$$A_2 \cdot (A_3 \cdot A_4)$$

$$4 \times 6 \quad 6 \times 2 \quad 2 \times 7$$

$$m[2,2] + m[3,4] + 4 \times 6 \times 7$$

$$0 + 84 + 168$$

$$252$$

$$(A_2 \cdot A_3) \cdot A_4$$

$$4 \times 6 \quad 6 \times 2 \quad 2 \times 7$$

$$m[2,3] + m[4,4] + 4 \times 2 \times 7$$

$$48 + 0 + 56$$

$$104$$

104 is minimum.

$$\text{So, } m[2,4] = 104 \quad S[2,4] = 3$$

So, $m[1,4] = 104$
 consider 4 matrices, possibilities $\Rightarrow A_1 \overset{5 \times 4}{\cdot} (A_2 \overset{4 \times 7}{\cdot} A_3 \cdot A_4)$, $(A_1 \overset{5 \times 6}{\cdot} A_2) \cdot (A_3 \overset{6 \times 7}{\cdot} A_4)$, $(A_1 \cdot A_2 \cdot A_3) \cdot A_4$

$$m[1,4] = \min \left\{ \begin{aligned} &m[1,1] + m[2,4] + 5 \times 4 \times 7, \\ &m[1,2] + m[3,4] + 5 \times 6 \times 7, \\ &m[1,3] + m[4,4] + 5 \times 2 \times 7 \end{aligned} \right\}$$

$$= 0 + 104 + 140, \quad 120 + 84 + 210, \quad 88 + 0 + 70$$

$$= \overset{244}{\cancel{244}}, \quad 414, \quad 158$$

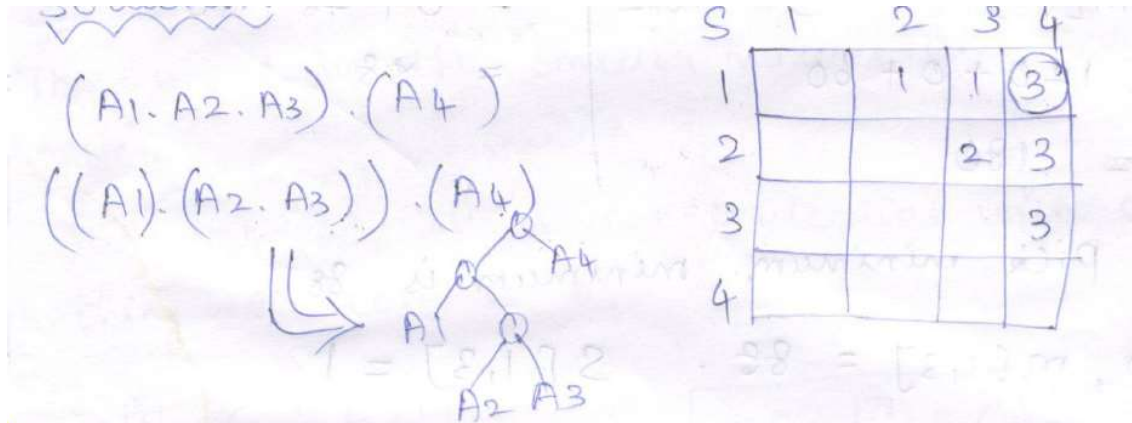
$$= 158$$

$$S[1,4] = 3$$

Solution:

Optimal cost = 158

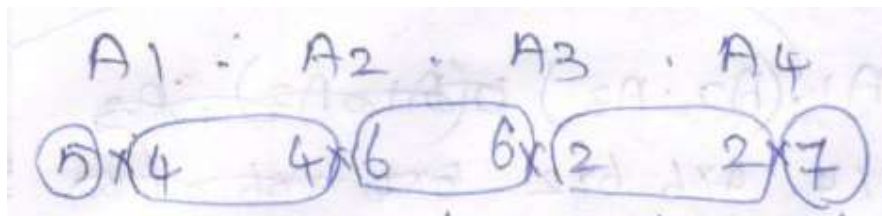
Sequence of multiplication:



Basic formula:

$$m[i, j] = \begin{cases} 0 & \text{if } i = j, \\ \min_{i \leq k < j} \{m[i, k] + m[k + 1, j] + p_{i-1} p_k p_j\} & \text{if } i < j. \end{cases}$$

Where, P denotes matrix dimension.



P0 = 5

P1 = 4

P2 = 6

P3 = 2

P4 = 7

Pseudocode:

MATRIX-CHAIN-ORDER(p)

```
1   $n \leftarrow \text{length}[p] - 1$ 
2  for  $i \leftarrow 1$  to  $n$ 
3      do  $m[i, i] \leftarrow 0$ 
4  for  $l \leftarrow 2$  to  $n$        $\triangleright l$  is the chain length.
5      do for  $i \leftarrow 1$  to  $n - l + 1$ 
6          do  $j \leftarrow i + l - 1$ 
7               $m[i, j] \leftarrow \infty$ 
8              for  $k \leftarrow i$  to  $j - 1$ 
9                  do  $q \leftarrow m[i, k] + m[k + 1, j] + p_{i-1}p_kp_j$ 
10                     if  $q < m[i, j]$ 
11                         then  $m[i, j] \leftarrow q$ 
12                              $s[i, j] \leftarrow k$ 
13  return  $m$  and  $s$ 
```

PRINT-OPTIMAL-PARENS(s, i, j)

```
1  if  $i = j$ 
2      then print " $A$ " $i$ 
3      else print "("
4          PRINT-OPTIMAL-PARENS( $s, i, s[i, j]$ )
5          PRINT-OPTIMAL-PARENS( $s, s[i, j] + 1, j$ )
6          print ")"
```

Time complexity:

There are three nested loops (loop l , i and k). Each loop executes maximum of n times.

Body of loop constant complexity

Total Complexity is $O(n^3)$