

Harmonic function:-

Any function ~~f(x)~~, which satisfy the Laplace's equation is known as harmonic function.

Note

4

Laplace eqⁿ is second order PDE, given as

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Theorem:- If $f(z) = u + iv$ is an analytic function, then u and v both are Harmonic functions.

Proof: $f(z) = u + iv$ is analytic i.e.

$$u_x = v_y \quad \text{--- (1)}$$

$$u_y = -v_x \quad \text{--- (2)}$$

from (1) $u_{xx} = v_{yx}$
" (2) $u_{yy} = -v_{xy}$

we know $v_{yx} = v_{xy}$

i.e. $u_{xx} + u_{yy} = 0$ i.e. u satisfy Laplace eqⁿ

Similarly we can show

$$v_{xx} + v_{yy} = 0 \text{ i.e. } v \text{ also " " "}$$

i.e. u and v are harmonic function.

i.e. if $f(z)$ is analytic then u & v both are ~~are~~ Harmonic functions but converse may or may not be true.

* such functions u & v are called harmonic conjugate or conjugate harmonic functions if $u+iv$ is also analytic functions.

Q:- Show that the function $u=e^x \sin y$ is harmonic.

Solⁿ $u = e^x \sin y$

To check $u_{xx} + u_{yy} = 0$ — (1)

$$\begin{aligned} u_x &= e^x \sin y, & u_{xx} &= e^x \sin y \\ u_y &= e^x \cos y, & u_{yy} &= -e^x \sin y \end{aligned}$$

adding

$$u_{xx} + u_{yy} = e^x \sin y - e^x \sin y = 0$$

ie u satisfy Laplace eqⁿ therefore u is harmonic function.

Q:- Prove that $u = x^2 - y^2$ and $v = \frac{4y}{x^2 + y^2}$ are harmonic functions of (x, y) but are not harmonic conjugates.

Solⁿ $u_x = 2x, u_{xx} = 2$
 $u_y = -2y, u_{yy} = -2$

$u_{xx} + u_{yy} = 0$ i.e. u satisfies Laplace eqⁿ
 i.e. u is harmonic function.

slly. $v = \frac{4y}{x^2 + y^2}$

$$v_x = \frac{-4(2x)}{(x^2 + y^2)^2}$$

$$v_{xx} = \frac{(x^2 + y^2)^2(-2y) - (-2xy)2(x^2 + y^2)(2x)}{(x^2 + y^2)^4}$$

$$\text{i.e. } v_{xx} = \frac{(x^2 + y^2)(-2y) + 8x^2y}{(x^2 + y^2)^3}$$

$$v_{xx} = \frac{-2yx^2 - 2y^3 + 8yx^2}{(x^2 + y^2)^3}$$

$$\text{i.e. } v_{xx} = \frac{6x^2y - 2y^3}{(x^2 + y^2)^3}$$

Now $v_y = \frac{x^2 - y^2}{(x^2 + y^2)^2}$

$$v_{yy} = \frac{-6x^2y + 2y^3}{(x^2 + y^2)^3}$$

adding

$$v_{xx} + v_{yy} = 0$$

ie v is also harmonic function.

Proved

Now u, v will be harmonic conjugate
if $u + iv$ is analytic function.
ie if u & v satisfy C-R eqⁿ.

we have

$$u_x = 2x \quad u_y = -2y$$

$$v_x = \frac{-2xy}{(x^2+y^2)^2} \quad v_y = \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$u_x \neq v_y$$

$$u_y \neq -v_x$$

ie u & v do not satisfy C-R eqⁿ
ie $f(z) = u + iv$ is not analytic
ie u & v are not harmonic
conjugate.

$$* = \cancel{24x} + x^2 - 3x +$$

Method to find conjugate function:-

If $f(z) = u + iv$ is analytic function, then u and v are called conjugate or conjugate harmonic of each other.

Method:- Let u is given and v is unknown then

$$\text{write } dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy \quad \text{--- (1)}$$

$$\text{replace } \begin{cases} v_x = -u_y \\ v_y = u_x \end{cases} \quad \text{CR eqns}$$

$$dv = \left(\frac{-\partial u}{\partial y} \right) dx + \left(\frac{\partial u}{\partial x} \right) dy \quad \text{--- (2)}$$

$$dv = M dx + N dy \quad \text{--- (3)} \quad \begin{array}{l} \text{say} \\ M = -\frac{\partial u}{\partial y} \end{array}$$

$\nabla =$ above differential eqn is exact differential eqn $\left| \begin{array}{l} N = \frac{\partial u}{\partial x} \end{array} \right.$

$$\text{as } \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad (\text{check})$$

Now solution of exact diff. eqn (3) will be

$$v = \int dv = \int M dx + \int N dx$$

treat y as constant

integrate only term not containing x

$$\begin{array}{l} M = -\frac{\partial u}{\partial y} \quad \frac{\partial M}{\partial y} = -\frac{\partial^2 u}{\partial y^2} \\ N = \frac{\partial u}{\partial x} \quad \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x^2} \end{array}$$

$$-\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial x^2} \quad \left[\text{since } u \text{ satisfy Laplace eqn} \right]$$

Date _____
Page _____

Method:- Similarly if v is given,
~~then~~ to find u , we start with

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$$

$$u = \int \frac{\partial u}{\partial x} dx + \int \frac{\partial u}{\partial y} dy$$

$$u = \int \frac{\partial v}{\partial y} dx + \int \left(-\frac{\partial v}{\partial x} \right) dy$$

$$u = \int M dx + \int N dy \quad \text{--- (1)}$$

diff eqⁿ (1) is exact differential eqⁿ.

check $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

condⁿ for
exact

Solⁿ will be

$$u = \int_{y \text{ const}} M dx + \int N dy$$

not terms
not containing
 x

once u is determine

$$\boxed{f(z) = u + i v}$$

Q1 → Prove that $u = x^2 - y^2 - 2xy - 2x + 3y$ is harmonic.
 find function v such that $f(z) = u + iv$ is analytic. Also express $f(z)$ in terms of z .

Solⁿ $u = x^2 - y^2 - 2xy - 2x + 3y$

$$u_x = 2x - 2y - 2, \quad u_{xx} = 2$$

$$u_y = -2y - 2x + 3, \quad u_{yy} = -2$$

$$u_{xx} + u_{yy} = 2 + (-2) = 0$$

u satisfy Laplace eqⁿ $\Rightarrow u$ is harmonic. Proved

To find v :-

$$dv = \left(\frac{\partial v}{\partial x}\right) dx + \left(\frac{\partial v}{\partial y}\right) dy$$

using C-R eq^{ns}

$$dv = \left(-\frac{\partial u}{\partial y}\right) dx + \left(\frac{\partial u}{\partial x}\right) dy$$

$$u_x = v_y$$

$$u_y = -v_x$$

$$dv = (2y + 2x - 3) dx + (2x - 2y - 2) dy$$

$$\int dv = \int (2y + 2x - 3) dx + \int (2x - 2y - 2) dy$$

$$dv = M dx + N dy$$

$$\frac{\partial M}{\partial y} = 2, \quad \frac{\partial N}{\partial x} = 2$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}, \quad \text{ie } \textcircled{1} \text{ is exact diff. eqⁿ .}$$

$$\text{Solⁿ is } \int dv = \int_{y \text{ const}} (2y + 2x - 3) dx + \int (-2y - 2) dy$$

only term not containing x

$$v = 2yx + x^2 - 3x - y^2 - 2y + C.$$

Now, $f(z) = u + iv$

$$f(z) = (x^2 - y^2 - 2xy - 2x + 3y) + i(2xy + x^2 - 3x - y^2 - 2y + C)$$

$$f(z) = (x^2 - y^2 + 2ixy) + i(x^2 - y^2 + 2ixy) + (-2x - 2yi) + (-3xi + 3y) + iC$$

$$f(z) = (x + iy)^2 + i(x + iy)^2 - 2(x + iy) - 3i(x + iy) + iC$$

$$f(z) = z^2 + iz^2 - 2z - 3iz + iC$$

$$f(z) = (1+i)z^2 - (2+3i)z + iC$$

Q:- find the imaginary part of analytic function whose real part is $x^3 - 3xy^2 + 3x^2 - 3y^2$.

Ans $v = 3x^2y + 6xy - y^3 + c$

Q:- find a function $w = u + iv$ such that w is analytic if $u = e^x \sin y$

Q. show that $u = \sin x \cosh y + 2 \cos x \sinh y + x^2 - y^2 + 4xy$ satisfies Laplace eqn and find analytic function $u + iv$.

In polar coordinates

Q: Let $f(z) = u + iv$ be an analytic function
 & $u = -r^3 \sin 3\theta$ then construct
 corresponding analytic function $f(z)$ in
 terms of z .

Soln

$$u = -r^3 \sin 3\theta$$

$$\frac{\partial u}{\partial r} = -3r^2 \sin 3\theta, \quad u_\theta = -3r^3 \cos 3\theta$$

$$dv = \frac{\partial v}{\partial r} dr + \frac{\partial v}{\partial \theta} d\theta$$

C-R eqn

$$dv = \left(-\frac{1}{r} \frac{\partial u}{\partial \theta} \right) dr + \left(r \frac{\partial u}{\partial r} \right) d\theta$$

$$\frac{\partial v}{\partial r} = \frac{1}{r} \frac{\partial u}{\partial \theta}$$

$$\frac{\partial u}{\partial \theta} = -r \frac{\partial v}{\partial r}$$

$$dv = -\frac{1}{r} (-3r^3 \cos 3\theta) dr + r (-3r^2 \sin 3\theta) d\theta$$

$$dv = \underbrace{3r^2 \cos 3\theta}_{M} dr + \underbrace{(-3r^3 \sin 3\theta)}_N d\theta$$

above diff-eqn is exact if

$$\frac{\partial M}{\partial \theta} = \frac{\partial N}{\partial r}$$

$$\frac{\partial M}{\partial \theta} = -9r^2 \sin 3\theta, \quad \frac{\partial N}{\partial r} = -9r^2 \sin 3\theta$$

$$\text{ie } \frac{\partial M}{\partial \theta} = \frac{\partial N}{\partial r}$$

Solⁿ is

$$\int dv = \int_{\theta \text{ const}} (3r^2 \cos 3\theta) dr + \int 0 + C$$

ignoring
r terms

$$v = r^3 \cos 3\theta + C$$

i.e

$$f(z) = u + i v$$

$$f(z) = -r^3 \sin 3\theta + i r^3 \cos 3\theta + i C$$

$$= i (r^3 \cos 3\theta + i r^3 \sin 3\theta) + i C$$

$$= i (z^3) + i C$$

also is

$$z = r e^{i\theta}$$

$$= r(\cos \theta + i \sin \theta)$$

$$f(z) = i z^3 + i C$$

Q:- find u such that $f(z) = u + i v$ is analytic
 given $v(r, \theta) = r^2 \cos 2\theta - r \cos \theta + 2$

Ans:- $u(r, \theta) = -r^2 \sin 2\theta + r \sin \theta + C$

$$f(z) = i(r^2 e^{2i\theta} - r e^{i\theta}) + 2iC$$

Page _____

Finding an analytic function $f(z) = u + iv$ when $u - v$ is given

Q1:- If $u - v = (x - y)(x^2 + 4xy + y^2)$ and $f(z) = u + iv$ is an analytic fun of $z = x + iy$ find $f(z)$ in terms of z .

Solⁿ: $f(z) = u + iv$

$$u + iv = f(z) \quad \text{--- (i)}$$

$$iu - v = i f(z) \quad \text{--- (ii) --- multiply (i) by } i$$

Adding

$$(u - v) + i(u + v) = (1 + i) f(z)$$

$$U + iV = F(z) \quad \text{--- (iii)}$$

$$\text{(Say } u - v = U$$

$$u + v = V$$

$$if(z) = F(z)$$

Now

$U = u - v = (x - y)(x^2 + 4xy + y^2)$ is given
 V can be obtained by $= x^3 - y^3 + 3x^2y - 3xy^2$

$$dV = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy$$

$$dV = -\frac{\partial U}{\partial y} dx + \frac{\partial U}{\partial x} dy \quad (\text{C-R eqn})$$

also $\frac{\partial U}{\partial x} = 3x^2 + 6xy - 3y^2$

$$\frac{\partial U}{\partial y} = 3x^2 - 6xy - 3y^2$$

$$dV = (-3x^2 + 6xy + 3y^2) dx + (3x^2 + 6xy - 3y^2) dy \quad \text{--- (A)}$$

on integrating

$$\int dV = \int (-3x^2 + 6xy + 3y^2) dx$$

y const

$$+ \int -3y^2 dy + C$$

ignoring x

diff. eqⁿ (A) is exact

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y} (-3x^2 + 6xy + 3y^2) = 6x + 6y$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x} (3x^2 + 6xy - 3y^2) = 6x + 6y$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$V = -\frac{3x^3}{3} + 6 \frac{x^2}{2} y + 3y^2 x - \frac{3y^3}{3} + C$$

$$V = -x^3 - y^3 + 3x^2 y + 3y^2 x + C$$

$$F(z) = U + iV$$

$$= (x^3 - y^3 + 3x^2 y - 3xy^2) + i(-x^3 - y^3 + 3x^2 y + 3xy^2) + iC$$

$$= (x^3 - iy^3 + 3ix^2 y - 3xy^2)$$

$$+ (ix^3 - y^3 + 3x^2 y + 3ixy^2) + iC$$

$$(x+iy)^3$$

$$= x^3 - iy^3 + 3x^2 iy - 3xy^2$$

$$= (x+iy)^3 - i(x^3 - iy^3 + 3ix^2 y - 3xy^2) + iC$$

$$= z^3 - iz^3 + iC$$

$$F(z) = (1-i)z^3 + iC$$

Now $F(z) = (1+i)f(z)$

$$(1+i) f(z) = (1-i) z^3 + i c$$

$$f(z) = \left(\frac{1-i}{1+i} \right) z^3 + i c$$

$$f(z) = \frac{(1-i)^2}{(1+i)(1-i)} z^3 + \frac{i(1-i)}{(1+i)(1-i)} c$$

$$f(z) = -i z^3 + \frac{(1+i)c}{2} \quad \underline{\underline{\text{Ans}}}$$