Module 6: FOURIER TRANSFORMS

Fourier Integral of $f(x)$ is given by

$$f(x) = \frac{1}{\pi} \int_0^\infty \int_{-\infty}^\infty \cos p(t-x)\, f(t)\, dt\, dp \longrightarrow (1)$$

FOURIER SINE and Cosine Integrals

Fourier Integral of $f(x)$ given by (1)

can be written as

$$f(x) = \frac{1}{\pi} \int_0^\infty \int_{-\infty}^\infty (\cos pt . \cos px + \sin pt . \sin px) f(t)\, dt\, dp.$$

$$= \frac{1}{\pi} \int_0^\infty \cos px \left(\int_{-\infty}^\infty \cos pt\, f(t)\, dt \right) dp$$

$$+ \frac{1}{\pi} \int_0^\infty \sin px \left(\int_{-\infty}^\infty \sin pt . f(t)\, dt \right) dp \longrightarrow (2)$$

when $f(t)$ is an odd function, from (2),

we have

$$f(x) = \frac{2}{\pi} \int_0^\infty \sin px \left(\int_0^\infty f(t) . \sin pt\, dt \right) dp$$

which is known as the Fourier Sine Integral.

When $f(t)$ is an even function, from ②,

we have

$$f(x) = \frac{2}{\pi} \int_0^\infty \cos px \left(\int_0^\infty f(t) \cos pt \, dt \right) dp$$

which is Know as the Fourier Cosine Integral.

FOURIER INTEGRAL IN COMPLEX FORM:

The complex for of Fourier integral of a function $f(x)$ is given by

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ipx} \left(\int_{-\infty}^{\infty} e^{ipt} f(t) \, dt \right) dp.$$

$\longrightarrow$ ①

Let $F(p) = \int_{-\infty}^{\infty} e^{ipt} f(t) \, dt$, then ① becomes

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(p) \, e^{-ipx} \, dp.$$

Here, $F(p)$ is called the Fourier transform of $f(x)$ and $f(x)$ is called the inverse Fourier transform of $F(p)$.

The infinite Fourier transform of $f(x)$:

The Fourier transform of a function $f(x)$ is given by

$$F\{f(x)\} = F(p) = \int_{-\infty}^{\infty} f(x)\, e^{ipx}\, dx$$

The inverse Fourier transform of $F(p)$ is given by

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(p)\, e^{-ipx}\, dp$$

Note: Some authors also defined as follows (Alternate Forms):

① $$F\{f(x)\} = F(p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{ipx}\, dx$$

and $$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(p)\, e^{-ipx}\, dp.$$

② $$F\{f(x)\} = F(p) = \int_{-\infty}^{\infty} f(x)\, e^{-ipx}\, dx$$

and $$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(p)\, e^{ipx}\, dp.$$

(i) We have

$$f(x) = \frac{2}{\pi} \int_0^\infty \sin(px) \left(\int_0^\infty f(t) \sin(pt)\, dt \right) dp$$

Taking $F_s(p) = \int_0^\infty f(t) \sin(pt)\, dt$, then we get

$$f(x) = \frac{2}{\pi} \int_0^\infty \sin(px)\, F_s(p)\, dp.$$

Here $F_s(p)$ is the infinite Fourier sine transform (or, Fourier sine transform) of (x) and $f(x)$ is called the inverse Fourier sine transform of $F_s(p)$.

(ii) we have

$$f(x) = \frac{2}{\pi} \int_0^\infty \cos(px) \left(\int_0^\infty f(t) \cos(pt)\, dt \right) dp$$

Taking $F_c(p) = \int_0^\infty f(t) \cos(pt)\, dt$, then we get

$$f(x) = \frac{2}{\pi} \int_0^\infty \cos(px) \cdot F_c(p)\, dp$$

The function $F_c(p)$ is called the infinite Fourier cosine transform (or, Fourier cosine transform) of $f(x)$ and $f(x)$ is called the inverse Fourier cosine transform of $F_c(p)$.

Problems:

1. Find the Fourier transform of

$$f(x) = \begin{cases} 1 & \text{if } |x| < a \\ 0 & \text{if } |x| > a \end{cases} \qquad (a>0)$$

and hence evaluate

(i) $\displaystyle\int_{-\infty}^{\infty} \frac{\sin(ap)\cos(px)}{p}\,dp$

and (ii) $\displaystyle\int_{0}^{\infty} \frac{\sin p}{p}\,dp$

Sol: we have

$$F\{f(x)\} = F(p) = \int_{-\infty}^{\infty} e^{ipx} f(x)\,dx$$

$$= \int_{-\infty}^{-a} e^{ipx}(0)\,dx + \int_{-a}^{a} e^{ipx}(1)\,dx + \int_{a}^{\infty} e^{ipx}(0)\,dx$$

$$= \int_{-a}^{a} e^{ipx}\,dx = \left[\frac{e^{ipx}}{ip}\right]_{-a}^{a} = \frac{e^{ipa} - e^{-ipa}}{ip}$$

$$= \frac{2i\sin pa}{ip} = \frac{2\sin(pa)}{p}$$

Therefore, Fourier transform of $f(x)$ is

$$F(P) = \frac{2\sin(pa)}{p}.$$

(i) By the inverse Fourier transform of $F(P)$ is $f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(P)\,e^{-ipx}\,dp$

Therefore, we have

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} F(P)\,e^{-ipx}\,dp = \begin{cases} 1 & \text{if } |x|<a \\ 0 & \text{if } |x|>a \end{cases}$$

$$\Rightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2\sin(pa)}{p}\left(\cos(px) - i\sin(px)\right)dp = \begin{cases} 1 & \text{if } |x|<a \\ 0 & \text{if } |x|>a \end{cases}$$

$$\Rightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2\sin(pa)}{p}\cos(px)\,dp = \begin{cases} 1 & \text{if } |x|<a \\ 0 & \text{if } |x|>a \end{cases}$$

$$\left(\text{since } \int_{-\infty}^{\infty} \underbrace{\frac{\sin(pa)}{p}\sin(px)}_{\text{odd}}\,dp = 0\right)$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\sin(pa)}{p} \cos(px)\, dp = \begin{cases} 1 & \text{if } |x| < a \\ 0 & \text{if } |x| > a \end{cases}$$

(ii) Taking $x = 0$ and $a = 1$, then we get

$$\int_{-\infty}^{\infty} \frac{\sin p}{p} dp = \pi$$

or, $2\int_0^{\infty} \frac{\sin p}{p} dp = \pi$ (since $\frac{1}{p} \cdot \sin p$ is odd)

Hence $\boxed{\int_0^{\infty} \frac{\sin p}{p} dp = \frac{\pi}{2}}$

2. Find the Fourier transform of $e^{-a|x|}$ $(a > 0)$ and hence show that

(i) $\int_0^{\infty} \frac{\cos(px)}{a^2 + p^2} dp = \frac{\pi}{2a} e^{-a|x|}$

and (ii) $\int_0^{\infty} \frac{1}{a^2 + p^2} dp = \frac{\pi}{2a}$

Answer: Fourier transform of $f(x)$ is

$$F\{f(x)\} = F(p) = \frac{2a}{a^2 + p^2}.$$

3. Obtain the Fourier transform of $f(x) = \begin{cases} 1-x^2 & \text{if } |x| \le 1 \\ 0 & \text{if } |x| > 1 \end{cases}$

hence evaluate

(i) $\int_0^\infty \left(\frac{x\cos x - \sin x}{x^3}\right) \cos\left(\frac{x}{2}\right) dx$

and (ii) $\int_0^\infty \frac{x\cos x - \sin x}{x^3} dx$.

Sol: Fourier transform of $f(x)$ is

$$F(p) = \int_{-\infty}^{\infty} e^{ipx} f(x)\, dx$$

$$= \int_{-1}^{1} e^{ipx} (1-x^2)\, dx$$

$$= \int_{-1}^{1} ((\cos px + i\sin px)(1-x^2)\, dx$$

$$= \int_{-1}^{1} \cos px\, (1-x^2)\, dx$$

(since $\sin px\,(1-x^2)$ is odd, $\int_{-1}^{1} \sin px\,(1-x^2)\,dx = 0$)

$$= 2\left\{\int_0^1 \cos px\, dx - \int_0^1 x^2 \cos px\, dx\right\}$$

$$= 2\left\{\left[\frac{\sin px}{p}\right]_0^1 - \left(\left[x^2\,\frac{\sin px}{p}\right]_0^1 - \left[2x\left(\frac{-\cos px}{p^2}\right)\right]_0^1 + \left[2\left(\frac{-\sin px}{p^3}\right)\right]_0^1\right)\right\}$$

$$= 2\left\{\frac{\sin p}{p} - \left(\frac{\sin p}{p} + \frac{2}{p^2}(\cos p) - \frac{2}{p^3}\sin p\right)\right\}$$

$$F[p] = \frac{4}{p^3}\left(\sin p - p\cos p\right)$$

By the inverse Fourier transform of $F(p)$, we have

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ipx} F(p)\,dp$$

or, $$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ipx}\left(\frac{4}{p^3}[\sin p - p\cos p]\right)dp = \begin{cases} 1-x^2 & \text{if } |x|\le 1 \\ 0 & \text{if } |x|>1 \end{cases} \longrightarrow (1)$$

(i) Taking $x = \frac{1}{2}$ in (1), we get

$$\frac{4}{2\pi}\int_{-\infty}^{\infty}\left(\cos(p/2) - i\sin(p/2)\right)\left(\frac{\sin p - p\cos p}{p^3}\right)dp = 1 - \frac{1}{4}$$

$$\Rightarrow \int_{-\infty}^{\infty}\left(\frac{\sin p - p\cos p}{p^3}\right)\cos\left(\frac{p}{2}\right)dp = \frac{3\pi}{8}$$

$$\Rightarrow 2\int_{0}^{\infty}\left(\frac{\sin p - p\cos p}{p^3}\right)\cos\left(\frac{p}{2}\right)dp = \frac{3\pi}{8}$$

Hence, $$2\int_{0}^{\infty}\left(\frac{\sin x - x\cos x}{x^3}\right)\cos\left(\frac{x}{2}\right)dx = \frac{3\pi}{8}$$

or, $$\boxed{\int_0^\infty \left(\frac{x\cos x - \sin x}{x^3}\right)\cos\left(\frac{x}{2}\right) dx = -\frac{3\pi}{16}}$$

(ii) Taking $x=0$ in ①, we get

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{4}{p^3}(\sin p - p\cos p)\, dp = 1$$

or, $$\int_{-\infty}^{\infty} \left(\frac{\sin p - p\cos p}{p^3}\right) dp = \frac{\pi}{2}$$

$$\Rightarrow 2\int_0^\infty \left(\frac{\sin p - p\cos p}{p^3}\right) dp = \frac{\pi}{2}$$

Hence, $$\boxed{\int_0^\infty \left(\frac{\sin x - x\cos x}{x^3}\right) dx = \frac{\pi}{4}}$$

4. obtain the Fourier transform of

$$f(x) = \frac{1}{\sqrt{|x|}}.$$

Sol: we have

$$F\{f(x)\} = F(p) = \int_{-\infty}^{\infty} e^{ipx} f(x)\, dx$$

$$= \underbrace{\int_{-\infty}^{0} e^{ipx} \frac{1}{\sqrt{(-x)}}\, dx}_{I_1} + \underbrace{\int_0^\infty e^{ipx} \frac{1}{\sqrt{x}}\, dx}_{I_2} \rightarrow (*)$$

Now,

$$I_1 = \int_{-\infty}^{0} e^{ipx} \frac{1}{\sqrt{(-x)}} dx$$

Taking $-x = t$, we get $dx = -dt$

and $x \to 0,\ t \to 0$

$x \to -\infty,\ t \to \infty$

So, $$I_1 = \int_{\infty}^{0} e^{-ipt} \frac{1}{\sqrt{t}} \cdot -dt$$

$$= \int_{0}^{\infty} e^{-ipt} \frac{1}{\sqrt{t}} dt$$

Therefore, from (*), we have

$$F(p) = \int_{0}^{\infty} e^{-ipx} \frac{1}{\sqrt{x}} dx + \int_{0}^{\infty} \frac{1}{\sqrt{x}} e^{ipx} dx$$

$$= \int_{0}^{\infty} \left(\frac{e^{-ipx} + e^{ipx}}{\sqrt{x}} \right) dx$$

$$= 2 \int_{0}^{\infty} \frac{\cos px \, dx}{\sqrt{x}}$$

$$= 2 \text{ Real Part of } \int_0^\infty \frac{e^{-ipx}}{\sqrt{x}} \frac{d(ipx)}{ip}$$

$$= 2 \text{ R.P. of } \int_0^\infty \frac{e^{-ipx}}{\sqrt{ipx}} \frac{d(ipx)}{\sqrt{ip}}$$

$$\left(e^{-ipx} = \underbrace{\cos px}_{\substack{\text{Real part} \\ (\cos px)}} - i \underbrace{\sin px}_{\substack{\text{imag. part} \\ (-\sin px)}} \right.$$

$$= 2 \text{ R.P. of } \frac{1}{\sqrt{ip}} \int_0^\infty \frac{e^{-ipx}}{\sqrt{ipx}} d(ipx)$$

$$= 2 \text{ R.P. of } \frac{1}{\sqrt{ip}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} du$$

Taking $ipx = u$, then $ip\,dx = du$ or, $d(ipx) = du$ and $x \to 0, u \to 0$; $x \to \infty, u \to \infty$

$$= \frac{2}{\sqrt{p}} \text{ R.P. of } i^{-1/2} \int_0^\infty e^{-u} u^{\frac{1}{2}-1} du$$

$$= \frac{2}{\sqrt{p}} \text{ R.P. of } \left(\cos\frac{\pi}{2} + i \sin\frac{\pi}{2}\right)^{-1/2} \Gamma\tfrac{1}{2}$$

$$= \frac{2}{\sqrt{p}} \text{ R.P. of } \left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)\sqrt{\pi}$$

$$= \frac{2}{\sqrt{p}} \left(\cos\frac{\pi}{4}\right)\sqrt{\pi} = \frac{2\sqrt{\pi}}{\sqrt{p}} \cdot \frac{1}{\sqrt{2}} = \sqrt{\frac{2\pi}{p}}$$

5. Find the Fourier transform of

$$f(x) = \begin{cases} \sin x & \text{if } 0 < x < \pi \\ 0 & \text{otherwise} \end{cases}$$

<u>Answer</u>: $F\{f(x)\} = F(p) = \int_{-\infty}^{\infty} e^{ipx} f(x)\,dx$

$$= \int_0^{\pi} e^{ipx} \sin x \, dx$$

$$= \frac{e^{ipx}}{(ip)^2 + 1^2} \Big[ip \sin x - 1 \cdot \cos x \Big] \Big]_0^{\pi}$$

$$= \frac{e^{ip\pi}(0+1) - 1(0-1)}{-p^2+1}$$

$$= \frac{1 + e^{ip\pi}}{1-p^2}$$

6. Find the Fourier transform of

$$f(x) = x e^{-x} \quad (0 < x < \infty)$$

<u>Answer</u>: $F\{f(x)\} = F(p) = \dfrac{(1+ip)^2}{(1+p^2)^2}$