

System (1) can be written in matrix form as

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

ie $\boxed{AX = b}$

here, $A_{m \times n}$ is coeff. matrix, $X_{n \times 1}$ is called solution vector, $b_{m \times 1}$ is called rhs vector.

if $b = \mathbf{0}$ (null vector)

$$b = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

ie $AX = \mathbf{0}$ is homogenous system

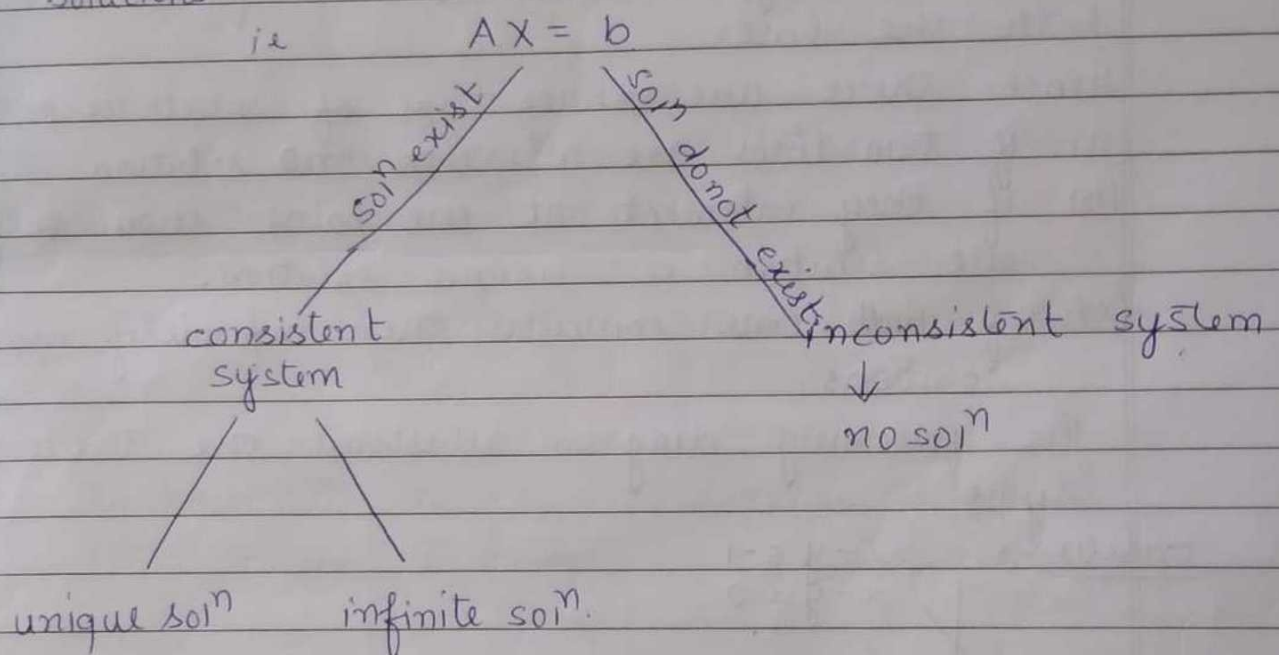
if $b \neq \mathbf{0}$ (null vector)

ie $AX = b$ is called non-homogenous system.

The We want to check, whether the system (1) of linear equation has a solution or not.

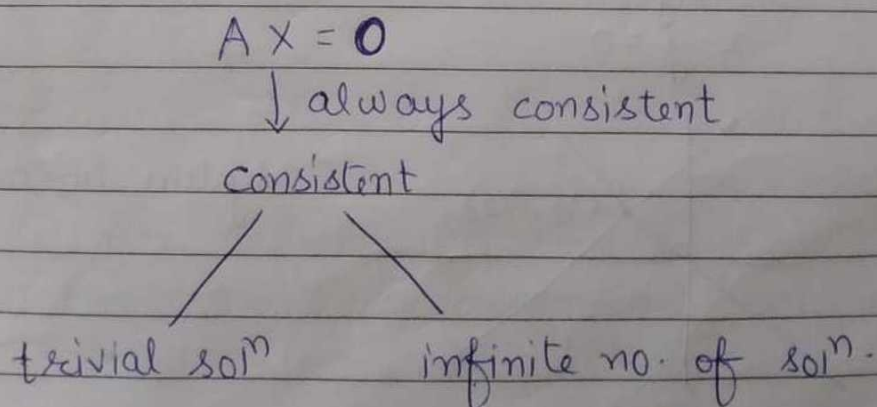
f :- consistent:- A system of linear equations is said to be consistent if it has at least one solution and inconsistent if it has no solution.

* Any nonhomogenous system $AX=b$, may be consistent or inconsistent. If it is consistent it may have unique solⁿ or infinite no. of solutions



is always consistent and

* Any homogenous system, always has at least one solution ie $x_1 = x_2 = \dots = x_n = 0$, called trivial solution.



Let us consider a particular case

when $m=n=2$

Ex - The system of 2 linear equations in two unknowns

$$a_1x + b_1y = c_1$$

$$a_2x + b_2y = c_2$$

A point $P(x, y)$ is a solution iff it lies on both the line.

Hence there are three type of solutions possible

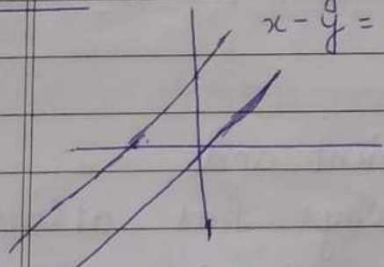
- (a) if two lines are $\parallel$ then no solution
- (b) if they intersect at one point then only one solution i.e. unique solution.
- (c) if both line coincide, then infinitely many solutions

the following diagram illustrate the three type

case (I) →

$$x - y = -1$$

$$x - y = 0$$

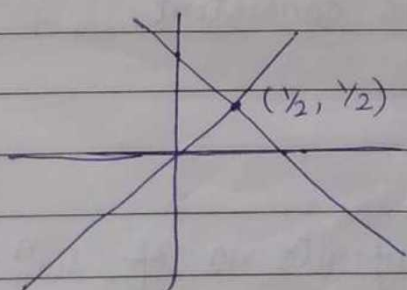


→ system has no solution.

case II:-

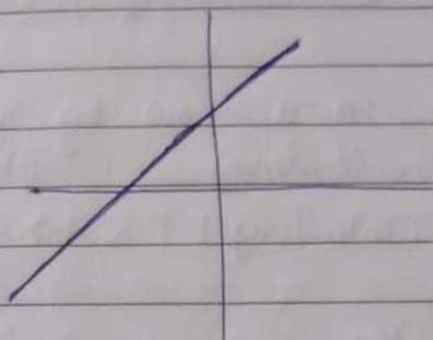
$$x + y = 1$$

$$x - y = 0$$



→ system has unique solⁿ

Case III:- $x - y = -1$
 $2x - 2y = -2$



infinitely many solutions

* In case (I) both lines have same slope
 i.e. if $-\frac{a_1}{b_1} = -\frac{a_2}{b_2}$

or $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$, then no solution

* In case IInd $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$, slope is different.
 i.e. unique solⁿ

* In case IIIrd $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$, i.e. both lines
 are same, then infinitely many solution.

Solving a linear system:-

The basic idea to develop a systematic procedure for solving linear system is to replace the system with an equivalent system which is easier to solve.

To change the given system into a simpler system, keeping the solution unchanged, the following three operations, called elementary ^{row} operations are used:-

- (1) Multiply all entries in a row by non zero constant (Scaling) $(R_i \rightarrow KR_i)$
- (2) interchange two rows (Interchange) $(R_i \leftrightarrow R_j)$
- (3) Replace one row by the sum of itself and multiple of another row. $(R_i \rightarrow R_i + KR_j)$
(Replacement)

Augmented matrix:-

In system (1), consider a matrix.

a_{11}	a_{12}	...	a_{1n}	b_1
a_{21}	a_{22}		a_{2n}	b_2
a_{31}	a_{32}		a_{3n}	b_3
...	...		...	...
a_{m1}	a_{m2}		a_{mn}	b_m

called augmented matrix for the system.

Defⁿ

Two matrices are said to be row-equivalent if one can be transformed to the other by a sequence of elementary row operations.

- * If a matrix B can be obtained from a matrix A , in this way, we can obviously recover A from B , by applying inverse elementary row operations in reverse order, so the elementary row operations will not change the solution of the system.

Theo^m

If two systems of linear equations are row equivalent, then they have the same set of solutions.

Echelon form or Row Echelon form

A rectangular matrix is in echelon form if it has following three properties:-

- (1) All non-zero rows are above any zero row
- (2) Each leading entry of a row is ~~in~~ a right to the leading entry of the row above it. (first nonzero entry of a row is called leading entry)
- (3) All entries in a column below a leading entry are zeros.

Ex:-

$$\begin{bmatrix} \boxed{2} & -3 & 2 & 1 \\ 0 & \boxed{1} & -4 & 8 \\ 0 & 0 & \boxed{0} & 5/2 \end{bmatrix}$$

* Echelon form is also called triangular form.

Reduced Echelon form or Row reduced echelon form:-

If a matrix in echelon form satisfy the following additional conditions, then it is in reduced echelon form:-

- (4) The leading entry in each nonzero row is 1.
- ~~Each leading entry is 1~~
- (5) Each column that contains a leading 1

has zeros everywhere else in that column.

examples of echelon and reduced echelon form

$$(i) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

reduced echelon form

$$(ii) \begin{bmatrix} 1 & 1 & 0 & 1 & 1 \\ 0 & 2 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}$$

echelon form

$$(iii) \begin{bmatrix} 1 & 3 & -4 & 5 \\ 0 & 1 & 7 & 2 \\ 0 & 0 & 1 & 6 \end{bmatrix}$$

echelon form

$$(iv) \begin{bmatrix} 0 & 1 & -3 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

reduced echelon form

$$(v) \begin{bmatrix} 0 & 1 & 3 & 5 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

echelon form

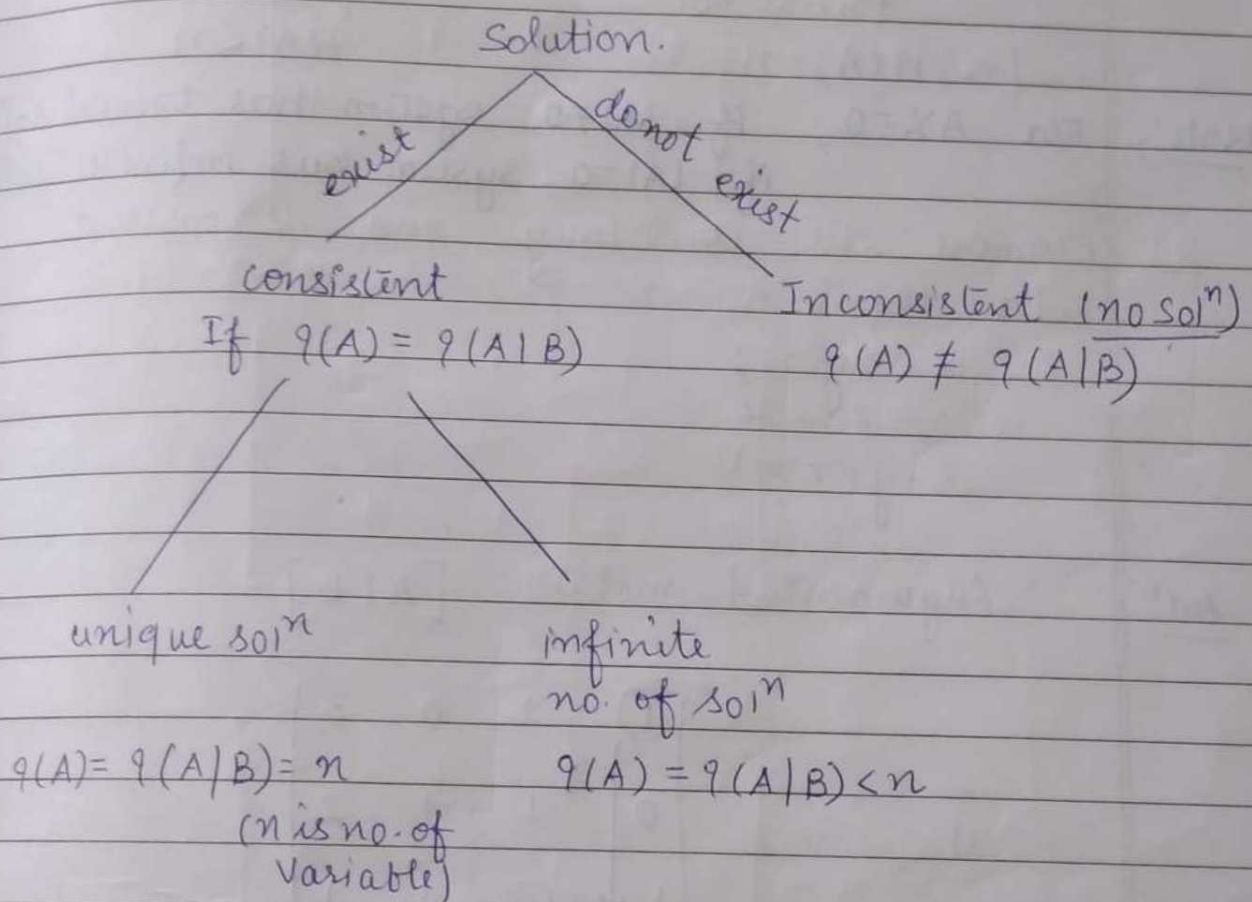
Pivot position:-

A pivot position in a matrix A , is a location that corresponds to a leading 1 in the reduced echelon form of A . A pivot column is a column that contains a pivot position.

Rank of a matrix:-

The no. of non-zero rows in a echelon form of matrix A is called rank of a matrix A . denoted by $\rho(A)$

* Rank of a matrix represents no. of linearly independent rows.
[linear system of eqⁿs]
 $AX=B$



Note:- for $AX=B$

if $|A| \neq 0$, then system has unique solⁿ.

if $|A| = 0$, system may have no solⁿ or infinite many solⁿ

For homogenous system of equations

$$AX = 0$$

↓ always consistent

consistent (solⁿ exist)

trivial solⁿ

$$\rho(A) = n$$

infinite solⁿ.

$$\rho(A) < n$$

note! For $AX=0$, if $|A| \neq 0$, system has trivial solⁿ.
if $|A| = 0$ system has infinite solⁿ.

Q: Examine the consistency and if consistent solve them.

$$x + 2y = 3$$

$$y - z = 2$$

$$x + y + z = 1$$

solⁿ:

Augmented matrix. $[A|b]$

$$= \left[\begin{array}{ccc|c} \textcircled{1} & 2 & 0 & 3 \\ 0 & 1 & -1 & 2 \\ 1 & 1 & 1 & 1 \end{array} \right]$$

$$R_3 \rightarrow R_3 - R_1$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 0 & \textcircled{1} & -1 & 2 \\ 0 & -1 & 1 & -2 \end{array} \right]$$

$$R_3 \rightarrow R_3 + R_2$$

$$\begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\rightarrow$ echelon form.

$$\rho(A) = 2$$

$$\rho(A|B) = 2$$

$\rho(A) = \rho(A|B)$ 'ie system is consistent.

$$\rho(A) = \rho(A|B) = 2 < 3 (\text{no. of unknown})$$

$$n=3$$

ie system has infinite no. of solutions.

to find solution, we write the system in matrix form as

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$$

$$\left. \begin{array}{l} x + 2y = 3 \\ y - z = 2 \end{array} \right\} \begin{array}{l} 2 \text{ eqn} \\ 3 \text{ unknown,} \\ 3 \text{ variable.} \end{array}$$

$$(3 - 2) = 1 \text{ free variable}$$

say $z = k$.

$$y = 2 + z = 2 + k$$

$$x = 3 - 2y = 3 - 2(2 + k)$$

$$= 3 - 4 - 2k = -1 - 2k$$

$$[x = -1 - 2k, y = 2 + k, z = k] \text{ infinite no. of sol}^n$$

Q: check for consistency and hence find the solution.

$$\begin{aligned} 3x + 3y + 2z &= 4 \\ x + 2y - z &= 4 \\ x + 3y + 2z &= 4 \end{aligned}$$

$$\begin{aligned} x + y + z &= 3 \\ x + 2y - z &= 4 \\ x + 3y + 2z &= 4 \end{aligned}$$

Soln.

Augmented matrix

$$[A|b] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & -1 & 4 \\ 1 & 3 & 2 & 4 \end{array} \right]$$

$$R_2 \rightarrow R_2 - R_1 \quad R_3 \rightarrow R_3 - R_1$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & -2 & 1 \\ 0 & 2 & 1 & 1 \end{array} \right]$$

$$R_3 \rightarrow R_3 - 2R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 5 & -1 \end{array} \right]$$

$$\rho(A) = 3$$

$$\rho(A|B) = 3$$

$\rho(A) = \rho(A|B)$ ie system is consistent
 $\rho(A) = \rho(A|B) = 3 = \text{no. of unknowns } (n=3)$
 ie unique soln.

writing it in matrix form

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$$

$$\boxed{z = -1/5}$$

$$5z = -1 \Rightarrow \boxed{z = -1/5}$$

$$y - 2z = 1$$

$$y - 2(-1/5) = 1 \Rightarrow y =$$

$$y - 2z = 1$$

$$y = 1 + 2z$$

$$\boxed{y = 3/5}$$

$$x + y + z = 3$$

$$x = 3 - y - z$$

$$\boxed{x = 13/5}$$

$$x = 13/5, y = 3/5, z = -1/5$$

∴ Solve the homogenous system

$$5x + 2y - 3z = 0$$

$$3x + y + z = 0$$

$$2x + y + 6z = 0$$

Q:- Test for consistency and solve

$$2x - 3y + 7z = 5, \quad 3x + y - 3z = 13, \quad 2x + 19y - 47z = 32$$

Soln

$$[A|b] = \begin{bmatrix} 2 & -3 & 7 & 5 \\ 3 & 1 & -3 & 13 \\ 2 & 19 & -47 & 32 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - \frac{3}{2}R_1, \quad R_3 \rightarrow R_3 - R_1$$

$$\sim \begin{bmatrix} 2 & -3 & 7 & 5 \\ 0 & \frac{11}{2} & -\frac{27}{2} & \frac{11}{2} \\ 0 & 22 & -54 & 27 \end{bmatrix}$$

$$R_2 \rightarrow 2R_2$$

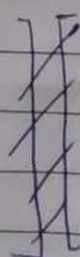
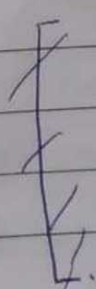
$$\sim \begin{bmatrix} 2 & -3 & 7 & 5 \\ 0 & (11) & -27 & 11 \\ 0 & 22 & -54 & 27 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - 2R_2$$

$$\sim \begin{bmatrix} 2 & -3 & 7 & 5 \\ 0 & 11 & -27 & 11 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

which is echelon form.

i.e.



$$\rho(A) = 2$$

$$\rho(A|B) = 3$$

$$\rho(A) \neq \rho(A|B)$$

no solution

Let us write down in matrix form

$$\begin{bmatrix} 2 & -3 & 7 \\ 0 & 11 & -27 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \\ 5 \end{bmatrix}$$

last eqⁿ

$$0 \cdot z = 5$$

$$0 = 5 \text{ (not possible)}$$

ie no solⁿ.

system is inconsistent.

Gauss elimination method to solve system of linear equations:-

- (1) write the system of equations in matrix form $AX=B$
- (2) Form Augmented matrix $= [A | B]$
- (3) Using elementary row operations, convert the system's Augmented matrix into row echelon form.
- (4) Now use back substitution to find solution.

Gauss Jordan method to solve system of linear equation:-

- (5) after reducing augmented matrix to row-echelon form (after step 3), reduce it to row reduced echelon form. Hence find the solⁿ, this method is called Gauss Jordan elimination method.

Example of Gauss elimination when unique sol exists!

2.1 Solve the system, using gauss elimination method

$$2y + 4z = 2$$

$$x + 2y + 2z = 3$$

$$3x + 4y + 6z = -1$$

Solⁿ. Augmented matrix $[A|B]$ is

$$\left[\begin{array}{ccc|c} 0 & 2 & 4 & 2 \\ 1 & 2 & 2 & 3 \\ 3 & 4 & 6 & -1 \end{array} \right]$$

Since the coeff. of x in first eqⁿ is zero, we interchange eqⁿ ① and ② or rows ① & ②

$$R_1 \leftrightarrow R_2$$

$$\left[\begin{array}{ccc|c} \textcircled{1} & 2 & 2 & 3 \\ 0 & 2 & 4 & 2 \\ 3 \downarrow & 4 & 6 & -1 \end{array} \right]$$

$$R_3 \rightarrow R_3 - 3R_1$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 3 \\ 0 & \textcircled{2} & 4 & 2 \\ 0 & -2 \downarrow & 0 & -10 \end{array} \right]$$

$$R_3 \rightarrow R_3 + R_2$$

$$\left[\begin{array}{ccc|c} \textcircled{1} & 2 & 2 & 3 \\ 0 & \textcircled{2} & 4 & 2 \\ 0 & 0 & \textcircled{4} & -8 \end{array} \right]$$

① → Pivot of first row

② → " " second row

④ → " " 3rd row

row echelon form.

divide each row by the pivot of that row

(optional)
$$\begin{bmatrix} 1 & 2 & 2 & | & 3 \\ 0 & 1 & 2 & | & 1 \\ 0 & 0 & 1 & | & -2 \end{bmatrix} \rightarrow \text{row echelon form}$$

we have

$$\begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix} \rightarrow \begin{aligned} x + y + z &= 3 \\ y + 2z &= 1 \\ z &= -2 \end{aligned}$$

using back substitution

$$\boxed{z = -2}$$

Note: $\begin{cases} \text{no free variable} \\ \Rightarrow \text{unique sol}^n \end{cases}$

$$y + 2z = 1$$

$$y + 2(-2) = 1$$

$$\boxed{y = 5}$$

$$x + 2y + 2z = 3$$

$$x = 3 - 2y - 2z$$

$$x = 3 - 2(5) + 4$$

$$\boxed{x = -3}$$

$$\boxed{x = -3}$$

we see that the system has unique solution.

ex of Gauss elimination if infinitely many solⁿ exist! / /

Q2 solve the linear system.

$$3x_1 + 5x_2 - 4x_3 = 7$$

$$-3x_1 - 2x_2 + 4x_3 = -1$$

$$6x_1 + x_2 - 8x_3 = -4$$

solⁿ Aug matrix $[A|B] = \left[\begin{array}{ccc|c} \textcircled{3} & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & -8 & -4 \end{array} \right]$

$$R_2 \rightarrow R_2 + R_1, R_3 \rightarrow R_3 - 2R_1$$

$$\sim \left[\begin{array}{ccc|c} 3 & 5 & -4 & 7 \\ 0 & \textcircled{3} & 0 & 6 \\ 0 & -9 & 0 & -18 \end{array} \right]$$

$$R_3 \rightarrow R_3 + 3R_2$$

$$\sim \left[\begin{array}{ccc|c} 3 & 5 & -4 & 7 \\ 0 & 3 & 0 & 6 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

row echelon form

we have

$$3x_1 + 5x_2 - 4x_3 = 7$$

$$3x_2 = 6$$

$$0 = 0$$

2 eqⁿ, 3 unknowns

check Substitution: $\rightarrow (3-2)=1$ free variable, say $x_3 = K$ (arbitrary)

$$\boxed{x_2 = 2}$$

$$3x_1 + 5x_2 - 4x_3 = 7$$

$$3x_1 = 7 - 5x_2 + 4x_3 = 7 - 5 \times 2 + 4K = -3 + 4K$$

$$\begin{bmatrix} x_1 = -1 + \frac{4}{3}k \\ x_2 = 2 \\ x_3 = k \end{bmatrix}$$

infinitely many solution.
k is arbitrary.

Gauss elimination if no solution exists:-

Q3 solve the system of eqⁿ.

$$3x_1 + 2x_2 + x_3 = 3$$

$$2x_1 + x_2 + x_3 = 0$$

$$6x_1 + 2x_2 + 4x_3 = 6$$

Solⁿ:-

$$\begin{bmatrix} \textcircled{3} & 2 & 1 & 3 \\ 2 & 1 & 1 & 0 \\ 6 & 2 & 4 & 6 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - \frac{2}{3}R_1, \quad R_3 \rightarrow R_3 - 2R_1$$

$$\sim \begin{bmatrix} 3 & 2 & 1 & 3 \\ 0 & -\frac{1}{3} & \frac{1}{3} & -2 \\ 0 & 0 & 0 & 12 \end{bmatrix}$$

row echelon form

back substitution

$$\begin{bmatrix} 3 & 2 & 1 \\ 0 & -\frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 12 \end{bmatrix}$$

$0 \cdot z = 12 \Rightarrow 0 = 12$ (not possible)
ie system has no solution.

Remarks:-

If after converting the system into row echelon form, if system has no free variable then unique soln.

✓ If in row echelon form, a row of the type $[0 \ 0 \ \dots \ 0 \ | \ b]$, with $b \neq 0$ represents an eqⁿ

$0x_1 + 0x_2 + \dots + 0x_n = b$, in this case system has no solⁿ.

Q:- solve the system of eqⁿ using Gauss-Jordan method

$$2y + 4z = 2$$

$$x + 2y + 2z = 3$$

$$3x + 4y + 6z = -1$$

$$[A|B] = \begin{bmatrix} 0 & 2 & 4 & | & 2 \\ 1 & 2 & 2 & | & 3 \\ 3 & 4 & 6 & | & -1 \end{bmatrix}$$

$$R_1 \leftrightarrow R_2$$

$$\sim \begin{bmatrix} \textcircled{1} & 2 & 2 & | & 3 \\ 0 & 2 & 4 & | & 2 \\ 3 & 4 & 6 & | & -1 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - 3R_1$$

$$\sim \begin{bmatrix} 1 & 2 & 2 & | & 3 \\ 0 & \textcircled{2} & 4 & | & 2 \\ 0 & -2 & 0 & | & -10 \end{bmatrix}$$

$$R_3 \rightarrow R_3 + R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 2 & 3 \\ 0 & 2 & 4 & 2 \\ 0 & 0 & 4 & -8 \end{array} \right]$$

row echelon form

Now to we ~~we~~ find reduced row echelon form
we use following steps:-

$$R_2 \rightarrow R_2/2 \quad R_4 \rightarrow R_4/4 \quad (\text{to make pivot of each row 1})$$

$$\sim \left[\begin{array}{ccc|c} \textcircled{1} & 2 & 2 & 3 \\ 0 & \textcircled{1} & 2 & 1 \\ 0 & 0 & \textcircled{1} & -2 \end{array} \right]$$

$$x + 2y + 2z = 3$$

$$\rightarrow y + 2z = 1$$

$$z = -2$$

$$R_2 \rightarrow R_2 - 2R_3, \quad R_1 \rightarrow R_1 - 2R_3$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 7 \\ 0 & \textcircled{1} & 0 & 5 \\ 0 & 0 & \textcircled{1} & -2 \end{array} \right]$$

$$x + 2y = 7$$

$$\rightarrow y = 5$$

$$z = -2$$

$$R_1 \rightarrow R_1 - 2R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & -2 \end{array} \right]$$

$$x = -3$$

$$y = 5$$

$$z = -2$$

this is row reduced echelon form

$$\text{i.e. } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 \\ 5 \\ -2 \end{bmatrix}$$

$$[x = -3, y = 5, z = -2]$$

Q:- Solve by Gauss-Jordan method

$$x_1 + 3x_2 - 2x_3 = 3$$

$$2x_1 + 6x_2 - 2x_3 + 4x_4 = 18$$

$$x_2 + x_3 + 3x_4 = 10$$

Soln:-

$$\begin{bmatrix} \boxed{1} & 3 & -2 & 0 & | & 3 \\ 2 & 6 & -2 & 4 & | & 18 \\ 0 & 1 & 1 & 3 & | & 10 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - 2R_1$$

$$\sim \begin{bmatrix} 1 & 3 & -2 & 0 & | & 3 \\ 0 & 0 & 2 & 4 & | & 12 \\ 0 & 1 & 1 & 3 & | & 10 \end{bmatrix}$$

Note that coeff of x_2 in second row is ~~not~~ zero but in the third row it is not. Thus we interchange second and third row.

$$R_2 \longleftrightarrow R_3$$

$$\sim \begin{bmatrix} \boxed{1} & 3 & -2 & 0 & | & -3 \\ 0 & \boxed{1} & 1 & 3 & | & 10 \\ 0 & 0 & \boxed{2} & 4 & | & 12 \end{bmatrix}$$

→ row echelon form

the pivot in third row is 2, i.e. divide Row 3 by 2 to make pivot 1

$$R_3 \rightarrow R_3/2$$

$$\sim \left[\begin{array}{cccc|c} 1 & 3 & -2 & 0 & 3 \\ 0 & 1 & 1 & 3 & 10 \\ 0 & 0 & 1 & 2 & 6 \end{array} \right]$$

to ~~reach~~ obtain reduced row echelon form, we proceed as follows:-

$$R_2 \rightarrow R_2 - R_3$$

$$R_1 \rightarrow R_1 + 2R_3$$

$$\sim \left[\begin{array}{cccc|c} 1 & 3 & 0 & 4 & 15 \\ 0 & 1 & 0 & 1 & 4 \\ 0 & 0 & 1 & 2 & 6 \end{array} \right]$$

$$R_1 \rightarrow R_1 - 3R_2$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 3 \\ 0 & 1 & 0 & 1 & 4 \\ 0 & 0 & 1 & 2 & 6 \end{array} \right]$$

this is row-reduced echelon form

$$\text{ie } x_1 + x_4 = 3$$

$$x_2 + x_4 = 4$$

$$x_3 + 2x_4 = 6$$

$\rightarrow 3 \text{ eq}^n, 4 \text{ unknowns}$

say x_4 is free variable, $x_4 = K$ (arbitrary)

$$x_1 = 3 - K$$

$$x_2 = 4 - K$$

$$x_3 = 6 - 2K$$

$$x_4 = K$$

$\rightarrow$ infinite many solutions

or solution set

$$(x_1, x_2, x_3, x_4) = (3-k, 4-k, 6-2k, k), k \in \mathbb{R}$$

k is arb.

Note: If no. of equations are less than the no. of variables in a linear system, then the system has either no solⁿ or infinitely many solutions.

* If no. of eqⁿ is greater than or equal to the no. of variable then the system may have unique solⁿ or no solⁿ or infinitely many solutions.

Q:- Solve the following systems using Gauss-elimination method

$$\begin{aligned} \text{(a)} \quad & 4x - 3y - 9z + 6w = 0 \\ & 2x + 3y + 3z + 6w = 6 \\ & 4x - 2y - 39z - 6w = -24 \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad & x + 2y - 2z = 1 \\ & 2x - 3y + z = 0 \\ & 5x + y - 5z = 1 \\ & 3x + 14y - 12z = 5 \end{aligned}$$

Q. 6 Solve by Gauss-Jordan method

$$x_1 + 3x_2 - 2x_3 + 2x_5 = 0$$

$$2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 = 1$$

$$5x_3 + 10x_4 + 15x_6 = 5$$

$$2x_1 + 6x_2 + 8x_4 + 4x_5 + 18x_6 = 6$$

Aug. matrix

Soln:

$$\left[\begin{array}{cccccc|c} \textcircled{1} & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & 1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 2\checkmark & 6 & 0 & 8 & 4 & 18 & 6 \end{array} \right]$$

$$R_2 \rightarrow R_2 - 2R_1, \quad R_3 \rightarrow R_3 - 2R_1$$

$$\left[\begin{array}{cccccc|c} \textcircled{1} & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & \textcircled{-1} & -2 & 0 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 0 & 4\checkmark & 8 & 0 & 18 & 6 \end{array} \right]$$

$$R_3 \rightarrow R_3 + 5R_2, \quad R_4 \rightarrow R_4 + 4R_2$$

$$\left[\begin{array}{cccccc|c} \textcircled{1} & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & \textcircled{-1} & -2 & 0 & -3 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \end{array} \right]$$

$$R_3 \leftrightarrow R_4$$

$$\left[\begin{array}{cccccc|c} \textcircled{1} & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & \textcircled{-1} & -2 & 0 & -3 & -1 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

row echelon form

$$R_2 \rightarrow R_2 - R_1 \quad R_3 \rightarrow R_3/6$$

$$\sim \left[\begin{array}{ccccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

$$R_2 \rightarrow R_2 - 3R_3$$

$$\sim \left[\begin{array}{ccccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

$$R_1 \rightarrow R_1 + 2R_2$$

$$\left[\begin{array}{ccccccc|c} 1 & 3 & 0 & 4 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

corresponding system of eqⁿ is

$$x_1 + 3x_2 + 4x_4 + 2x_5 = 0$$

$$x_3 + 2x_4 = 0$$

$$x_6 = \frac{1}{3}$$

3 eqⁿ, 6 unknown

$(6-3) = 3$ free variable

Say x_2, x_4, x_5 are free variables

$$x_1 = -3x_2 - 4x_4 - 2x_5$$

$$x_3 = -2x_4$$

$$x_6 = \frac{1}{3}$$

$$\text{Say } x_2 = K_1$$

$$x_4 = K_2$$

$$x_5 = K_3$$

$$x_1 = -3K_1 - 4K_2 - 2K_3$$

$$x_2 = K_1$$

$$x_3 = -2K_2$$

$$x_4 = K_2$$

$$x_5 = K_3$$

$$x_6 = 1/3$$

What conditions must b_1, b_2, b_3 must satisfy in order to be consistent for

$$x_1 + 2x_2 + 3x_3 = b_1$$

$$2x_1 + 5x_2 + 3x_3 = b_2$$

$$x_1 + 8x_3 = b_3, \text{ to be consistent.}$$

Determine the values of λ for which the following equations are consistent. Also, solve the system for these values of λ .

$$x + 2y + z = 3$$

$$x + y + z = \lambda$$

$$3x + y + 3z = \lambda^2$$

Investigate for what values of λ and μ the equations

$$x + y + z = 6$$

$$x + 2y + 3z = 10$$

$$x + 2y + \lambda z = \mu$$

have

① no solution ② unique solⁿ ③ infinite solⁿ

Q:- Find the values of k , so that the equations
 $x + y + 3z = 0$
 $4x + 3y + kz = 0$
 $2x + y + 2z = 0$
 have non-trivial solution. Also find solution.

Solⁿ

$$AX = 0$$

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 4 & 3 & k \\ 2 & 1 & 2 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - 4R_1, \quad R_3 \rightarrow R_3 - 2R_1$$

$$\sim \begin{bmatrix} 1 & 1 & 3 \\ 0 & -1 & k-12 \\ 0 & -1 & -4 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 3 \\ 0 & -1 & k-12 \\ 0 & 0 & 8-k \end{bmatrix}$$

For non-trivial solⁿ

$$R(A) < 3 \quad (n)$$

n is no of variable

ie for non-trivial solⁿ

$$8-k = 0$$

$$\boxed{k=8}$$

also when $k \neq 8$

system has trivial solⁿ.

Solution

$$\begin{bmatrix} 1 & 1 & 3 \\ 0 & -1 & k-12 \\ 0 & 0 & 8-k \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$k=8$$

$$\begin{bmatrix} 1 & 1 & 3 \\ 0 & -1 & -4 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-y - 4z = 0$$

$$x + y + 3z = 0$$

2 eqⁿs, 3 variables.
i.e. 1 free variable.

Say. $z = x$

$$-y = 4x \Rightarrow \boxed{y = -4x}$$

$$x = -y - 3z$$

$$x = 4x - 3x$$

$$\boxed{x = x, y = -4x, z = x} \quad x \in \mathbb{R}, x \neq 0$$

Q:- Investigate for what values of λ and μ the equations

$$x + 2y + z = 8$$

$$2x + 2y + 2z = 13$$

$$3x + 4y + \lambda z = \mu$$

have

(i) no solⁿ (ii) a unique solⁿ (iii) many solⁿ

Solⁿ

Aug. matrix $[A|B] = \begin{bmatrix} 1 & 2 & 1 & | & 8 \\ 2 & 2 & 2 & | & 13 \\ 3 & 4 & \lambda & | & \mu \end{bmatrix}$

$$R_2 \rightarrow R_2 - 2R_1, \quad R_3 \rightarrow R_3 - 3R_1$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & | & 8 \\ 0 & -2 & 0 & | & -3 \\ 0 & -2 & \lambda-3 & | & \mu-24 \end{bmatrix}$$

Elementary matrices:-

$$R_3 \rightarrow R_3 - R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 0 & -2 & 0 & -3 \\ 0 & 0 & \lambda-3 & \mu-21 \end{array} \right]$$

(i) if $\lambda=3$ and $\mu \neq 21$, ~~say~~

$$R(A)=2, R(A|B)=3$$

$$\text{i.e. } R(A) \neq R(A|B)$$

i.e. system is inconsistent and has no solution.

(ii) if $\lambda \neq 3$ and μ has any value

$$R(A)=3 \quad R(A|B)=3$$

i.e. $R(A) = R(A|B) = \text{no. of unknown}$

i.e. system is consistent and has unique solⁿ.

(iii) if $\lambda=3$ & $\mu=21$

$$R(A)=2 \quad R(A|B)=2$$

i.e. $R(A) = R(A|B) < n$,

system is consistent and has infinite solutions.

Q:- Determine the values of λ for which the following equations are consistent. Also solve the system for these values.

$$x+2y+z=3$$

$$x+y+z=\lambda$$

$$3x+y+3z=\lambda^2$$

Echelon form is

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & 0 & 3-\lambda \\ 0 & 0 & 0 & \lambda^2-5\lambda+6 \end{array} \right]$$

equation will be consistent if.
 $\lambda^2-5\lambda+6=0$ i.e. $\lambda=3$ or $\lambda=2$

Case (i):- when $\lambda=3$, we have

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow y=0 \text{ \& } x+2y+z=0$$

1 free variable.

Say z is free variable, $z=K$.
 $x = 3-K$, $y=0$, $z=K$ i.e. system has infinite solⁿ.

Case (ii):- when $\lambda=2$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right] \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$$

$$y=1, \quad x+2y+z=3$$

Say z is free variable i.e. $z=K$

$$x = 3-2y-z \Rightarrow x = 3-2-1-K$$

$$x = 1-K$$

Solⁿ is

$$x=1-K, \quad y=1, \quad z=K$$

i.e. system has infinite solutions.