Module:5 Multiple integrals

- Evaluation of double integrals
- Change of order of integration
- Change of variables between Cartesian and polar co-ordinates
- Evaluation of triple integrals
- Change of variables between Cartesian and cylindrical and spherical co-ordinates
- Evaluation of multiple integrals using gamma and beta functions

Evaluation of double integrals

double integral of f over R, written as

$$\iint_R f(x, y)\, dA \qquad \text{or} \qquad \iint_R f(x, y)\, dx\, dy.$$

THEOREM If $f(x, y)$ is continuous throughout the rectangular region R: $a \le x \le b$, $c \le y \le d$, then

$$\iint_R f(x, y)\, dA = \int_c^d \int_a^b f(x, y)\, dx\, dy = \int_a^b \int_c^d f(x, y)\, dy\, dx.$$

EXAMPLE Calculate $\iint_R f(x, y)\, dA$ for

$$f(x, y) = 100 - 6x^2y \qquad \text{and} \qquad R: \quad 0 \le x \le 2, \quad -1 \le y \le 1.$$

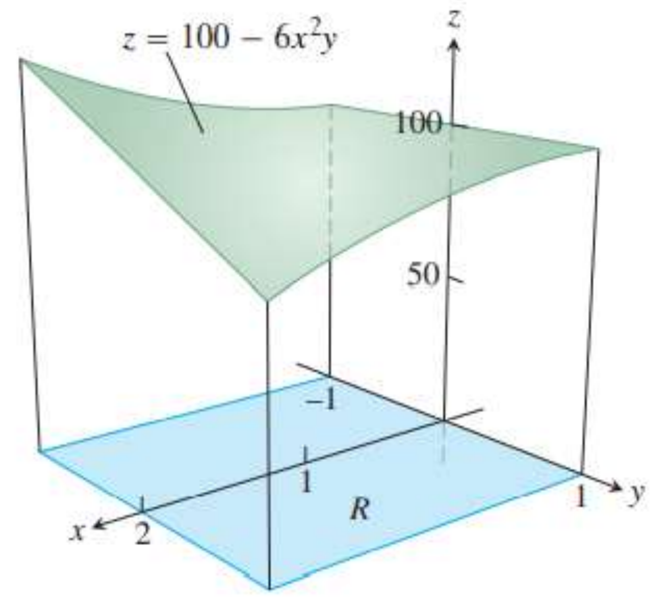
Solution

$$\iint_R f(x, y)\, dA = \int_{1}^{1} \int_0^2 (100 - 6x^2y)\, dx\, dy$$

$$= \int_{1}^{1} \Big[100x - 2x^3y\Big]_{x=0}^{x=2} dy$$

$$= \int_{-1}^{1} (200 - 16y)\, dy$$

$$= \Big[200y - 8y^2\Big]_{-1}^{1} = 400.$$



The double integral $\iint_R f(x, y)\, dA$ gives the volume under this surface over the rectangular region R

Reversing the order of integration gives the same answer:

$$\int_0^2 \int_{-1}^{1} (100 - 6x^2y)\, dy\, dx = \int_0^2 \Big[100y - 3x^2y^2\Big]_{y=-1}^{y=1} dx$$

$$= \int_0^2 [(100 - 3x^2) - (-100 - 3x^2)]\, dx$$

$$= \int_0^2 200\, dx = 400.$$

Evaluate the following integrals

Answer 1)16 and 2) 14

1) $$\int_0^3 \int_0^2 (4 - y^2)\, dy\, dx$$

2) $$\iint_R (6y^2 - 2x)\, dA, \qquad R: \quad 0 \le x \le 1, \quad 0 \le y \le 2$$

Find the volume of the region bounded above by the paraboloid $z = x^2 + y^2$ and below by the square $R: -1 \le x \le 1$, $-1 \le y \le 1$.

Answer 8/3

THEOREM Let $f(x, y)$ be continuous on a region R.

1. If R is defined by $a \leq x \leq b$, $g_1(x) \leq y \leq g_2(x)$, with g_1 and g_2 continuous on $[a, b]$, then

$$\iint_R f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx.$$

2. If R is defined by $c \leq y \leq d$, $h_1(y) \leq x \leq h_2(y)$, with h_1 and h_2 continuous on $[c, d]$, then

$$\iint_R f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy.$$

Finding Limits of Integration

Using Vertical Cross-sections When faced with evaluating $\iint_R f(x, y)\, dA$, integrating first with respect to y and then with respect to x, do the following three steps:

1. *Sketch*. Sketch the region of integration and label the bounding curves
2. *Find the y-limits of integration*. Imagine a vertical line L cutting through R in the direction of increasing y. Mark the y-values where L enters and leaves. These are the y-limits of integration and are usually functions of x (instead of constants)
3. *Find the x-limits of integration*. Choose x-limits that include all the vertical lines through R. The integral

$$\iint_R f(x, y)\, dA$$

Using Horizontal Cross-sections To evaluate the same double integral as an iterated integral with the order of integration reversed, use horizontal lines instead of vertical lines in Steps 2 and 3. The integral is

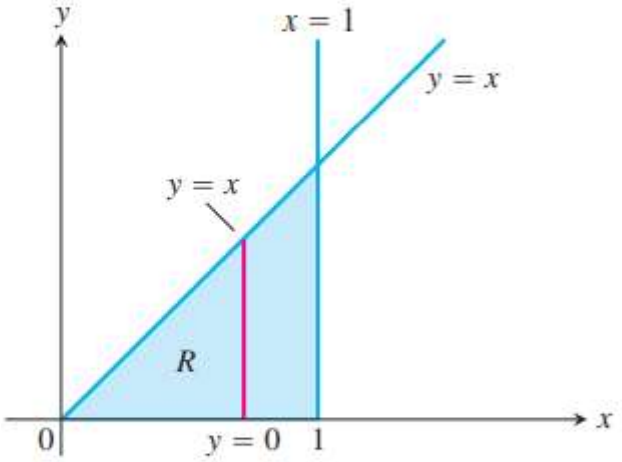
$$\iint_R f(x, y)\, dA = \iint_R f(x, y)\, dx\, dy.$$

EXAMPLE Find the volume of the prism whose base is the triangle in the xy-plane bounded by the x-axis and the lines $y = x$ and $x = 1$ and whose top lies in the plane

$$z = f(x, y) = 3 - x - y.$$

$$V = \int_0^1 \int_0^x (3 - x - y)\, dy\, dx = \int_0^1 \left[3y - xy - \frac{y^2}{2} \right]_{y=0}^{y=x} dx$$

$$= \int_0^1 \left(3x - \frac{3x^2}{2} \right) dx = \left[\frac{3x^2}{2} - \frac{x^3}{2} \right]_{x=0}^{x=1} = 1.$$

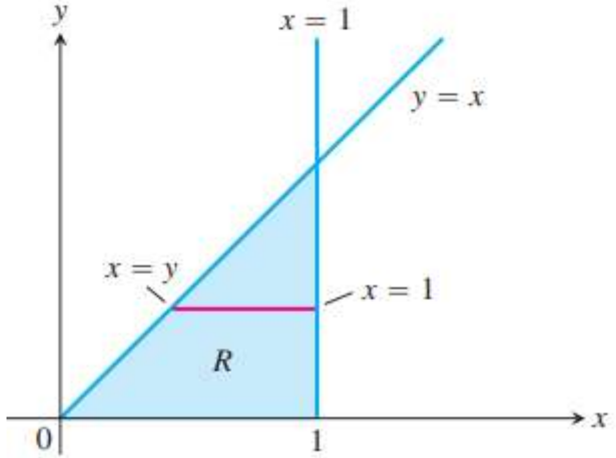


When the order of integration is reversed , the integral for the volume is

$$V = \int_0^1 \int_y^1 (3 - x - y)\, dx\, dy = \int_0^1 \left[3x - \frac{x^2}{2} - xy \right]_{x=y}^{x=1} dy$$

$$= \int_0^1 \left(3 - \frac{1}{2} - y - 3y + \frac{y^2}{2} + y^2 \right) dy$$

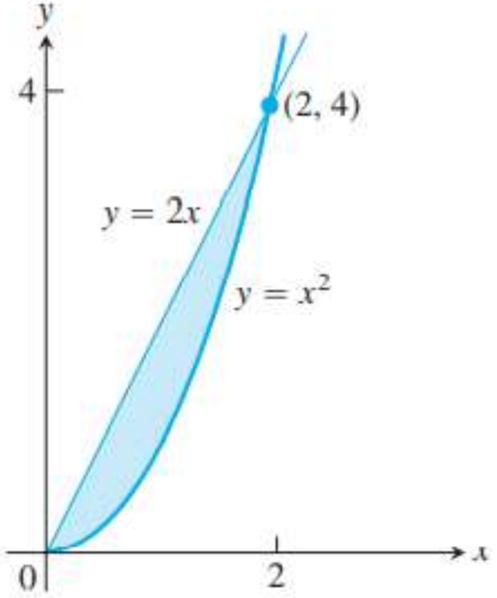
$$= \int_0^1 \left(\frac{5}{2} - 4y + \frac{3}{2} y^2 \right) dy = \left[\frac{5}{2} y - 2y^2 + \frac{y^3}{2} \right]_{y=0}^{y=1} = 1.$$



Sketch the region of integration for the integral

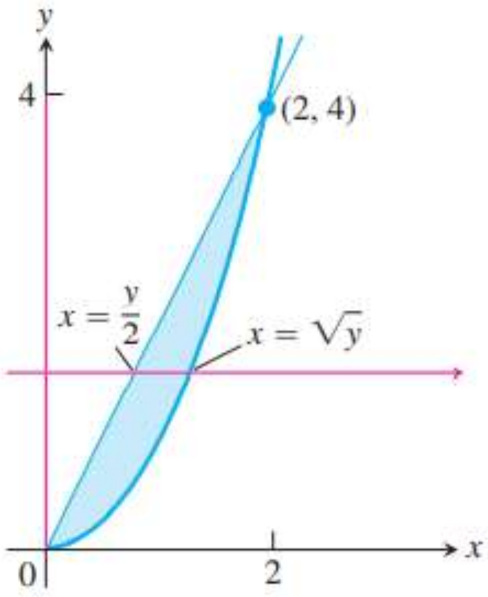
$$\int_0^2 \int_{x^2}^{2x} (4x + 2)\, dy\, dx$$

and write an equivalent integral with the order of integration reversed.



$$\int_0^4 \int_{y/2}^{\sqrt{y}} (4x + 2)\, dx\, dy.$$

integrals is 8.



Solution The region of integration is given by the inequalities $x^2 \le y \le 2x$ and $0 \le x \le 2$. It is therefore the region bounded by the curves $y = x^2$ and $y = 2x$ between $x = 0$ and $x = 2$

To find limits for integrating in the reverse order, we imagine a horizontal line passing from left to right through the region. It enters at $x = y/2$ and leaves at $x = \sqrt{y}$. To include all such lines, we let y run from $y = 0$ to $y = 4$

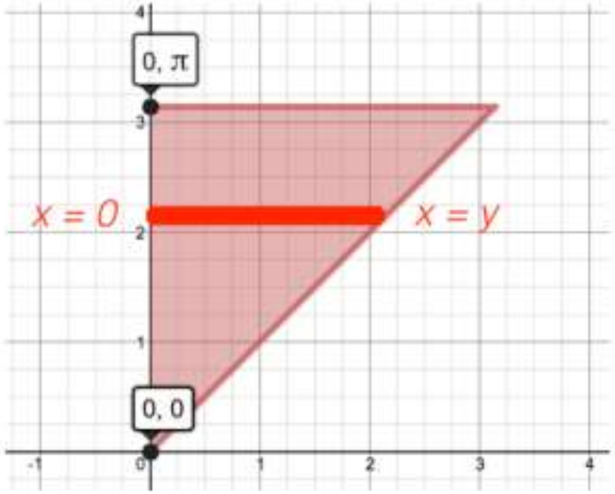
The integral is

$$\int_0^4 \int_{y/2}^{\sqrt{y}} (4x + 2)\, dx\, dy = \int_0^4 (2x^2 + 2x)\Big|_{y/2}^{\sqrt{y}}\, dy$$

$$= \int_0^4 (y + 2\sqrt{y} - y^2/2)\, dy$$

$$= 8$$

2. Sketch the region of integration, reverse the order of integration, and evaluate the integral $\int_0^{\pi}\int_x^{\pi} \frac{\sin y}{y}\, dy\, dx$



The region of Integration can be defined as

$$R = \{(x,y) \mid x \leq y \leq \pi,\ 0 \leq x \leq \pi\}$$

Sketch of the region is shown :

Now the region by using horizontal cross-sections

After reversing the order of integration we get

$$\int_0^{\pi}\int_0^{y} \frac{\sin y}{y}\, dx\, dy = \int_0^{\pi} \left[\frac{x \sin y}{y}\right]_0^{y} dy$$

$$= \int_0^{\pi} \sin y\, dy$$

$$= \Big[-\cos y\Big]_0^{\pi}$$

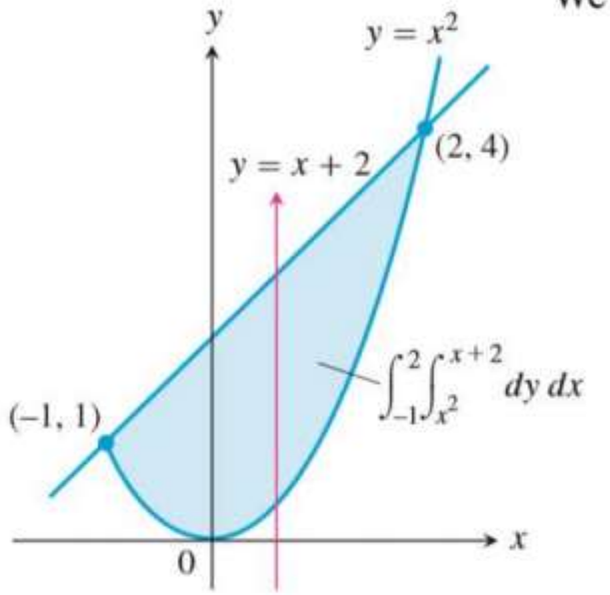
$$= -\cos(\pi) + \cos(0)$$

$$= 1 + 1 = 2$$

DEFINITION The **area** of a closed, bounded plane region R is

$$A = \iint_R dA.$$

Find the area of the region R enclosed by the parabola $y = x^2$ and the line $y = x + 2$?



We sketch the region and calculate the area as

$$\begin{aligned} A &= \iint_R dA \\ &= \int_{-1}^{2} \Big[y \Big]_{x^2}^{x+2} dx \\ &= \int_{-1}^{2} (x + 2 - x^2)\, dx \\ &= \left[\frac{x^2}{2} + 2x - \frac{x^3}{3} \right]_{-1}^{2} \\ &= \frac{9}{2} \end{aligned}$$

Double Integrals in Polar Form

$$\iint_R f(r,\theta)\,dA = \int_{\theta=\alpha}^{\theta=\beta}\int_{r=g_1(\theta)}^{r=g_2(\theta)} f(r,\theta)\,r\,dr\,d\theta.$$

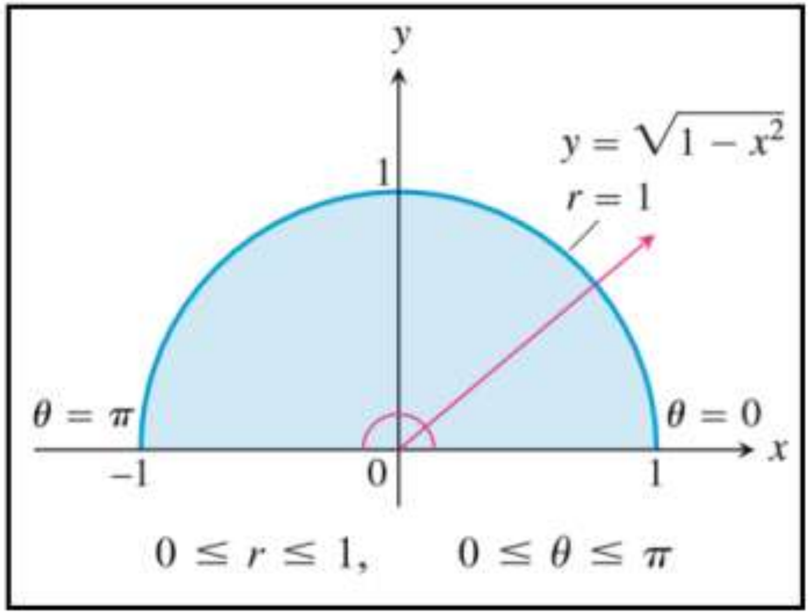
Finding Limits of Integration

The procedure for finding limits of integration in rectangular coordinates also works for polar coordinates. To evaluate $\iint_R f(r,\theta)\,dA$ over a region R in polar coordinates, integrating first with respect to r and then with respect to θ, take the following steps.

1. *Sketch*. Sketch the region and label the bounding curves
2. *Find the r-limits of integration*. Imagine a ray L from the origin cutting through R in the direction of increasing r. Mark the r-values where L enters and leaves R. These are the r-limits of integration. They usually depend on the angle θ that L makes with the positive x-axis
3. *Find the θ-limits of integration*. Find the smallest and largest θ-values that bound R. These are the θ-limits of integration . The polar iterated integral is

$$\iint_R f(r,\theta)\,dA = \int_{\theta=\alpha}^{\theta=\beta}\int_{r=g_1(\theta)}^{r=g_2(\theta)} f(r,\theta)\,r\,dr\,d\theta.$$

Evaluate $\iint_R e^{x^2+y^2}\,dy\,dx$, where R is the semicircular region bounded by the X-axis and the curve $y=\sqrt{1-x^2}$ by using polar coordinates



Substituting $x = r\cos\theta, y = r\sin\theta$ and replacing $dy\,dx$ by $r\,dr\,d\theta$ enables to evaluate

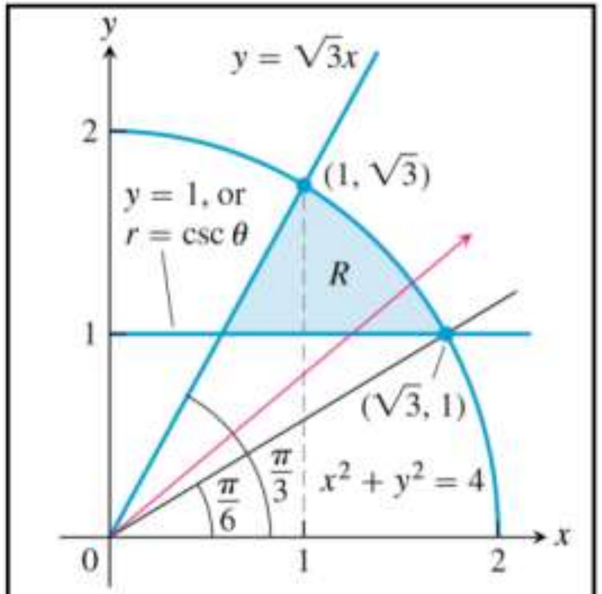
$$\iint_R e^{x^2+y^2}\,dy\,dx = \int_0^{\pi}\int_0^1 e^{r^2}\,r\,dr\,d\theta$$

$$= \int_0^{\pi}\left[\frac{1}{2}e^{r^2}\right]_0^1 d\theta$$

$$= \int_0^{\pi}\frac{1}{2}(e-1)\,d\theta$$

$$= \frac{\pi}{2}(e-1)$$

Using polar integration, find the area of the region R in the xy – plane enclosed by the circle $x^2 + y^2 = 4$, above the line y = 1 and below the line $y = \sqrt{3}\ x$

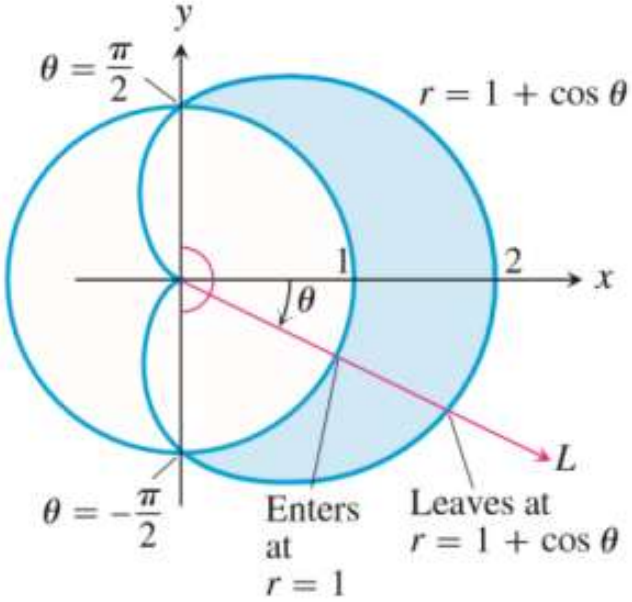


θ varies from $\pi/6$ to $\pi/3$
r varies from $r = \csc\theta$ to $r = 2$.

$y = \sqrt{3}x$ has slope $\sqrt{3} = \tan\theta$, so $\theta = \pi/3$. Next the line $y = 1$ intersects the circle $x^2 + y^2 = 4$ when $x^2 + 1 = 4$, or $x = \sqrt{3}$.
giving its angle of inclination as $\theta = \pi/6$.
In the polar coordinate r varies from the horizontal line $y = 1$ to the circle $x^2 + y^2 = 4$.

$$\begin{aligned}
\iint_R dA &= \int_{\pi/6}^{\pi/3}\int_{\csc\theta}^{2} r\,dr\,d\theta \\
&= \int_{\pi/6}^{\pi/3}\left[\frac{1}{2}r^2\right]_{r=\csc\theta}^{r=2} d\theta \\
&= \int_{\pi/6}^{\pi/3}\frac{1}{2}\left[4 - \csc^2\theta\right] d\theta \\
&= \frac{1}{2}\left[4\theta + \cot\theta\right]_{\pi/6}^{\pi/3} \\
&= \frac{1}{2}\left(\frac{4\pi}{3} + \frac{1}{\sqrt{3}}\right) - \frac{1}{2}\left(\frac{4\pi}{6} + \sqrt{3}\right) = \frac{\pi - \sqrt{3}}{3}.
\end{aligned}$$

Find the area of the region that lies inside the cardioid $r = 1 + \cos\theta$ and outside the circle $r = 1$.



r-limits of integration $r = 1$ and leaves where $r = 1 + \cos\theta$

θ-limits of integration from $\theta = -\pi/2$ to $\theta = \pi/2$

$$A = 2\left\{\int_0^{\pi/2}\int_1^{1+\cos\theta} r\,dr\,d\theta\right\}$$

$$= 2\int_0^{\pi/2} [(1+\cos\theta)^2 - 1]\,d\theta$$

$$= 2\int_0^{\pi/2} (2\cos\theta + \cos^2\theta)\,d\theta$$

$$= \int_0^{\pi/2} 4\cos\theta\,d\theta + \int_0^{\pi/2} 2.\tfrac{1}{2}(1+\cos 2\theta)\,d\theta$$

$$= 4\Big[\sin\theta\Big]_0^{\pi/2} + \Big[\theta + \tfrac{1}{2}\sin 2\theta\Big]_0^{\pi/2}$$

$$= 4\left[\sin\tfrac{1}{2}\pi - \sin 0\right] + \left[\left(\tfrac{1}{2}\pi + \tfrac{1}{2}\sin\pi\right) - \left(0 + \tfrac{1}{2}\sin 0\right)\right]$$

$$= 4[1] + \quad [\tfrac{1}{2}\pi]$$

$$= 4 + \tfrac{1}{2}\pi$$

Triple Integration

$$\boxed{\iiint\limits_D F(x, y, z)\, dx\, dy\, dz.}$$

DEFINITION The **volume** of a closed, bounded region D in space is

$$V = \iiint\limits_D dV.$$

Evaluate the integrals

1. $\displaystyle\int_0^1 \int_0^1 \int_0^1 (x^2 + y^2 + z^2)\, dz\, dy\, dx$ **Ans 1**

2. $\displaystyle\int_1^e \int_1^{e^2} \int_1^{e^3} \frac{1}{xyz}\, dx\, dy\, dz$ **Ans 6**

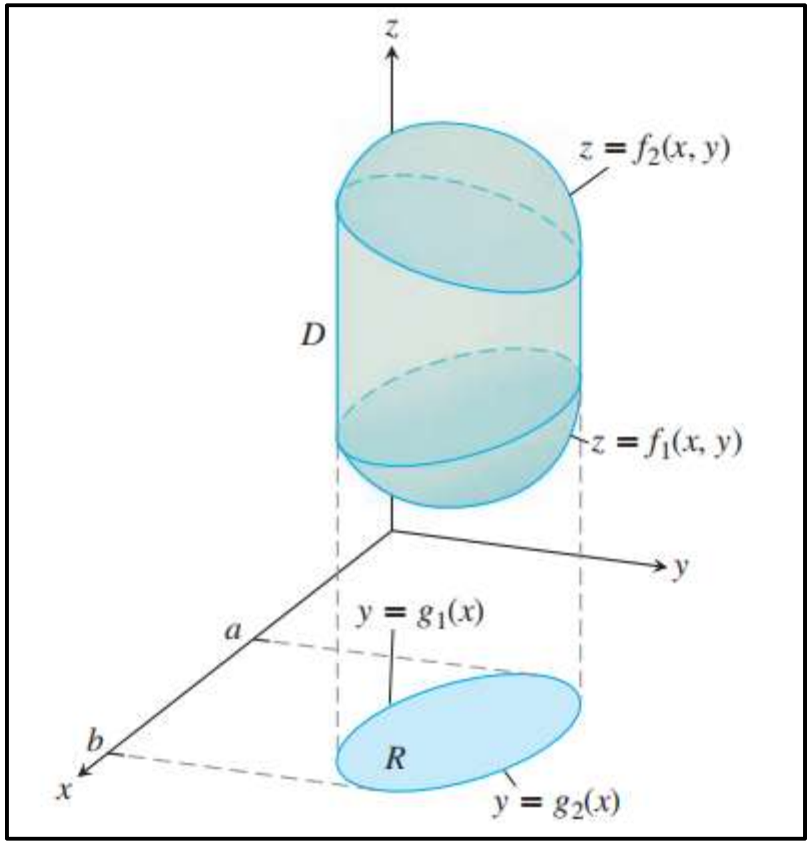
3. $\displaystyle\int_0^3 \int_0^{\sqrt{9-x^2}} \int_0^{\sqrt{9-x^2}} dz\, dy\, dx$ **Ans 18**

Finding Limits of Integration in the Order *dz dy dx*

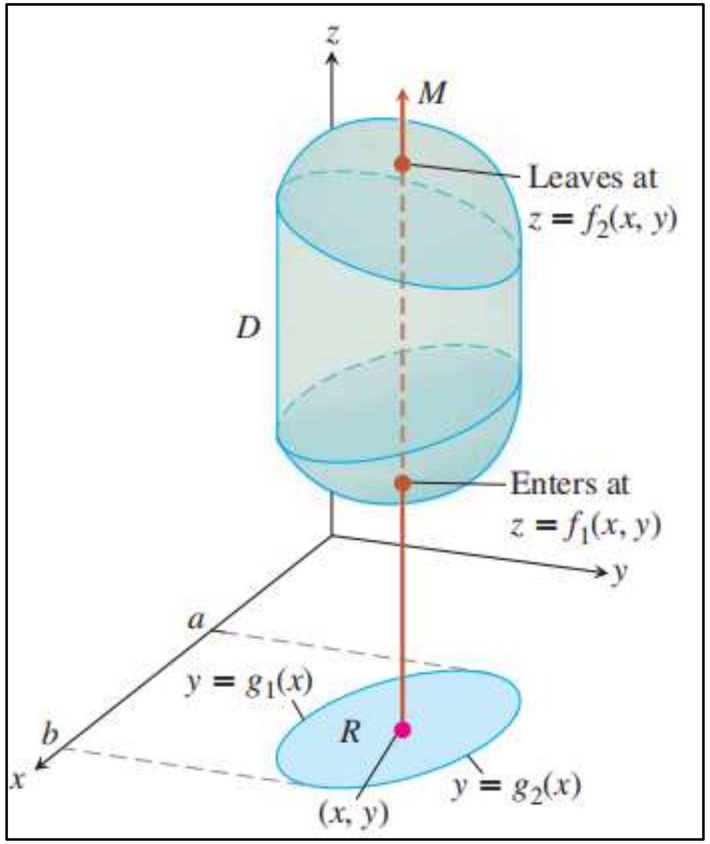
To evaluate $$\iiint_D F(x, y, z)\, dV$$

over a region D, integrate first with respect to z, then with respect to y, and finally with respect to x.

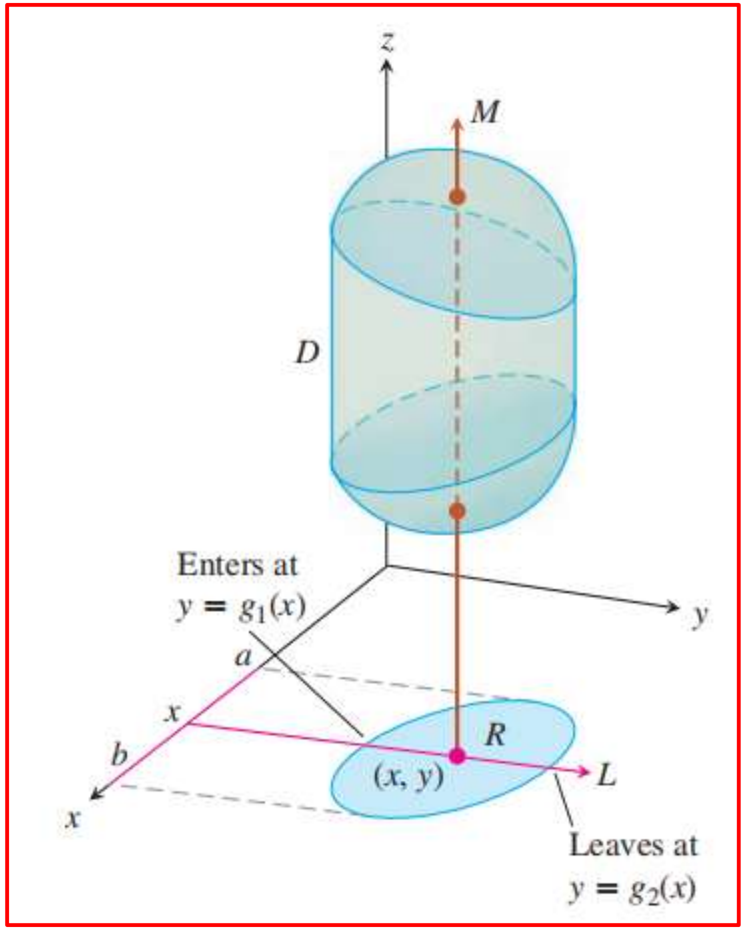
1. *Sketch*. Sketch the region D along with its "shadow" R (vertical projection) in the xy-plane. Label the upper and lower bounding surfaces of D and the upper and lower bounding curves of R.



2. *Find the z-limits of integration*. Draw a line M passing through a typical point (x, y) in R parallel to the z-axis. As z increases, M enters D at $z = f_1(x, y)$ and leaves at $z = f_2(x, y)$. These are the z-limits of integration.



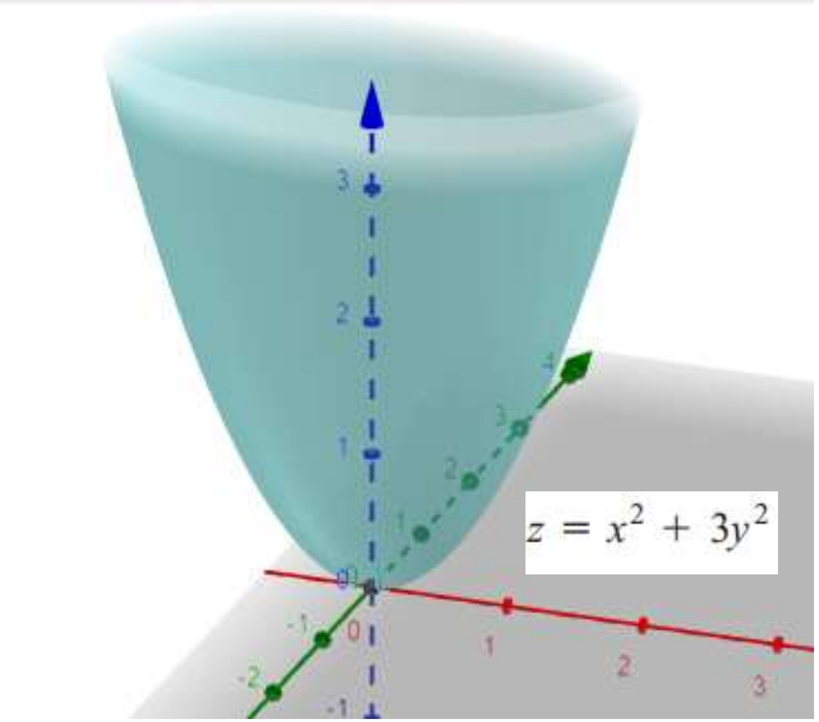
3. *Find the y-limits of integration*. Draw a line L through (x, y) parallel to the y-axis. As y increases, L enters R at $y = g_1(x)$ and leaves at $y = g_2(x)$. These are the y-limits of integration.

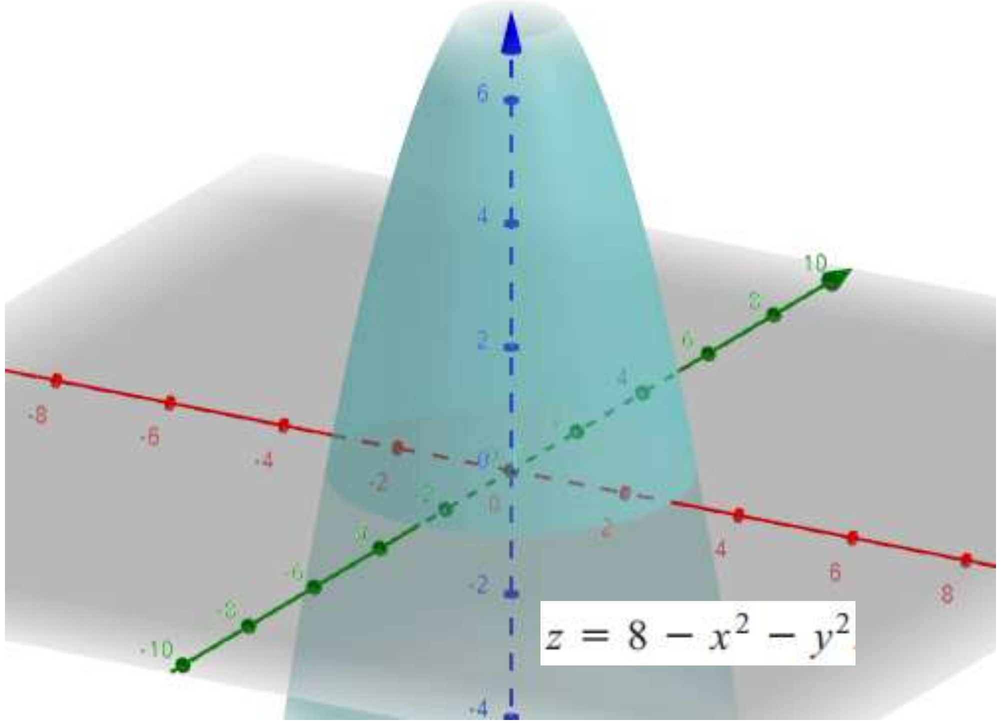


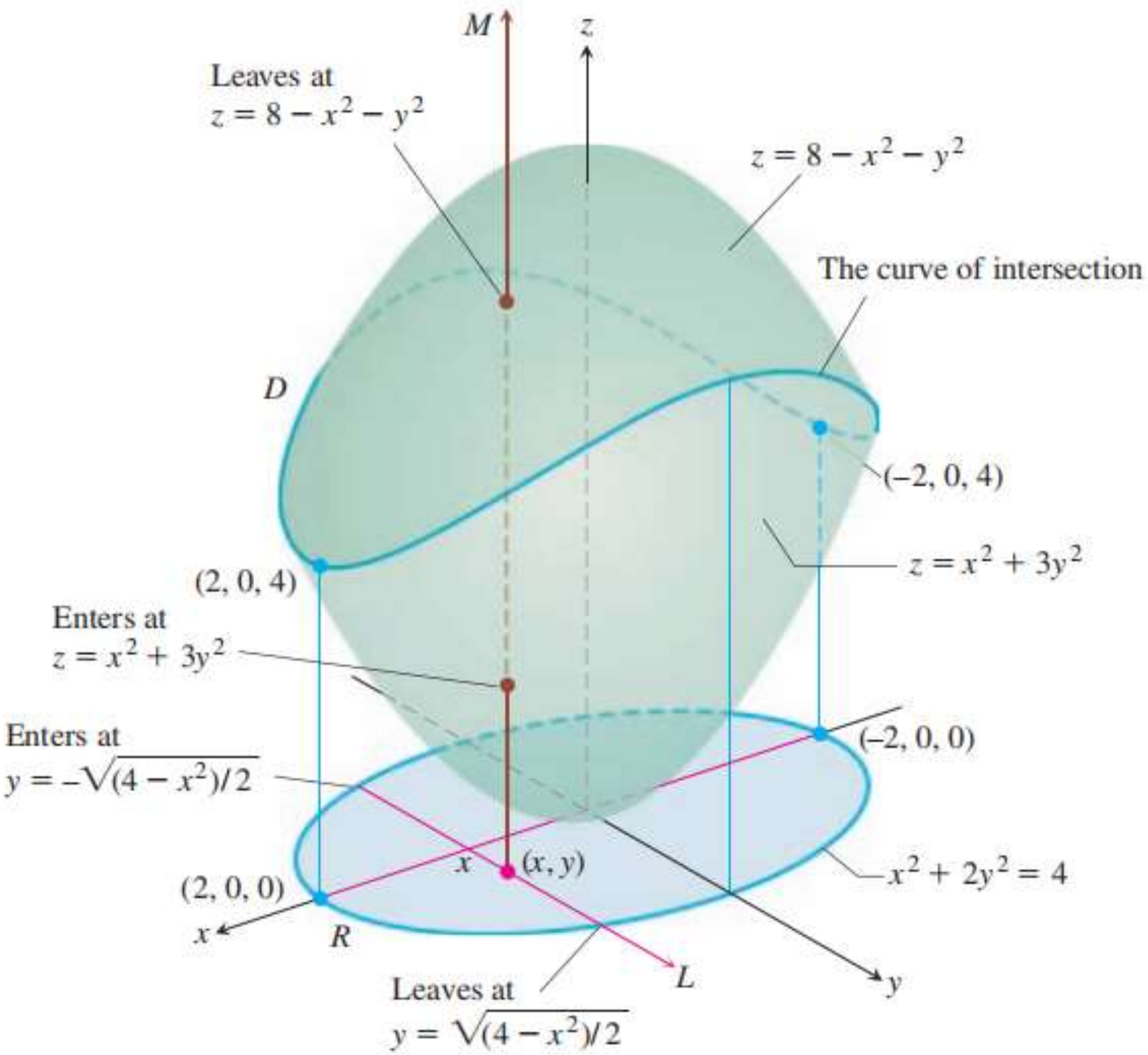
4. *Find the x-limits of integration.* Choose x-limits that include all lines through R parallel to the y-axis ($x = a$ and $x = b$ in the preceding figure). These are the x-limits of integration. The integral is

$$\int_{x=a}^{x=b} \int_{y=g_1(x)}^{y=g_2(x)} \int_{z=f_1(x,y)}^{z=f_2(x,y)} F(x, y, z)\, dz\, dy\, dx.$$

EXAMPLE Find the volume of the region D enclosed by the surfaces $z = x^2 + 3y^2$ and $z = 8 - x^2 - y^2$.







M
z
Leaves at
$z = 8 - x^2 - y^2$
$z = 8 - x^2 - y^2$
The curve of intersection
D
(−2, 0, 4)
$z = x^2 + 3y^2$
(2, 0, 4)
Enters at
$z = x^2 + 3y^2$
Enters at
$y = -\sqrt{(4 - x^2)/2}$
(−2, 0, 0)
x
(x, y)
$x^2 + 2y^2 = 4$
(2, 0, 0)
x
R
L
y
Leaves at
$y = \sqrt{(4 - x^2)/2}$

$$
\begin{aligned}
V &= \iiint_D dz\,dy\,dx \\
&= \int_{-2}^{2}\int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}}\int_{x^2+3y^2}^{8-x^2-y^2} dz\,dy\,dx \\
&= \int_{-2}^{2}\int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} (8-2x^2-4y^2)\,dy\,dx \\
&= \int_{-2}^{2}\left[(8-2x^2)y-\frac{4}{3}y^3\right]_{y=-\sqrt{(4-x^2)/2}}^{y=\sqrt{(4-x^2)/2}} dx \\
&= \int_{-2}^{2}\left(2(8-2x^2)\sqrt{\frac{4-x^2}{2}}-\frac{8}{3}\left(\frac{4-x^2}{2}\right)^{3/2}\right) dx \\
&= \int_{-2}^{2}\left[8\left(\frac{4-x^2}{2}\right)^{3/2}-\frac{8}{3}\left(\frac{4-x^2}{2}\right)^{3/2}\right] dx = \frac{4\sqrt{2}}{3}\int_{-2}^{2}(4-x^2)^{3/2}\,dx \\
&= 8\pi\sqrt{2}. \qquad \text{After integration with the substitution } x = 2\sin u
\end{aligned}
$$

Let D be the region bounded by the paraboloids $z = 8 - x^2 - y^2$ and $z = x^2 + y^2$. Write six different triple iterated integrals for the volume of D. Evaluate one of the integrals.

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{x^2+y^2}^{8-x^2-y^2} 1\,dz\,dy\,dx \qquad \int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}\int_{x^2+y^2}^{8-x^2-y^2} 1\,dz\,dx\,dy,$$

$$\int_{-2}^{2}\int_{4}^{8-y^2}\int_{-\sqrt{8-z-y^2}}^{\sqrt{8-z-y^2}} 1\,dx\,dz\,dy + \int_{-2}^{2}\int_{y^2}^{4}\int_{-\sqrt{z-y^2}}^{\sqrt{z-y^2}} 1\,dx\,dz\,dy,$$

$$\int_{4}^{8}\int_{-\sqrt{8-z}}^{\sqrt{8-z}}\int_{-\sqrt{8-z-y^2}}^{\sqrt{8-z-y^2}} 1\,dx\,dy\,dz + \int_{0}^{4}\int_{-\sqrt{z}}^{\sqrt{z}}\int_{-\sqrt{z-y^2}}^{\sqrt{z-y^2}} 1\,dx\,dy\,dz,$$

$$\int_{-2}^{2}\int_{4}^{8-x^2}\int_{-\sqrt{8-z-x^2}}^{\sqrt{8-z-x^2}} 1\,dy\,dz\,dx + \int_{-2}^{2}\int_{x^2}^{4}\int_{-\sqrt{z-x^2}}^{\sqrt{z-x^2}} 1\,dy\,dz\,dx,$$

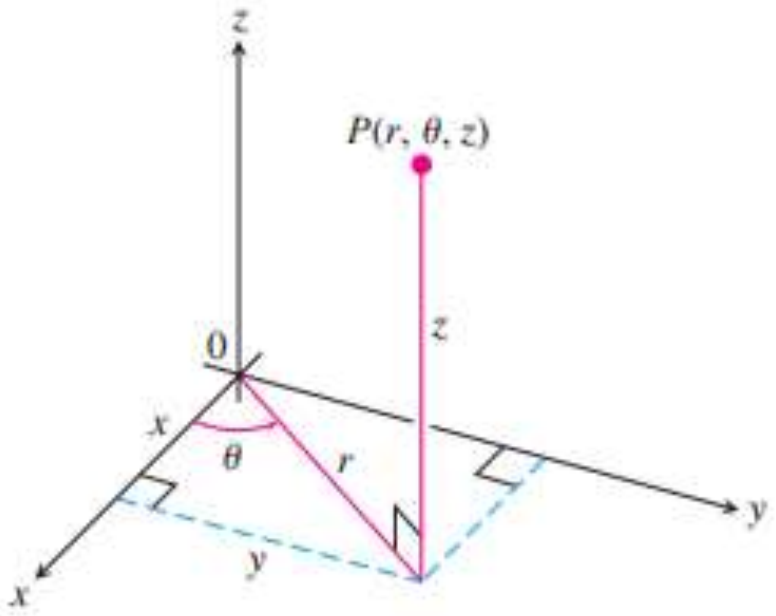
$$\int_{4}^{8}\int_{-\sqrt{8-z}}^{\sqrt{8-z}}\int_{-\sqrt{8-z-x^2}}^{\sqrt{8-z-x^2}} 1\,dy\,dx\,dz + \int_{0}^{4}\int_{-\sqrt{z}}^{\sqrt{z}}\int_{-\sqrt{z-x^2}}^{\sqrt{z-x^2}} 1\,dy\,dx\,dz.$$

The value of all six integrals is 16π.

Triple Integrals in Cylindrical and Spherical Coordinates

DEFINITION **Cylindrical coordinates** represent a point P in space by ordered triples (r, θ, z) in which

1. r and θ are polar coordinates for the vertical projection of P on the xy-plane
2. z is the rectangular vertical coordinate.



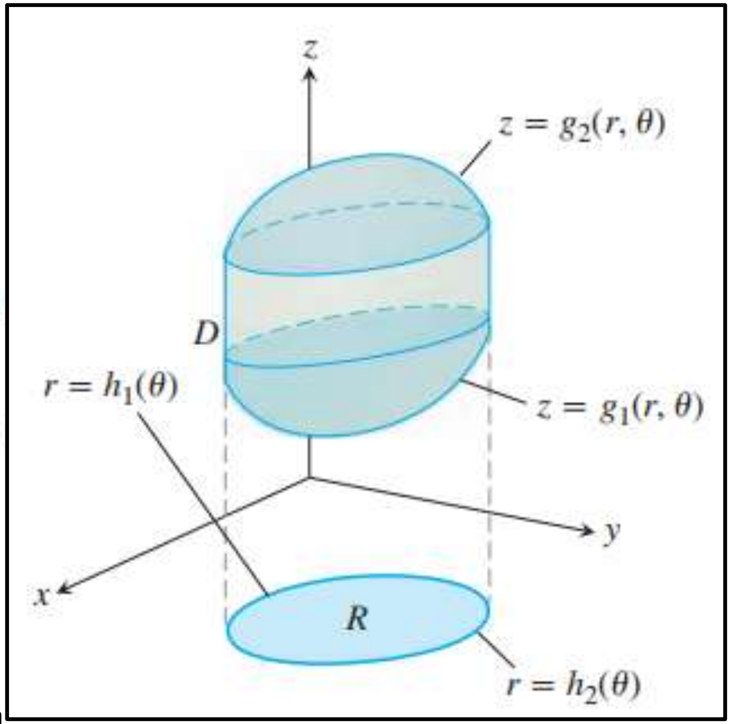
Equations Relating Rectangular (x, y, z) and Cylindrical (r, θ, z) Coordinates

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z,$$
$$r^2 = x^2 + y^2, \qquad \tan\theta = y/x$$

How to Integrate in Cylindrical Coordinates

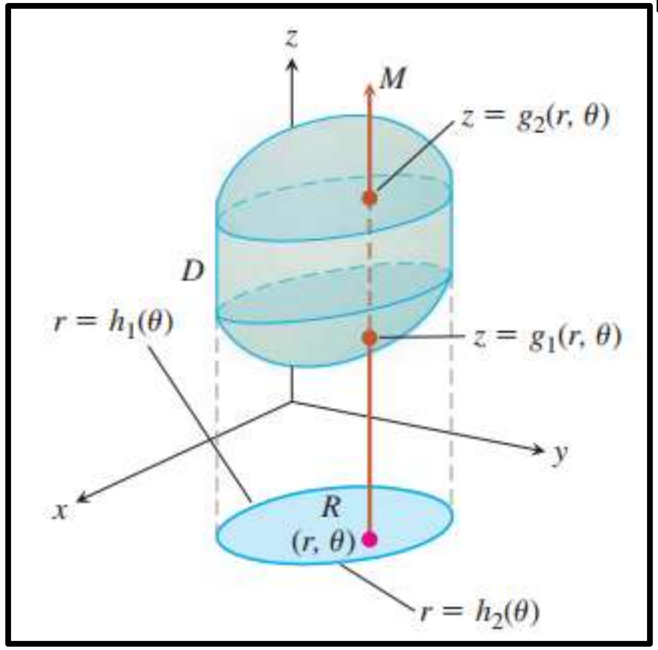
To evaluate $\boxed{\iiint\limits_D f(r, \theta, z)\, dV}$

over a region D in space in cylindrical coordinates, integrating first with respect to z, then with respect to r, and finally with respect to θ, take the following steps.

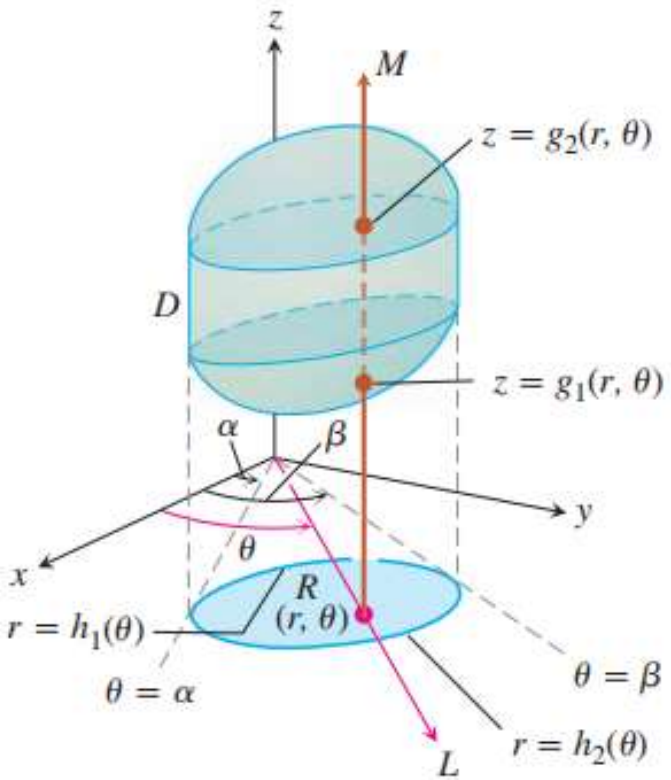


1. *Sketch*. Sketch the region D along with its projection R on the xy-plane. Label the surfaces and curves that bound D and R.

2. *Find the z-limits of integration*. Draw a line M through a typical point (r, θ) of R parallel to the z-axis. As z increases, M enters D at $z = g_1(r, \theta)$ and leaves at $z = g_2(r, \theta)$. These are the z-limits of integration.



3. *Find the r-limits of integration.* Draw a ray L through (r, θ) from the origin. The ray enters R at $r = h_1(\theta)$ and leaves at $r = h_2(\theta)$. These are the r-limits of integration.



4. *Find the θ-limits of integration.* As L sweeps across R, the angle θ it makes with the positive x-axis runs from $\theta = \alpha$ to $\theta = \beta$. These are the θ-limits of integration. The integral is

$$\boxed{\iiint_D f(r, \theta, z)\, dV = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=h_1(\theta)}^{r=h_2(\theta)} \int_{z=g_1(r,\theta)}^{z=g_2(r,\theta)} f(r, \theta, z)\, dz\, r\, dr\, d\theta.}$$

Evaluate the cylindrical coordinate integrals

1. $$\int_0^{2\pi}\int_0^1\int_r^{\sqrt{2-r^2}} dz\, r\, dr\, d\theta$$ **Ans** $$\frac{4\pi\left(\sqrt{2}-1\right)}{3}$$

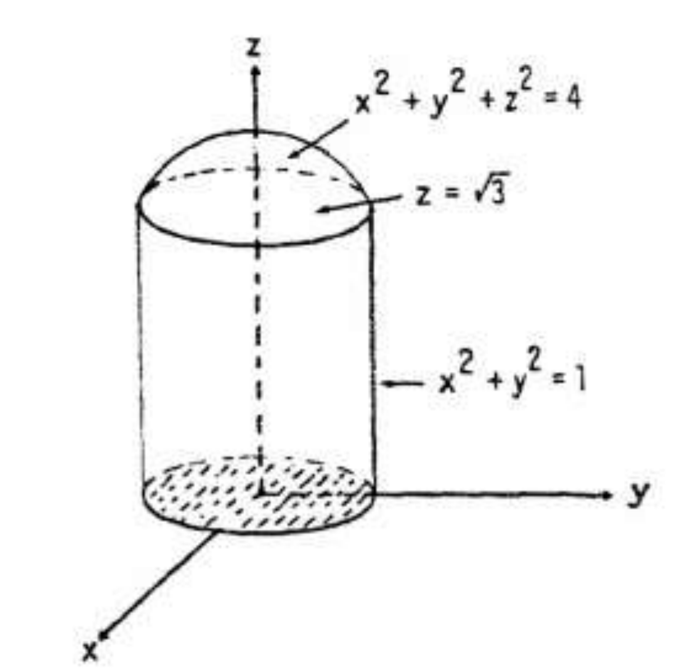
3. $$\int_0^{2\pi}\int_0^{\theta/2\pi}\int_0^{3+24r^2} dz\, r\, dr\, d\theta$$ **Ans.** $$\frac{17\pi}{5}$$

Let D be the region bounded below by the plane $z = 0$, above by the sphere $x^2 + y^2 + z^2 = 4$, and on the sides by the cylinder $x^2 + y^2 = 1$. Set up the triple integrals in cylindrical coordinates that give the volume of D using the following orders of integration.

a. $dz\,dr\,d\theta$ **b.** $d\theta\,dz\,dr$

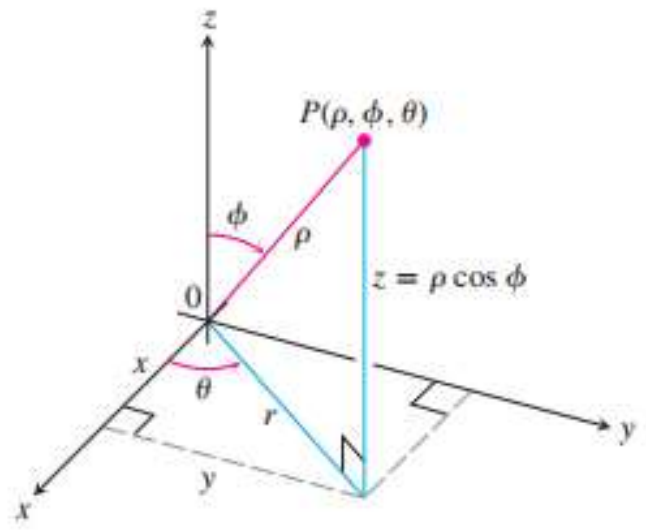
(a) $\int_0^{2\pi}\int_0^1\int_0^{\sqrt{4-r^2}} dz\, r\, dr\, d\theta$

(b) $\int_0^1\int_0^{\sqrt{4-r^2}}\int_0^{2\pi} r\, d\theta\, dz\, dr$



DEFINITION **Spherical coordinates** represent a point P in space by ordered triples (ρ, ϕ, θ) in which

1. ρ is the distance from P to the origin.
2. ϕ is the angle $\overrightarrow{OP}$ makes with the positive z-axis $(0 \le \phi \le \pi)$.
3. θ is the angle from cylindrical coordinates $(0 \le \theta \le 2\pi)$.



Equations Relating Spherical Coordinates to Cartesian and Cylindrical Coordinates

$$
\begin{aligned}
r &= \rho \sin \phi, \qquad x = r \cos \theta = \rho \sin \phi \cos \theta, \\
z &= \rho \cos \phi, \qquad y = r \sin \theta = \rho \sin \phi \sin \theta, \\
\rho &= \sqrt{x^2 + y^2 + z^2} = \sqrt{r^2 + z^2}.
\end{aligned}
\tag{1}
$$

Coordinate Conversion Formulas

Cylindrical to Rectangular	Spherical to Rectangular	Spherical to Cylindrical
$x = r\cos\theta$	$x = \rho\sin\phi\cos\theta$	$r = \rho\sin\phi$
$y = r\sin\theta$	$y = \rho\sin\phi\sin\theta$	$z = \rho\cos\phi$
$z = z$	$z = \rho\cos\phi$	$\theta = \theta$

Corresponding formulas for dV in triple integrals:

$$\begin{aligned} dV &= dx\,dy\,dz \\ &= dz\,r\,dr\,d\theta \\ &= \rho^2\sin\phi\,d\rho\,d\phi\,d\theta \end{aligned}$$

$$\boxed{\rho^2 = x^2 + y^2 + z^2}$$

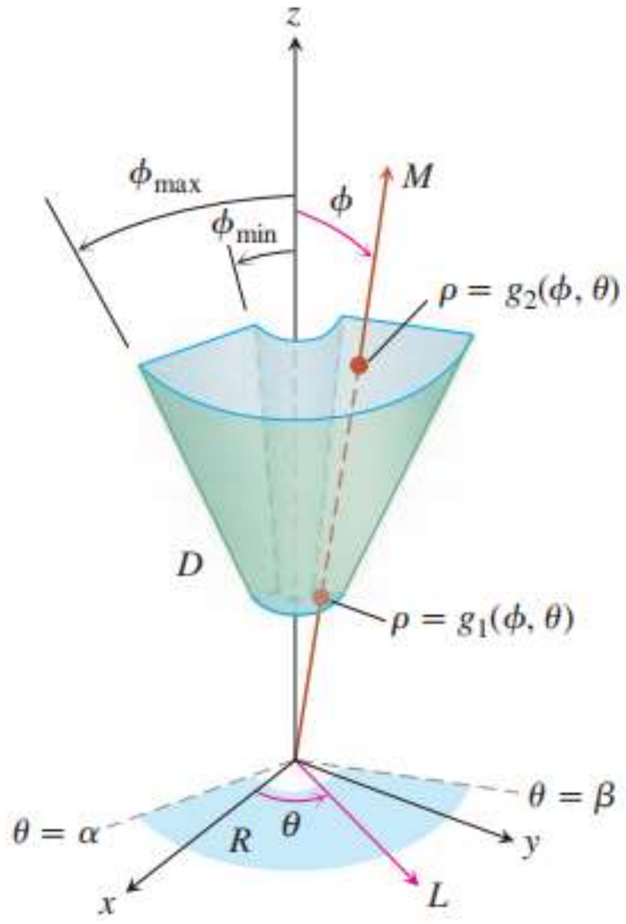
How to Integrate in Spherical Coordinates

$$\text{To evaluate} \qquad \iiint_D f(\rho, \phi, \theta)\, dV$$

over a region D in space in spherical coordinates, integrating first with respect to ρ, then with respect to ϕ, and finally with respect to θ, take the following steps.

1. *Sketch*. Sketch the region D along with its projection R on the xy-plane. Label the surfaces that bound D.

2. *Find the ρ-limits of integration*. Draw a ray M from the origin through D making an angle ϕ with the positive z-axis. Also draw the projection of M on the xy-plane (call the projection L). The ray L makes an angle θ with the positive x-axis. As ρ increases, M enters D at $\rho = g_1(\phi, \theta)$ and leaves at $\rho = g_2(\phi, \theta)$. These are the ρ-limits of integration.



3. *Find the ϕ-limits of integration*. For any given θ, the angle ϕ that M makes with the z-axis runs from $\phi = \phi_{\min}$ to $\phi = \phi_{\max}$. These are the ϕ -limits of integration.

4. *Find the θ-limits of integration*. The ray L sweeps over R as θ runs from α to β. These are the θ-limits of integration. The integral is

$$\iiint_D f(\rho, \phi, \theta)\, dV = \int_{\theta=\alpha}^{\theta=\beta} \int_{\phi=\phi_{\min}}^{\phi=\phi_{\max}} \int_{\rho=g_1(\phi, \theta)}^{\rho=g_2(\phi, \theta)} f(\rho, \phi, \theta)\, \rho^2 \sin\phi \; d\rho \; d\phi \; d\theta.$$

Evaluate $\int_0^{2\pi}\int_0^{1}\int_{-1/2}^{1/2}(r^2\sin^2\theta+z^2)\,dz\;r\,dr\,d\theta$

$$\int_0^{2\pi}\int_0^{1}\int_{-1/2}^{1/2}(\mathrm{r}^2\sin^2\theta+\mathrm{z}^2)\,\mathrm{dz}\,\mathrm{r}\,\mathrm{dr}\,\mathrm{d}\theta$$

$$=\int_0^{2\pi}\int_0^{1}\left(\mathrm{r}^3\sin^2\theta+\tfrac{\mathrm{r}}{12}\right)\mathrm{dr}\,\mathrm{d}\theta$$

$$=\int_0^{2\pi}\left(\tfrac{\sin^2\theta}{4}+\tfrac{1}{24}\right)\mathrm{d}\theta$$

$$=\tfrac{\pi}{3}$$

Evaluate $\int_0^{\pi}\int_0^{\pi}\int_0^{2\sin\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

$$\int_0^{\pi}\int_0^{\pi}\int_0^{2\sin\phi} \rho^2 \sin\phi \, \mathrm{d}\rho \, \mathrm{d}\phi \, \mathrm{d}\theta = \tfrac{8}{3}\int_0^{\pi}\int_0^{\pi} \sin^4\phi \, \mathrm{d}\phi \, \mathrm{d}\theta$$

$$= \tfrac{8}{3}\int_0^{\pi}\left(\left[-\frac{\sin^3\phi\cos\phi}{4}\right]_0^{\pi} + \tfrac{3}{4}\int_0^{\pi}\sin^2\phi \, \mathrm{d}\phi\right)\mathrm{d}\theta$$

$$= 2\int_0^{\pi}\int_0^{\pi} \sin^2\phi \, \mathrm{d}\phi \, \mathrm{d}\theta$$

$$= \int_0^{\pi}\left[\theta - \tfrac{\sin 2\theta}{2}\right]_0^{\pi} \mathrm{d}\theta$$

$$= \int_0^{\pi} \pi \, \mathrm{d}\theta$$

$$= \pi^2$$

$$\int_0^{\pi/2} \sin^n\theta \cos^m\theta \, d\theta = \frac{\Gamma\frac{n+1}{2}\,\Gamma\frac{m+1}{2}}{2\,\Gamma\frac{m+n+2}{2}}$$

$$\therefore \int_0^{\pi} \sin^4\phi \, d\phi = 2\int_0^{\pi/2} \sin^4\phi \, d\phi$$

$$= 2 \times \frac{\Gamma 5/2 \, \Gamma 1/2}{2\,\Gamma\frac{4+0+2}{2}}$$

$$= \frac{\cancel{2} \times \frac{3}{\cancel{2}} \times \frac{1}{2}\,\Gamma 1/2 \times \Gamma 1/2}{2 \cdot \Gamma 3}$$

$$= \frac{3/2 \times \sqrt{\pi}\,\sqrt{\pi}}{2 \cdot 2}$$

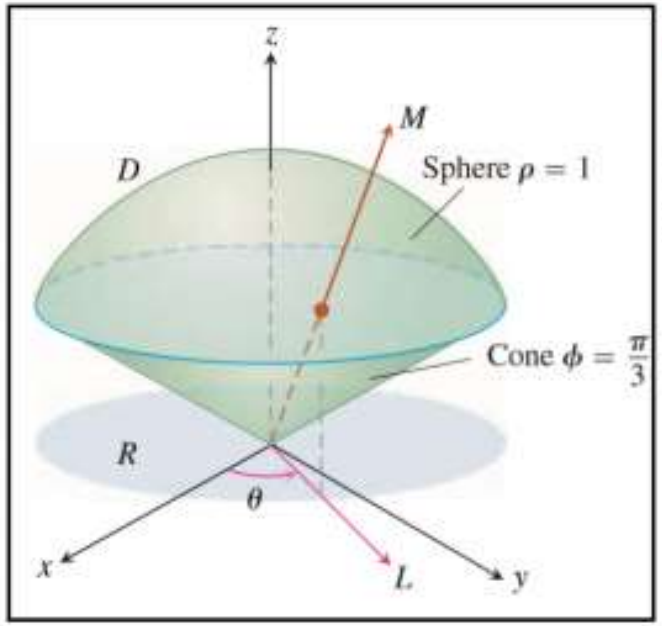
$$= \frac{3}{8}\pi$$

$$\therefore \frac{8}{3}\int_0^{\pi}\int_0^{\pi} \sin^4\phi \, d\phi \, d\theta$$

$$= \frac{8}{3} \times \frac{3}{8}\pi \int_0^{\pi} d\theta$$

$$= \pi(\pi - 0) = \pi^2$$

Find the volume of the " ice cream cone" D cut from the solid sphere $\rho \le 1$ by the cone $\phi = \pi/3$



Solution The volume is $V = \iiint_D \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$ over D.

The ρ-limits of integration. Ray M enters D at $\rho = 0$ and leaves at $\rho = 1$

The ϕ-limits of integration. the angle ϕ can run from $\phi = 0$ to $\phi = \pi/3$

The θ-limits of integration. θ runs from 0 to 2π

$$V = \iiint_D \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi/3} \int_0^1 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi/3} \left[\frac{\rho^3}{3}\right]_0^1 \sin\phi \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi/3} \frac{1}{3} \sin\phi \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \left[-\frac{1}{3}\cos\phi\right]_0^{\pi/3} d\theta$$

$$= \int_0^{2\pi} \left(-\frac{1}{6} + \frac{1}{3}\right) d\theta$$

$$= \frac{1}{6}(2\pi) = \frac{\pi}{3}$$

Convert $\int_0^3 \int_0^{\sqrt{9-y^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{18-x^2-y^2}} x^2+y^2+z^2\,dz\,dx\,dy$ into spherical coordinates.

Given

$$0 \le y \le 3$$
$$0 \le x \le \sqrt{9-y^2}$$
$$\sqrt{x^2+y^2} \le z \le \sqrt{18-x^2-y^2}$$

Range of z lower bound, $z=\sqrt{x^2+y^2}$, is the upper half of a cone

upper bound, $z=\sqrt{18-x^2-y^2}$, is the upper half of the sphere,

$$x^2+y^2+z^2=18$$

and so from this we now have the following range for ρ

$$\boxed{0 \le \rho \le \sqrt{18} = 3\sqrt{2}}$$

range for φ. *is intersection of* the cone and the sphere

$$\left(\sqrt{x^2+y^2}\right)^2+z^2=18$$
$$z^2+z^2=18$$
$$z^2=9$$
$$z=3$$

$$\rho\cos\varphi = 3$$
$$3\sqrt{2}\cos\varphi = 3$$
$$\cos\varphi = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \qquad \Rightarrow \qquad \varphi = \frac{\pi}{4}$$

Hence $\quad 0 \le \varphi \le \dfrac{\pi}{4}$

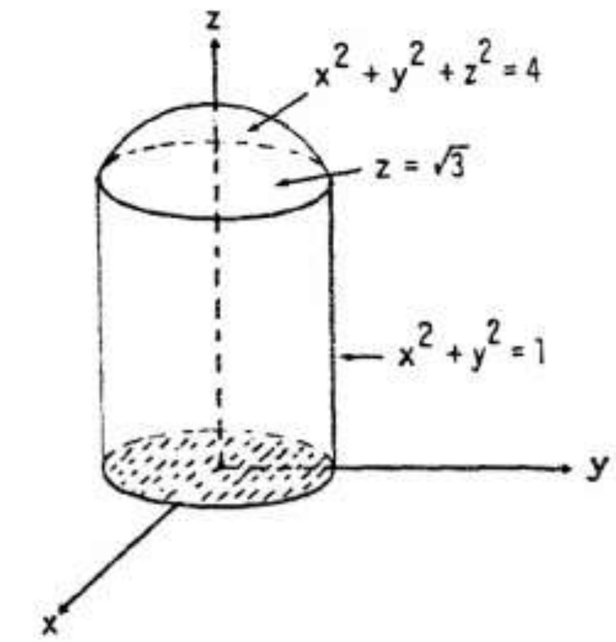
Now
$$z = r$$
$$\rho\cos\varphi = \rho\sin\varphi$$
$$1 = \tan\varphi \qquad \Rightarrow \qquad \varphi = \frac{\pi}{4}$$

So, recalling that $\rho^2 = x^2 + y^2 + z^2$, the integral is then,

$$\int_0^3 \int_0^{\sqrt{9-y^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{18-x^2-y^2}} x^2 + y^2 + z^2 \, dz \, dx \, dy = \int_0^{\pi/4} \int_0^{\pi/2} \int_0^{3\sqrt{2}} \rho^4 \sin\varphi \, d\rho \, d\theta \, d\varphi$$

Let D be the region in Figure. Set up the triple integrals in spherical coordinates that give the volume of D using the following orders of integration.

a. $d\rho\, d\phi\, d\theta$ **b.** $d\phi\, d\rho\, d\theta$



(a) $x^2 + y^2 = 1 \Rightarrow \rho^2 \sin^2 \phi = 1$, and $\rho \sin \phi = 1 \Rightarrow \rho = \csc \phi$; thus

$$\int_0^{2\pi} \int_0^{\pi/6} \int_0^2 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta + \int_0^{2\pi} \int_{\pi/6}^{\pi/2} \int_0^{\csc \phi} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$

(b) $$\int_0^{2\pi} \int_1^2 \int_{\pi/6}^{\sin^{-1}(1/\rho)} \rho^2 \sin \phi \, d\phi \, d\rho \, d\theta + \int_0^{2\pi} \int_0^2 \int_0^{\pi/6} \rho^2 \sin \phi \, d\phi \, d\rho \, d\theta$$

find the spherical coordinate limits for the integral that calculates the volume of the given solid

solid enclosed by the cardioid of revolution $\rho = 1 - \cos\phi$

$$V = \int_0^{2\pi}\int_0^{\pi}\int_0^{1-\cos\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \frac{1}{3}\int_0^{2\pi}\int_0^{\pi} (1-\cos\phi)^3 \sin\phi \, d\phi \, d\theta$$

$$= \frac{1}{3}\int_0^{2\pi} \left[\frac{(1-\cos\phi)^4}{4}\right]_0^{\pi} d\theta$$

$$= \frac{1}{12}(2)^4 \int_0^{2\pi} d\theta$$

$$= \frac{4}{3}(2\pi)$$

$$= \frac{8\pi}{3}$$