Evaluation of improper integral

Type I :-

Evaluation of $\displaystyle\int_{-\infty}^{\infty} \frac{f_1(x)}{f_2(x)}\,dx$, where $f_1(x)$ and $f_2(x)$ are polynomial in x

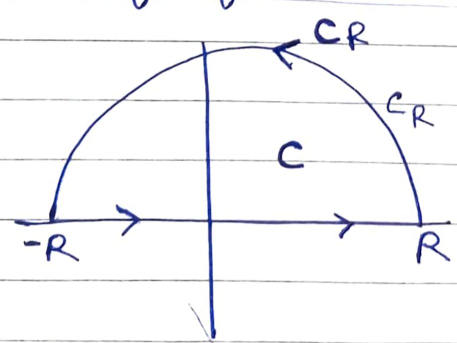
Such integrals can be reduced to contour integrals if

(1) $f_2(x)$ has no real roots

(2) degree of $f_2(x)$ is greater than the degree of $f_1(x)$ by at least 2.

Procedure :- To evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{f_1(x)}{f_2(x)}\,dx$, we evaluate $\displaystyle\oint_C \frac{f_1(z)}{f_2(z)}\,dz$, where C is closed contour consisting of real axis from $-R$ to R and upper half of the circle $|z|=R$

i.e



contour c

* Let $F(z) = \dfrac{f_1(z)}{f_2(z)}$

then

$$\oint_C F(z)\,dz = 2\pi i \times (\text{sum of residues of } F(z) \text{ at poles within } C)$$

(~~analytic inside and on c~~)

$$\int_{-R}^{R} F(x)\,dx + \int_{C_R} F(z)\,dz = 2\pi i \times (\text{sum of residues within } C)$$

$$z = Re^{i\theta}$$

$$\int_{-R}^{R} F(x)\,dx + \int_{C_R} F(Re^{i\theta})\,Rie^{i\theta}\,d\theta = 2\pi i \times (\text{sum of residues})$$

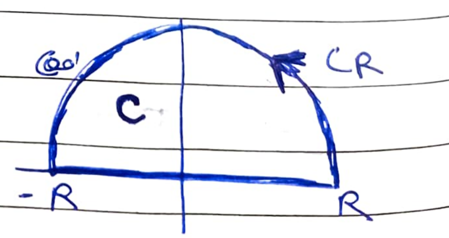
as $R \to \infty$

$$\lim_{R\to\infty}\int_{-R}^{R} F(x)\,dx + \underbrace{\lim_{R\to\infty}\int F(Re^{i\theta})\,Rie^{i\theta}\,d\theta}_{=0} = 2\pi i\,(\text{sum of residues})$$

$$\int_{-\infty}^{\infty} F(x)\,dx = 2\pi i \times \text{sum of residues within } C$$

$$\boxed{\int_{-\infty}^{\infty} \frac{f_1(x)}{f_2(x)}\,dx = 2\pi i \times \text{sum of residues within } C}$$

where C is upper half of circle $|z| = R$, $R \to \infty$ and the line from $-R$ to R.



Jordan Lemma :-

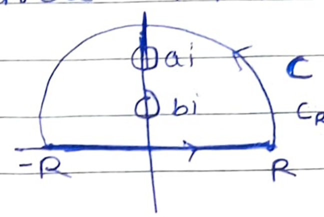
If $\lim_{z \to \infty} f(z) = 0$ then $\lim_{R \to \infty} \int_{C_R} f(z)\, dz = 0$

If $\lim_{z \to \infty} f(z) = 0$ then $\lim_{R \to \infty} \int_{C_R} e^{imz} \cdot f(z)\, dz = 0$

Evaluation of Improper integrals

(1) Evaluate $\int_{-\infty}^{\infty} \frac{x^2\,dx}{(x^2+a^2)(x^2+b^2)}$ using contour integration, where $a > b > 0$.

Soln consider $\oint_C \frac{z^2\,dz}{(z^2+a^2)(z^2+b^2)}$, where C is contour consisting of real axis from $-R$ to R and upper half of circle $|z| = R$



singularities $z = \pm ai$, $z = \pm bi$ all are simple poles.

of these poles $z = ia$ and $z = ib$ lies inside C.

$$R_1 = \underset{z=ia}{\text{Res.}}\, f(z) = \lim_{z \to ia} (z - ia)\, f(z)$$

$$= \lim_{z \to ia} \frac{z^2}{(z+ai)(z^2+b^2)}$$

$$R_1 = \frac{i^2 a^2}{(2ai)(i^2 a^2 + b^2)}$$

$$R_1 = \frac{-a^{\cancel{2}}}{2\cancel{a}i\,(b^2-a^2)} = -\frac{a}{2i(b^2-a^2)} = \frac{a}{2i(a^2-b^2)}$$

$$R_2 = \underset{z=ib}{\text{Res.}}\, f(z) = \lim_{z \to ib} (z - ib)\, f(z)$$

$$= \lim_{z \to ib} \frac{z^2}{(z+ib)(z^2+a^2)} = \frac{-b^2}{2ib(a^2-b^2)}$$

$$R_2 = \frac{-b}{2i(a^2-b^2)}$$

By cauchy's Residue theorem

$$\oint_C \frac{z^2}{(z^2+a^2)(z^2+b^2)}\,dz = 2\pi i \times (\text{sum of Residue within } C)$$

$$= 2\pi i \times \left[\frac{a}{2i(a^2-b^2)} - \frac{b}{2i(a^2-b^2)}\right]$$

$$= \frac{2\pi i}{2i}\,\frac{(a-b)}{(a^2-b^2)}$$

$$\boxed{\oint_C \frac{z^2}{(z^2+a^2)(z^2+b^2)}\,dz = \frac{\pi}{a+b}}$$

$$\int_{-R}^{R} f(x)\,dx + \int_{C_R} f(z)\,dz = \frac{\pi}{a+b}$$

as $R \to \infty$

$$\lim_{R\to\infty}\int_{-R}^{R} f(x)\,dx + \lim_{R\to\infty}\int_{C_R} f(z)\,dz = \frac{\pi}{a+b}$$

$$\int_{-\infty}^{\infty} f(x)\,dx + 0 = \frac{\pi}{a+b}$$

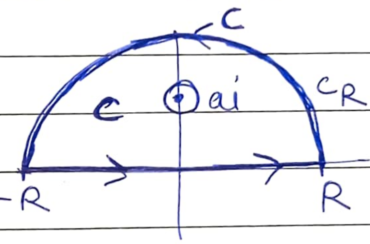
as $R \to \infty$, $\int_{C_R} f(z)\,dz = 0$ (using Jordan lemma)

ie $$\int_{-\infty}^{\infty} f(x)\,dx = \frac{\pi}{a+b} \Rightarrow \int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)(x^2+b^2)}\,dx = \frac{\pi}{a+b}$$

Q:- $\displaystyle\int_{-\infty}^{\infty} \frac{x^2\,dx}{(x^2+1)(x^2+4)}$, solve using Residue theorem.

Q:- Solve $\displaystyle\int_{0}^{\infty} \frac{dx}{(x^2+a^2)^3}$

Soln:- consider $\displaystyle\oint_C \frac{dz}{(z^2+a^2)^3}$ where c is closed curve consisting of real axis from $-R$ to R and upper half of circle $|z| = R$



poles are

$$(z^2+a^2)^3 = 0$$

$$(z+ai)^3 (z-ai)^3 = 0$$

$z = ai$ is a pole of order 3

$z = -ai$ is a pole of order 3.

clearly only $z = ai$ lies inside the contourc

Residue at $z = ai$:-

$$\operatorname*{Res}_{z=ai} f(z) = \frac{1}{2!} \lim_{z\to ai} \frac{d^2}{dz^2}\left(\cancel{(z-ai)^3} \cdot \frac{1}{(z+ai)^3\cancel{(z-ai)^3}}\right)$$

$$= \frac{1}{2!} \lim_{z\to ai} \frac{d^2}{dz^2}\left(\frac{1}{(z+ai)^3}\right)$$

$$= \frac{1}{2!} \lim_{z\to ai} \frac{d}{dz}\left(\frac{-3}{(z+ai)^4}\right)$$

Date
Page

$$= \frac{1}{2!} \lim_{z \to ai} \left(\frac{12}{(z+ai)^5} \right)$$

$$= \frac{1}{2!} \frac{12}{(2ai)^5} = \frac{12}{2 \times 32 a^5 i^5}$$

$(i^5 = i)$

$$= \frac{3}{16 a^5 i}$$

by cauchy's Residue theorem

$$\oint_C f(z)\,dz = 2\pi i \times (\text{sum of Residue within } C)$$

$$= 2\pi \cancel{i} \times \frac{3}{16 a^5 \cancel{i}}$$

$$\oint_C f(z)\,dz = \frac{3\pi}{8a^5}$$

$$\int_{-R}^{R} f(x)\,dx + \int_{CR} f(z)\,dz = \frac{3\pi}{8a^5}$$

as $R \to \infty$ $\quad \int_{CR} f(z)\,dz = 0$

$$\int_{-\infty}^{\infty} f(x)\,dx = \frac{3\pi}{8a^5}$$

$$\boxed{\int_{-\infty}^{\infty} \frac{1}{(x^2+a^2)^3}\,dx = \frac{3\pi}{8a^5}}$$

integrand is even function

$$2\int_0^{\infty} \frac{1}{(x^2+a^2)^3}\,dx = \frac{3\pi}{8a^5}$$

$$\left[\int_0^{\infty} \frac{1}{(x^2+a^2)^3}\,dx = \frac{3\pi}{16a^5}\right]$$

3. $\displaystyle\int_0^{\infty} \frac{1}{1+x^2}\,dx$ Ans $\left(\frac{\pi}{2}\right)$

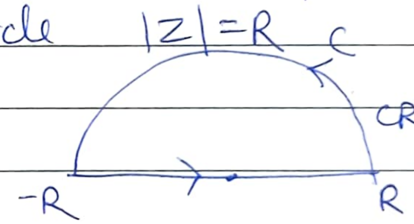
(4) solve using cauchy Residue theorem

(a) $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^4}\,dx$

(b) $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^6}\,dx$

$n = 0, 1, 2, 3, 4, 5$

Sol^n consider $\displaystyle\oint_C \frac{1}{1+z^4}\,dz$, where C is closed contour consisting of real axis $-R$ to R and upper half of circle $|z| = R$



poles → $1+z^4=0$

$$z^4 = -1$$

$$z = (-1)^{1/4} = (\cos\pi + i\sin\pi)^{1/4}$$

$$z = \left[\cos(2n+1)\pi + i\sin(2n+1)\pi\right]^{\frac{1}{4}}$$

$$z = \cos(2n+1)\frac{\pi}{4} + i\sin(2n+1)\frac{\pi}{4}$$

$n = 0, 1, 2, 3$

$n=0$

$$z_1 = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = e^{i\pi/4} = \left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)$$

$n=1$

$$z_2 = \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4} = e^{i\frac{3\pi}{4}} = \left(-\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)$$

$n=2$

$$z_3 = \cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4} = e^{i5\pi/4} = \left(\frac{-1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)$$

$n=3$

$$z_4 = \cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4} = \left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)$$

all are simple poles.

Out of four poles only z_1 and z_2 lies in the contour C.

$$\underset{z=e^{i\pi/4}}{\text{Res}}\, f(z) = \left[\frac{1}{\frac{d}{dz}(z^4+1)}\right]_{z=e^{i\pi/4}}$$

Using

$$\underset{z=a}{\text{Res}}\, f(z) = \frac{P(a)}{Q'(a)}$$

$$= \left[\frac{1}{4z^3}\right]_{z=e^{i\pi/4}} = \frac{1}{4e^{3i\pi/4}} = \frac{1}{4}e^{-\frac{3i\pi}{4}}$$

$$= \cancel{\frac{1}{4}\left(-\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)}$$

$$= \frac{1}{4}\left[\cos\frac{3\pi}{4} - i\sin\frac{3\pi}{4}\right] = \frac{1}{4}\left[\frac{-1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right]$$

classmate
Date
Page

$$\operatorname{Res}_{z=e^{3i\pi/4}} f(z) = \left[\frac{1}{\frac{d}{dz}(z^4+1)}\right]_{z=e^{i3\pi/4}}$$

$$= \left[\frac{1}{4z^3}\right]_{z=e^{i3\pi/4}}$$

$$= \frac{1}{4e^{i9\pi/4}} = \frac{1}{4} e^{-i9\pi/4}$$

$$= \frac{1}{4}\left[\cos\left(\frac{9\pi}{4}\right) - i\sin\left(\frac{9\pi}{4}\right)\right]$$

$$= \frac{1}{4}\left[\cos\left(2\pi+\frac{\pi}{4}\right) - i\sin\left(2\pi+\frac{\pi}{4}\right)\right]$$

$$= \frac{1}{4}\left[\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right] = \frac{1}{4}\left[\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right]$$

$$\oint_C f(z)\,dz = 2\pi i \times \text{sum of Residue at poles}$$

$$\oint_C \frac{1}{1+z^4}\,dz = 2\pi i\left[\frac{1}{4}\left(\frac{-1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right) + \frac{1}{4}\left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)\right]$$

$$\oint_C \frac{1}{1+z^4}\,dz = 2\pi i\left[\frac{-1\, i}{2\sqrt{2}}\right] = \frac{\pi}{\sqrt{2}}$$

Now

$$\int_{-R}^{R} \frac{1}{1+x^4}\,dx + \int_{C_R} \frac{1}{1+z^4}\,dz = \pi/\sqrt{2}$$

as $R \to \infty$

$$\boxed{\int_{-\infty}^{\infty} \frac{1}{1+x^4}\,dx = \pi/\sqrt{2}}$$

II^{nd} form

Integral of the form

$$\int_{-\infty}^{\infty} \frac{f_1(x)}{f_2(x)} \cos mx \, dx \quad \text{or} \quad \int_{-\infty}^{\infty} \frac{f_1(x)}{f_2(x)} \sin mx \, dx$$

$f_1(x)$ and $f_2(x)$ are poly. in x.

To solve start with

So

$$\oint_C \frac{f_1(z)}{f_2(z)} e^{imz} \, dz,$$ after solving

take real part or imaginary part as required.

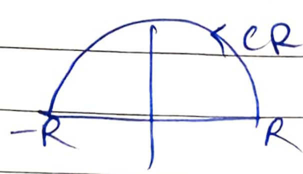
Note:- using Jordan lemma if $\lim_{z \to \infty} f(z) = 0$ then $\lim_{R \to \infty} \int_{C_R} f(z) \cdot e^{miz} = 0$

(1) Solve $$\int_0^{\infty} \frac{\cos x}{x^2+a^2} \, dx.$$

Sol^n consider

$$\oint_C \frac{\cancel{\cos}\, e^{iz}}{z^2+a^2} \, dz,$$ where c is closed curve containing real axis from $-R$ to R and upper half or circle $|z| = R$.

pole are $z = \pm ai$



clearly only $z = ai$ lies inside the contour c.

$$\operatorname*{Res}_{z=ai} f(z) = \lim_{z \to ai} \cancel{(z-ai)} \frac{e^{iz}}{(z+ai)\cancel{(z-ai)}}$$

$$= \frac{e^{iai}}{2ai} = \frac{e^{-a}}{2ai}$$

$$\oint_C \frac{e^{iz}}{z^2+a^2}\,dz = 2\pi i \times \text{sum of residue within } C$$

$$= 2\pi i \times \frac{e^{-a}}{2ai} = \frac{\pi e^{-a}}{a}$$

$$\oint_C \frac{e^{iz}}{z^2+a^2}\,dz = \frac{\pi}{a} e^{-a}$$

$$\int_{-R}^{R} \frac{e^{ix}}{x^2+a^2}\,dx + \int_{C_R} \frac{e^{iz}}{z^2+a^2}\,dz = \frac{\pi}{a} e^{-a}$$

as $R \to \infty$ $\quad \displaystyle\int_{C_R} \frac{e^{iz}}{z^2+a^2}\,dz \to 0$

$z = Re^{i}$

i.e $\quad \displaystyle \lim_{R\to\infty} \int_{-R}^{R} \frac{e^{ix}}{x^2+a^2}\,dx = \frac{\pi}{a} e^{-a}$

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+a^2}\,dx = \frac{\pi}{a} e^{-a}$$

Real part

$$\int_{-\infty}^{\infty} \frac{\cos x + i\sin x}{x^2+a^2}\,dx = \frac{\pi}{a} e^{-a}$$

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2+a^2}\,dx + i\int_{-\infty}^{\infty} \frac{\sin x}{x^2+a^2}\,dx = \frac{\pi e^{-a}}{a}$$

comparing real and imaginary part

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2+a^2}\,dx = \frac{\pi e^{-a}}{a}$$

$\frac{\cos x}{x^2+a^2}$ is even function

i.e

$$2\int_{0}^{\infty} \frac{\cos x}{x^2+a^2}\,dx = \frac{\pi e^{-a}}{a}$$

$$\boxed{\int_{0}^{\infty} \frac{\cos x}{x^2+a^2}\,dx = \frac{\pi}{2} e^{-a}}$$

$$\int_{-\infty}^{\infty} \frac{\sin x}{x^2+a^2}\,dx = 0.$$

↓

Obvious as integrand is odd function.

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(2) Solve $\int_{-\infty}^{\infty} \frac{\sin mx}{x(x^2+a^2)}\,dx$, $m>0$ by contour integration.

Ans $\frac{\pi}{a^2}\left(1 - e^{-am}\right)$

(3) Evaluate $\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+a^2)(x^2+b^2)}\,dx$ by contour integration

(3) Evaluate $\int_{-\infty}^{\infty} \frac{\sin x}{x^2+2x+2}\,dx$