4.7 SINGULAR POINTS

A point at which the function $f(z)$ is not analytic is known as a singular point or singularity of the function.

Types of Singularities

1. Isolated Singularity

A singular point $z = a$ is known as an isolated singularity if there is no other singularity within a small circle surrounding the point $z = a$; otherwise it is called a non-isolated singularity.

For example, $f(z) = \dfrac{z^2}{z-2}$ has isolated singularity at $z = 2$.

2. Removable Singularity

An isolated singular point $z = a$ is known as a removable singularity if $\lim_{z\to a} f(z)$ exists or there is no negative power term of $(z - a)$ in Laurent's series of $f(z)$, i.e., $b_n = 0$.

For example, $f(z) = \dfrac{\sin z}{z}$ **[Summer 2013]**

$z = 0$ is a singularity of $f(z)$.

$$\lim_{z\to 0} \frac{\sin z}{z} = 1$$

$\therefore$ $f(z)$ has a removable singularity at $z = 0$.

3. Essential Singularity

A singular point $z = a$ is known as an essential singularity if the number of negative power terms of $(z - a)$ in Laurent's series is infinite.

For example, $f(z) = e^{\frac{1}{z-2}}$

$$= 1 + \frac{1}{z-2} + \frac{1}{2!}\frac{1}{(z-2)^2} + \cdots$$

Since the number of negative power terms of $(z - 2)$ is infinite, $z = 2$ is an essential singularity.

4. Pole of Order *m*

An isolated singularity $z = a$ is known as a pole of order m if in Laurent's series, all the negative powers of $(z - a)$ after the m^{th} power are zero, i.e., highest power of $\dfrac{1}{z-a}$ is m.

For example, $$f(z) = \frac{1}{(z-1)(z-3)^2}$$

$z = 1$ is a pole of order 1 and $z = 3$ is a pole of order 2.

Note A pole of order one is also known as a simple pole.

Meromorphic Function A function $f(z)$ which is analytic everywhere in the finite plane except at a finite number of poles is called a meromorphic function.

Example 1

Define the singular point of the function and state whether the function $f(z) = \dfrac{z^2+1}{(z+2)(z^2+2z+2)}$ *is analytic?* **[Winter 2014]**

Solution

A point at which the function $f(z)$ is not analytic is known as a singular point or a singularity of the function.

$f(z)$ is analytic everywhere except at the points given by

$$(z+2)(z^2+2z+2) = 0$$

$$z+2 = 0, \quad z^2+2z+2 = 0$$

$$z = -2, \quad z = \frac{-2 \pm \sqrt{4-8}}{2}$$

$$= \frac{-2 \pm 2i}{2}$$

$$= -1 \pm i$$

The singular points are $z = -2$ and $z = -1 \pm i$.

Hence, z is analytic everywhere except at $z = -2$ and $z = -1 \pm i$.

Example 2

Identify the type of singularities of the function $f(z) = e^{\frac{1}{z-1}}$.

Solution

$$f(z) = e^{\frac{1}{z-1}}$$

$$= 1 + \frac{1}{(z-1)} + \frac{1}{2!(z-1)^2} + \cdots \infty$$

Since the number of terms of negative powers of $(z - 1)$ are infinite, $z = 1$ is an essential singularity.

Example 3

Classify the poles of $f(z) = \dfrac{1}{z^2 - z^6}$.

Solution

$$f(z) = \frac{1}{z^2(1 - z^4)}$$

The poles are given by

$$z^2(1 - z^4) = 0$$

$$z^2(1 + z^2)(1 - z^2) = 0$$

$$z = 0, 0, \pm 1, \pm i$$

$z = 0$ is a pole of order 2 and $z = \pm 1, \pm i$ are poles of order one or simple poles.

Example 4

Find the residue at $z = 0$ *of* $f(z) = z\cos\dfrac{1}{z}$.

Solution

$$\begin{aligned} f(z) &= z\cos\frac{1}{z} \\ &= z\left[1 - \frac{\left(\frac{1}{z}\right)^2}{2!} + \frac{\left(\frac{1}{z}\right)^4}{4!} - \cdots\right] \\ &= z - \frac{1}{2z} + \frac{1}{24z^3} - \cdots \end{aligned}$$

$$\text{Res}\,[f(z); z = 0] = \text{coefficient of } \frac{1}{z}$$

$$= -\frac{1}{2}$$

Example 5

Find the pole and its order of the function $f(z) = \dfrac{\sin z}{z^4}$. **[Winter 2013]**

Solution

$$\begin{aligned} f(z) &= \frac{\sin z}{z^4} \\ &= \frac{1}{z^4}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots\right) \\ &= \frac{1}{z^3} - \frac{1}{6z} + \frac{z}{120} - \frac{z^3}{5040} + \cdots \end{aligned}$$

Since the highest power of $\frac{1}{z}$ is 3, $z = 0$ is a pole of order 3.

Example 6

Expand $f(z) = \dfrac{z - \sin z}{z^2}$ *at* $z = 0$*, classify the singular point* $z = 0$*.*

[Summer 2015]

Solution

$$\begin{aligned} f(z) &= \frac{z - \sin z}{z^2} \\ &= \frac{1}{z^2}\left[z - \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots\right)\right] \\ &= \frac{z}{3!} - \frac{z^3}{5!} + \frac{z^5}{7!} + \cdots \\ \lim_{z\to 0} f(z) &= \lim_{z\to 0}\left(\frac{z}{3!} - \frac{z^3}{5!} + \frac{z^5}{7!} - \cdots\right) \\ &= 0 \end{aligned}$$

Since the limit is finite, $z = 0$ is a removable singularity.

Example 7

What is the nature of the singularity at $z = 0$ *of the function* $f(z) = \dfrac{\sin z - z}{z^3}$*?*

Solution

$$f(z) = \frac{\sin z - z}{z^3}$$

$$\lim_{z\to 0} f(z) = \lim_{z\to 0} \frac{\sin z - z}{z^3} \qquad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{z\to 0} \frac{\cos z - 1}{3z^2} \qquad [\text{Using L'Hospital's rule}]$$

$$= \lim_{z\to 0} \frac{-\sin z}{6z} \qquad [\text{Using L'Hospital's rule}]$$

$$= -\frac{1}{6} \qquad \left[\because \lim_{z\to 0} \frac{\sin z}{z} = 1\right]$$

Since the limit is finite, $z = 0$ is a removable singularity.

Example 8

Identify the type of singularity of $f(z) = \dfrac{2 - e^z}{z^3}$.

Solution

$$f(z) = \frac{2 - e^z}{z^3}$$

$$= \frac{1}{z^3}\left[2 - \left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \frac{z^5}{5!} + \cdots\right)\right]$$

$$= \frac{1}{z^3} - \frac{1}{z^2} - \frac{1}{2z} - \frac{1}{6} - \frac{z}{24} - \frac{z^2}{120} - \cdots$$

Since the highest power of $\dfrac{1}{z}$ is 3, $z = 0$ is a pole of order 3.

Example 9

Write the singularity of the function $\dfrac{1}{1 - e^z}$.

Solution

Let

$$f(z) = \frac{1}{1 - e^z}$$

$$= \frac{1}{1 - \left(1 + z + \dfrac{z^2}{2!} + \dfrac{z^3}{3!} + \dfrac{z^4}{4!} + \cdots\right)}$$

$$= \frac{1}{-\left(z+\frac{z^2}{2}+\frac{z^3}{6}+\frac{z^4}{24}+\cdots\right)}$$

$$= \frac{1}{-z\left(1+\frac{z}{2}+\frac{z^2}{6}+\frac{z^3}{24}+\ldots\right)}$$

$$= -\frac{1}{z}\left(1+\frac{z}{2}+\frac{z^2}{6}+\frac{z^3}{24}+\cdots\right)^{-1}$$

$$= -\frac{1}{z}\left[1+\left(\frac{z}{2}+\frac{z^2}{6}+\frac{z^3}{24}+\cdots\right)\right]^{-1}$$

$$= -\frac{1}{z}\left[1-\left(\frac{z}{2}+\frac{z^2}{6}+\frac{z^3}{24}+\cdots\right)+\frac{\left(\frac{z}{2}+\frac{z^2}{6}+\cdots\right)^2}{2!}-\cdots\right]$$

$$= -\frac{1}{z}+\frac{1}{2}+\frac{z}{24}-\frac{z^2}{24}+\cdots$$

Since the highest power of $\frac{1}{z}$ is 1, $z = 0$ is a pole of order 1.

Example 10

Identify the type of singularity of $f(z) = e^{e^{\frac{1}{z^2}}}$.

Solution

$$f(z) = e^{e^{\frac{1}{z^2}}}$$

$$= e^{\left(1+\frac{1}{z^2}+\frac{1}{2!z^4}+\cdots\right)}$$

$$= e^1 e^{\left(\frac{1}{z^2}+\frac{1}{2!z^4}+\cdots\right)}$$

$$= e\left[1+\left(\frac{1}{z^2}+\frac{1}{2z^4}+\cdots\right)+\frac{\left(\frac{1}{z^2}+\frac{1}{2z^4}+\cdots\right)^2}{2!}+\cdots\right]$$

Since the number of terms of negative powers of z are infinite, $z = 0$ is an essential singularity.

Example 11

Find the singularities of the function $f(z) = \dfrac{\tan z}{z}$.

Solution

$$f(z) = \frac{\tan z}{z}$$

$$\lim_{z\to 0} f(z) = \lim_{z\to 0} \frac{\tan z}{z}$$

$$= \lim_{z\to 0} \frac{z + \dfrac{z^3}{3} + \dfrac{2z^5}{15} + \cdots}{z}$$

$$= \lim_{z\to 0} \left(1 + \frac{z^2}{3} + \frac{2z^4}{15} + \cdots\right)$$

$$= 1$$

Since the limit is finite, $z = 0$ is a removable singularity.

Example 12

Find the singularities of the function $f(z) = \dfrac{\cot \pi z}{(z-a)^3}$.

Solution

$$f(z) = \frac{\cot \pi z}{(z-a)^3}$$

$$= \frac{\cos \pi z}{(z-a)^3 \sin \pi z}$$

Singularities are at $(z - a)^3 \sin \pi z = 0$.

$$z - a = 0, \qquad \sin \pi z = 0$$

$$z = a, \qquad \pi z = n\pi, \qquad n = 0, \pm 1, \pm 2, \ldots$$

$$z = n$$

$$= 0, \pm 1, \pm 2, \pm 3, \ldots$$

Since in the neighbourhood of $z = a$, there are infinite number of singularities, $z = a$, is a non-isolated singularity.

Example 13

Identify the type of singularity of $f(z)=\dfrac{z-2}{4}\sin\left(\dfrac{1}{z-1}\right)$.

Solution

$$f(z)=\frac{z-2}{4}\sin\left(\frac{1}{z-1}\right)$$

Let $z-1=t$ then $z=t+1$.

$$f(z)=\frac{t+1-2}{4}\sin\left(\frac{1}{t}\right)$$

$$=\frac{t-1}{4}\left[\frac{1}{t}-\frac{1}{3!t^3}+\frac{1}{5!t^5}-\cdots\right]$$

$$=\frac{1}{4}\left[\left(1-\frac{1}{3!t^2}+\frac{1}{5!t^4}-\cdots\right)-\left(\frac{1}{t}-\frac{1}{3!t^3}+\frac{1}{5!t^5}-\cdots\right)\right]$$

$$=\frac{1}{4}\left[1-\frac{1}{t}-\frac{1}{3!t^2}+\frac{1}{3!t^3}+\cdots\right]$$

$$=\frac{1}{4}\left[1-\frac{1}{z-1}-\frac{1}{3!(z-1)^2}+\frac{1}{3!(z-1)^3}+\cdots\right]$$

Since the number of terms of negative powers of $(z-1)$ are infinite, $z=1$ is an essential singularity.

Example 14

Find the singular points of $f(z)=\dfrac{1}{\sin\dfrac{1}{z-a}}$. *State their nature.*

Solution

$$f(z)=\frac{1}{\sin\dfrac{1}{z-a}}$$

Singularity is at $\sin\dfrac{1}{z-a}=0$.

$$\frac{1}{z-a} = n\pi, \quad n = \pm 1, \pm 2, \ldots$$

$$z - a = \frac{1}{n\pi}$$

$$z = a + \frac{1}{n\pi}$$

Since in the neighbourhood of $z = a$, there are infinite number of singularities, $z = a$ is non-isolated singularity.

Example 15

Find the singularities of the function $f(z) = \dfrac{z^2+1}{e^z}$.

Solution

$$f(z) = \frac{z^2+1}{e^z}$$

$$= (z^2+1)e^{-z}$$

$$= (z^2+1)\left(1 - z + \frac{z^2}{2!} - \frac{z^3}{3!} + \cdots\right)$$

$f(z)$ is analytic everywhere.
Hence, there is no singularity.

EXERCISE 4.3

Determine the nature of singularities of the following functions:

1. $z^2 e^{\frac{1}{z}}$

[**Ans.:** $z = 0$ is an essential singularity.]

2. $\dfrac{\sin 3z}{z}$

[**Ans.:** $z = 0$ is a removable singularity.]

3. $e^{-\frac{1}{z^2}}$

[**Ans.:** No singularity]

4. $\dfrac{2 - e^z}{z^3}$

[**Ans.:** $z = 0$ is a pole of order 3.]

5. $z^3 e^{\frac{1}{z-1}}$

[**Ans.:** $z = 1$ is an essential singularity.]

6. $\dfrac{1}{(z-5)^2(z^2-4)}$

[**Ans.:** $z = 2, -2$ are simple poles and $z = 5$ is a pole of order 2.]

7. $\sin\dfrac{1}{z-2}$

[**Ans.:** $z = 2$ is an essential singularity.]

8. $z e^{\frac{1}{z^2}}$

[**Ans.:** $z = 0$ is an essential singularity.]

4.8 RESIDUES

The coefficient of $\dfrac{1}{z-a}$ in the Laurent series expansion of $f(z)$ is known as the residue of $f(z)$ at $z = a$.

The Laurent series expansion about $z = a$ is

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n}$$

$$\begin{aligned}
\text{Residue (at } z = a) &= \text{Coefficient of } \frac{1}{z-a} \\
&= b_1 \\
&= \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^{-1+1}}\,dz \quad \left[\because \quad b_n = \frac{1}{2\pi i}\int \frac{f(z)}{(z-a)^{-n+1}}\,dz\right] \\
&= \frac{1}{2\pi i}\int_C f(z)\,dz
\end{aligned}$$

Evaluation of Residues

1. Residue at a Simple Pole

(i) If $f(z)$ has a simple pole at $z = a$ then

$$\text{Res}\left[f(z); z = a\right] = \lim_{z\to a}(z-a)f(z)$$

(ii) If $f(z)$ is of the form $f(z)=\dfrac{g(z)}{h(z)}$, where $g(z)$ and $h(z)$ are analytic at $z = a$.

If $h(a)=0$ but $h'(a)\neq 0$ then $z=a$ is a simple pole.

If $h(a)=0$ but $g(a)\neq 0$ then

$$\text{Res}\left[f(z);\, z=a\right]=\lim_{z\to a}\frac{g(z)}{h'(z)}$$

2. Residue at a Pole of Order *m*

If $f(z)$ has a pole of order m at $z = a$ then

$$\text{Res}\left[f(z); z=a\right]=\frac{1}{(m-1)!}\lim_{z\to a}\left[\frac{d^{m-1}}{dz^{m-1}}(z-a)^m f(z)\right]$$

Example 1

If $f(z)=\dfrac{-1}{z-1}-2\left[1+(z-1)+(z-1)^2+\cdots\right]$, *find the residue of* $f(z)$ *at* $z = 1$.

Solution

$$\text{Res }\left[f(z); z=1\right] = \text{Coefficient of } \frac{1}{z-1} \text{ in the Laurent's series of } f(z) \text{ about } z = 1$$

$$= -1$$

Example 2

Find the residue of $f(z)=z^2\sin\left(\dfrac{1}{z}\right)$ at $z=0$.

Solution

$$f(z)=z^2\sin\left(\frac{1}{z}\right)$$

$$=z^2\left[\frac{1}{z}-\frac{\left(\frac{1}{z}\right)^3}{3!}+\cdots\right]$$

$$=z-\frac{1}{6z}+\cdots$$

$$\text{Res}\left[f(z);\, z=0\right] = \text{Coefficient of } \frac{1}{z}$$

$$= -\frac{1}{6}$$

Example 3

Obtain the residue of $f(z) = \dfrac{z-3}{(z+1)(z+2)}$ *at its poles.*

Solution

$$f(z) = \frac{z-3}{(z+1)(z+2)}$$

The poles are given by

$$(z+1)(z+2) = 0$$

$$z = -1, -2$$

$z = -1$ is a simple pole.

$$\text{Res}\left[f(z);\, z=-1\right] = \lim_{z\to -1}(z+1)f(z)$$

$$= \lim_{z\to -1}\frac{z-3}{z+2}$$

$$= \frac{-1-3}{-1+2}$$

$$= -4$$

$z = -2$ is a simple pole.

$$\text{Res}\left[f(z);\, z=-2\right] = \lim_{z\to -2}(z+2)f(z)$$

$$= \lim_{z\to -2}\frac{z-3}{z+1}$$

$$= \frac{-2-3}{-2+1}$$

$$= 5$$

Example 4

Test for singularity of $\dfrac{1}{z^2+1}$ *and, hence, find the corresponding residues.*

Solution

Let $$f(z) = \frac{1}{z^2+1}$$

$$= \frac{1}{(z+i)(z-i)}$$

$z = -i$ and $z = i$ are simple poles.

$$\text{Res}\left[f(z); z=-i\right] = \lim_{z\to -i} (z+i)f(z)$$

$$= \lim_{z\to -i} \frac{1}{z-i}$$

$$= \frac{1}{-i-i}$$

$$= -\frac{1}{2i}$$

$$\text{Res}\left[f(z); z=i\right] = \lim_{z\to i} (z-i)f(z)$$

$$= \lim_{z\to i} \frac{1}{z+i}$$

$$= \frac{1}{i+i}$$

$$= \frac{1}{2i}$$

Example 5

Find the residue of $f(z) = \dfrac{z}{z^2+1}$ about $z = i$.

Solution

$$f(z) = \frac{z}{z^2+1}$$

The poles are given by

$$z^2+1 = 0$$

$$z^2 = -1$$

$$z = \pm i$$

$$\therefore \quad f(z) = \frac{z}{(z+i)(z-i)}$$

$z = i$ is a simple pole.

$$\begin{aligned}\text{Res}\left[f(z);\, z = i\right] &= \lim_{z\to i} (z-i) f(z) \\ &= \lim_{z\to i} \frac{z}{z+i} \\ &= \frac{i}{i+i} \\ &= \frac{1}{2}\end{aligned}$$

Example 6

Find the residue of the function $f(z) = \dfrac{4}{z^3(z-2)}$ *at a simple pole.*

Solution

$$f(z) = \frac{4}{z^3(z-2)}$$

The poles are given by

$$\begin{aligned} z^3(z-2) &= 0 \\ z &= 0, 0, 0, 2 \end{aligned}$$

$z = 2$ is a simple pole.

$$\begin{aligned}\text{Res}\left[f(z);\, z = 2\right] &= \lim_{z\to 2} (z-2) f(z) \\ &= \lim_{z\to 2} \frac{4}{z^3} \\ &= \frac{4}{8} \\ &= \frac{1}{2}\end{aligned}$$

Example 7

Find the residue of $f(z) = \dfrac{z^2}{(z-1)^2(z+2)}$ *at* $z = -2.$

Solution

$$f(z) = \frac{z^2}{(z-1)^2(z+2)}$$

$z = -2$ is a simple pole.

$$\begin{aligned}\text{Res}\left[f(z); z=-2\right] &= \lim_{z\to -2}(z+2)f(z)\\ &= \lim_{z\to -2}\frac{z^2}{(z-1)^2}\\ &= \frac{(-2)^2}{(-2-1)^2}\\ &= \frac{4}{9}\end{aligned}$$

Example 8

Find the residue of $\tan z$ at $z = \dfrac{\pi}{2}$.

Solution

$$f(z) = \tan z$$

$z = \dfrac{\pi}{2}$ is a simple pole.

$$\begin{aligned}\text{Res}\left[f(z); z=\frac{\pi}{2}\right] &= \lim_{z\to\frac{\pi}{2}}\left(z-\frac{\pi}{2}\right)\tan z\\ &= \lim_{z\to\frac{\pi}{2}}\frac{z-\frac{\pi}{2}}{\cot z} && \left[\frac{0}{0}\text{ form}\right]\\ &= \lim_{z\to\frac{\pi}{2}}\frac{1}{-\text{cosec}^2 z} && \left[\text{Using L'Hospital's rule}\right]\\ &= -\frac{1}{\text{cosec}^2\frac{\pi}{2}}\\ &= -1\end{aligned}$$

Example 9

Find the residue of $f(z)=\dfrac{z}{(z-1)^2}$ *at its pole.*

Solution

$$f(z)=\frac{z}{(z-1)^2}$$

$z=1$ is a pole of order 2.

$$\begin{aligned}\text{Res}\left[f(z);\, z=1\right] &= \frac{1}{(2-1)!}\lim_{z\to 1}\frac{\mathrm{d}}{\mathrm{d}z}\left[(z-1)^2 f(z)\right]\\ &= \lim_{z\to 1}\frac{\mathrm{d}}{\mathrm{d}z}(z)\\ &= \lim_{z\to 1}(1)\\ &= 1\end{aligned}$$

Example 10

Find the residue at the pole of the function $f(z)=\dfrac{2z+3}{(z+2)^2}$.

Solution

$$f(z)=\frac{2z+3}{(z+2)^2}$$

$z=2$ is a pole of order 2.

$$\begin{aligned}\text{Res}\left[f(z);\, z=-2\right] &= \frac{1}{(2-1)!}\lim_{z\to -2}\frac{\mathrm{d}}{\mathrm{d}z}\left[(z+2)^2 f(z)\right]\\ &= \lim_{z\to -2}\frac{\mathrm{d}}{\mathrm{d}z}(2z+3)\\ &= \lim_{z\to -2}(2)\\ &= 2\end{aligned}$$

Example 11

Calculate the residue of $f(z)=\dfrac{e^{2z}}{(z+1)^2}$ *at its pole.*

Solution

$$f(z)=\frac{e^{2z}}{(z+1)^2}$$

The poles are given by

$$(z+1)^2=0$$
$$z+1=0$$
$$z=-1$$

$z=-1$ is a pole of order 2.

$$\begin{aligned}\operatorname{Res}\left[f(z);\, z=-1\right]&=\frac{1}{(2-1)!}\lim_{z\to -1}\frac{\mathrm{d}}{\mathrm{d}z}\left[(z+1)^2 f(z)\right]\\&=\frac{1}{1!}\lim_{z\to -1}\frac{\mathrm{d}}{\mathrm{d}z}\left[e^{2z}\right]\\&=\lim_{z\to -1}2e^{2z}\\&=2e^{-2}\end{aligned}$$

Example 12

Find the residue of $f(z)=\dfrac{1-e^z}{z^3}$ *at* $z=0$.

Solution

$$f(z)=\frac{1-e^z}{z^3}$$

$z=0$ is a pole of order 3.

$$\begin{aligned}\operatorname{Res}\left[f(z);\, z=0\right]&=\frac{1}{(3-1)!}\lim_{z\to 0}\frac{\mathrm{d}^2}{\mathrm{d}z^2}\left[(z-0)^3 f(z)\right]\\&=\frac{1}{2!}\lim_{z\to 0}\frac{\mathrm{d}^2}{\mathrm{d}z^2}(1-e^z)\\&=\frac{1}{2}\lim_{z\to 0}\frac{\mathrm{d}}{\mathrm{d}z}(-e^z)\\&=\frac{1}{2}\lim_{z\to 0}(-e^z)\\&=\frac{1}{2}(-e^0)\\&=-\frac{1}{2}\end{aligned}$$

Example 13

Find the residue of $\dfrac{1-e^{2z}}{z^4}$ at $z=0$.

Solution

$$f(z)=\frac{1-e^{2z}}{z^4}$$

$z=0$ is a pole of order 4.

$$\begin{aligned}\operatorname{Res}\left[f(z);z=0\right]&=\frac{1}{(4-1)!}\lim_{z\to 0}\frac{\mathrm{d}^3}{\mathrm{d}z^3}\left[(z-0)^4 f(z)\right]\\&=\frac{1}{3!}\lim_{z\to 0}\frac{\mathrm{d}^3}{\mathrm{d}z^3}(1-e^{2z})\\&=\frac{1}{6}\lim_{z\to 0}\frac{\mathrm{d}^2}{\mathrm{d}z^2}(-2e^{2z})\\&=\frac{1}{6}\lim_{z\to 0}\frac{\mathrm{d}}{\mathrm{d}z}(-4e^{2z})\\&=\frac{1}{6}\lim_{z\to 0}(-8e^{2z})\\&=\frac{1}{6}(-8e^{0})\\&=-\frac{4}{3}\end{aligned}$$

Example 14

Find the residue of $f(z)=\dfrac{50z}{(z+4)(z-1)^2}$ *at* $z=1$.

Solution

$$f(z)=\frac{50z}{(z+4)(z-1)^2}$$

$z=1$ is a pole of order 2.

$$\begin{aligned}\operatorname{Res}\left[f(z);\ z=1\right]&=\frac{1}{(2-1)!}\lim_{z\to 1}\frac{\mathrm{d}}{\mathrm{d}z}\left[(z-1)^2 f(z)\right]\\&=\lim_{z\to 1}\frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{50z}{z+4}\right]\end{aligned}$$

$$= \lim_{z\to 1}\left[\frac{(z+4)50-50z}{(z+4)^2}\right]$$

$$= \frac{5(50)-50}{(5)^2}$$

$$= 8$$

Example 15

Find the residues of the function $f(z)=\dfrac{z^2-2z}{(z+1)^2(z^2+4)}$ *at each of its poles.*

[Winter 2014]

Solution

$$f(z)=\frac{z^2-2z}{(z+1)^2(z^2+4)}$$

The poles are given by

$$(z+1)^2(z^2+4)=0$$

$$z=-1,-1,\pm 2i$$

$$\therefore \qquad f(z)=\frac{z^2-2z}{(z+1)^2(z+2i)(z-2i)}$$

$z=-1$ is a pole of order 2.

$$\text{Res}\left[f(z);\, z=-1\right]=\frac{1}{(2-1)!}\lim_{z\to -1}\frac{\mathrm{d}}{\mathrm{d}z}\left[(z+1)^2 f(z)\right]$$

$$= \lim_{z\to -1}\frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{z^2-2z}{z^2+4}\right]$$

$$= \lim_{z\to -1}\left[\frac{(2z-2)(z^2+4)-(z^2-2z)\cdot 2z}{(z^2+4)^2}\right]$$

$$= \lim_{z\to -1}\left[\frac{2z^3+8z-2z^2-8-2z^3+4z^2}{(z^2+4)^2}\right]$$

$$= \lim_{z\to -1}\left[\frac{2z^2+8z-8}{(z^2+4)^2}\right]$$

$$= \frac{2-8-8}{25}$$

$$= -\frac{14}{25}$$

$z = 2i$ is a simple pole,

$$\begin{aligned}
\text{Res}\left[f(z);\, z = 2i\right] &= \lim_{z\to 2i}\left[(z-2i)\, f(z)\right] \\
&= \lim_{z\to 2i} \frac{z^2 - 2z}{(z+1)^2 (z+2i)} \\
&= \frac{(2i)^2 - 2(2i)}{(2i+1)^2\, 4i} \\
&= \frac{-4-4i}{(-4+1+4i)\, 4i} \\
&= \frac{-(1+i)}{i(4i-3)} \\
&= \frac{(i-1)}{(4i-3)} \times \frac{(4i+3)}{(4i+3)} \qquad \left[\because \quad \frac{1}{i} = -i\right] \\
&= \frac{(4i^2 + 3i - 4i - 3)}{-25} \\
&= \frac{(7+i)}{25}
\end{aligned}$$

$z = -2i$ is a simple pole.

$$\begin{aligned}
\text{Res}\left[f(z);\, z = -2i\right] &= \lim_{z\to -2i}\left[(z+2i) f(z)\right] \\
&= \lim_{z\to -2i} \frac{z^2 - 2z}{(z+1)^2 (z-2i)} \\
&= \frac{(-2i)^2 + 4i}{(-2i+1)^2(-4i)} \\
&= \frac{-4+4i}{(-4+1-4i)(-4i)} \\
&= \frac{-(1+i)}{-(3+4i)} \times \frac{(3-4i)}{(3-4i)} \\
&= \frac{3-4i+3i-4i^2}{25} \\
&= \frac{7-i}{25}
\end{aligned}$$

Example 16

Determine poles and their orders for the function $\dfrac{z+2}{(z+1)^2(z-2)}$. *Find the residues at the poles.*

Solution

$$f(z) = \frac{z+2}{(z+1)^2(z-2)}$$

The poles are given by

$$(z+1)^2(z-2) = 0$$
$$z = -1, -1, 2$$

$z = -1$ is a pole of order 2.

$$\begin{aligned}\text{Res}\left[f(z); z=-1\right] &= \frac{1}{(2-1)!}\lim_{z\to -1}\frac{\text{d}}{\text{d}z}\left[(z+1)^2 f(z)\right]\\ &= \lim_{z\to -1}\frac{\text{d}}{\text{d}z}\left[\frac{z+2}{z-2}\right]\\ &= \lim_{z\to -1}\left[\frac{(z-2)(1)-(z+2)(1)}{(z-2)^2}\right]\\ &= \lim_{z\to -1}\left[\frac{-4}{(z-2)^2}\right]\\ &= \frac{-4}{(-1-2)^2}\\ &= -\frac{4}{9}\end{aligned}$$

$z = 2$ is a simple pole.

$$\begin{aligned}\text{Res}\left[f(z); z=2\right] &= \lim_{z\to 2}(z-2)f(z)\\ &= \lim_{z\to 2}\frac{z+2}{(z+1)^2}\\ &= \frac{2+2}{(2+1)^2}\\ &= \frac{4}{9}\end{aligned}$$

Example 17

Obtain the Laurent's expansion of the function $\frac{e^z}{(z-1)^2}$ *in the neighbourhood of its singular point. Hence, find the residue at that point.*

Solution

Let $$f(z) = \frac{e^z}{(z-1)^2}$$

$f(z)$ is not analytic at $z = 1$. But it is analytic in the region $|z-1| > 0$ about $z = 1$.
Let $z - 1 = t$, then $z = t + 1$.

$$\begin{aligned} f(z) &= \frac{e^z}{(z-1)^2} \\ &= \frac{e^{t+1}}{t^2} \\ &= \frac{e}{t^2}(e^t) \\ &= \frac{e}{t^2}\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots\right) \\ &= \frac{e}{t^2} + \frac{e}{t} + \frac{e}{2} + e\frac{t}{6} + \cdots \\ &= \frac{e}{(z-1)^2} + \frac{e}{z-1} + \frac{e}{2} + e\frac{(z-1)}{6} + \cdots \end{aligned}$$

$$\begin{aligned} \text{Res}\left[f(z); z=1\right] &= \text{Coefficient of } \frac{1}{z-1} \\ &= e \end{aligned}$$

Example 18

Find the residues of $f(z) = \frac{z^2}{(z-1)^2(z+2)^2}$ *at its isolated singularities using Laurent's series expansion. Also, state the valid region.*

Solution

$$f(z)=\frac{z^2}{(z-1)^2(z+2)^2}$$
$$=\frac{A}{(z-1)}+\frac{B}{(z-1)^2}+\frac{C}{z+2}+\frac{D}{(z+2)^2}$$
$$z^2=A(z-1)(z+2)^2+B(z+2)^2+C(z+2)(z-1)^2+D(z-1)^2$$

Putting $z = 1$,

$$1=9B$$
$$B=\frac{1}{9}$$

Putting $z = -2$,

$$4=9D$$
$$D=\frac{4}{9}$$

Equating the coefficient of z^3,

$$0 = A + C$$
$$A = -C$$

Putting $z = 0$,

$$0=-4A+4B+2C+D$$
$$0=-4A+\frac{4}{9}+2C+\frac{4}{9}$$
$$4A-2C=\frac{8}{9}$$
$$-4C-2C=\frac{8}{9}$$
$$-6C=\frac{8}{9}$$
$$C=-\frac{4}{27}$$
$$A=\frac{4}{27}$$
$$\therefore\quad f(z)=\frac{4}{27}\frac{1}{z-1}+\frac{1}{9}\frac{1}{(z-1)^2}-\frac{4}{27}\frac{1}{z+2}+\frac{4}{9}\frac{1}{(z+2)^2}$$

$z = 1$ and $z = -2$ are poles of order 2. Since poles are isolated singularities, $z = 1$ and $z = -2$ are isolated singularities.

(i) For Laurent's series about $z = 1$,

Let $z - 1 = t$ then $z = t + 1$

$$f(z) = \frac{4}{27}\frac{1}{t} + \frac{1}{9}\frac{1}{t^2} - \frac{4}{27}\frac{1}{(t+1+2)} + \frac{4}{9}\frac{1}{(t+1+2)^2}$$

$$= \frac{4}{27t} + \frac{1}{9t^2} - \frac{4}{27}\frac{1}{3\left(1+\frac{t}{3}\right)} + \frac{4}{9}\frac{1}{3^2\left(1+\frac{t}{3}\right)^2}$$

$$= \frac{4}{27t} + \frac{1}{9t^2} - \frac{4}{81}\left(1+\frac{t}{3}\right)^{-1} + \frac{4}{81}\left(1+\frac{t}{3}\right)^{-2}$$

$$= \frac{4}{27t} + \frac{1}{9t^2} - \frac{4}{81}\left(1-\frac{t}{3}+\frac{t^2}{9}-\cdots\right) + \frac{4}{81}\left(1-\frac{2t}{3}+\frac{3t^2}{9}-\cdots\right)$$

$$= \frac{4}{27(z-1)} + \frac{1}{9(z-1)^2} - \frac{4}{81}\left[1-\frac{z-1}{3}+\frac{(z-1)^2}{9}-\cdots\right]$$

$$+ \frac{4}{81}\left[1-\frac{2(z-1)}{3}+\frac{3(z-1)^2}{9}-\cdots\right]$$

$$\text{Res}\,[f(z);\, z = 1] = \text{Coefficient of } \frac{1}{z-1}$$

$$= \frac{4}{27}$$

(ii) For Laurent's series about $z = -2$,

Let $z + 2 = t$ then $z = t - 2$.

$$f(z) = \frac{4}{27}\frac{1}{(t-2-1)} + \frac{1}{9}\frac{1}{(t-2-1)^2} - \frac{4}{27}\frac{1}{t} + \frac{4}{9}\frac{1}{t^2}$$

$$= \frac{4}{27}\left[\frac{1}{-3\left(1-\frac{t}{3}\right)}\right] + \frac{1}{9}\left[\frac{1}{(-3)^2\left(1-\frac{t}{3}\right)^2}\right] - \frac{4}{27t} + \frac{4}{9t^2}$$

$$= -\frac{4}{81}\left(1-\frac{t}{3}\right)^{-1} + \frac{1}{81}\left(1-\frac{t}{3}\right)^{-2} - \frac{4}{27t} + \frac{4}{9t^2}$$

$$= -\frac{4}{81}\left(1+\frac{t}{3}+\frac{t^2}{9}+\cdots\right) + \frac{1}{81}\left(1+\frac{2t}{3}+\frac{3t^2}{9}+\cdots\right) - \frac{4}{27t} + \frac{4}{9t^2}$$

$$= -\frac{4}{81}\left[1+\frac{z+2}{3}+\frac{(z+2)^2}{9}+\cdots\right]+\frac{1}{81}\left[1+\frac{2(z+2)}{3}+\frac{(z+2)^2}{3}+\cdots\right]$$

$$-\frac{4}{27(z+2)}+\frac{4}{9(z+2)^2}$$

$$\text{Res}\,[f(z);\, z=-2] = \text{Coefficient of } \frac{1}{z+2}$$

$$= -\frac{4}{27}$$

EXERCISE 4.4

Find the residues of $f(z)$ at the singular points:

1. $\dfrac{z}{(z-1)^2}$

$\left[\textbf{Ans.:} -\dfrac{1}{2i}\right]$

2. $\dfrac{z+2}{(z+1)^2}$

[**Ans.:** 1]

3. $\dfrac{z^2-z}{(z+1)^2(z^2+4)}$

$\left[\textbf{Ans.:} -\dfrac{11}{25}, \dfrac{11 \pm 2i}{50}\right]$

4. $\dfrac{e^z}{(z-1)^3}$

$\left[\textbf{Ans.:} \dfrac{e}{2}\right]$

5. $\dfrac{\sin^2 z}{z^3}$

$\left[\textbf{Ans.:} \dfrac{1}{4}\right]$

6. $\dfrac{z+3}{z(z-1)(z+2)}$

$\left[\textbf{Ans.:} -\dfrac{3}{2}, \dfrac{4}{3}, \dfrac{1}{6}\right]$

7. $\dfrac{e^{2z}}{z^2+\pi^2}$

$$\left[\textbf{Ans.: } \frac{1}{2\pi i}, -\frac{1}{2\pi i}\right]$$

8. $\dfrac{z^3}{(z+a)^2}$

[Ans.: $3a^2$**]**

9. $\dfrac{z^2+1}{z(z-2)}$

$$\left[\textbf{Ans.: } -\frac{1}{2}, \frac{5}{2}\right]$$

10. $z\cos\dfrac{1}{z}$

[Ans.: -4**]**

4.9 CAUCHY'S RESIDUE THEOREM

If $f(z)$ is analytic inside and on a simple closed curve C except at a finite number of singular points inside C then

$$\oint_C f(z)\,\mathrm{d}z = 2\pi i \text{ (sum of residues)}$$

where the integral is taken in the anticlockwise direction around C.

Proof

Let $z_1, z_2, \ldots, z_n$ be finite numbers of singular points inside C (Fig. 4.36). Enclose each of the singular point in a small circle such that no other singular point lies inside this circle. The curve C along with these small circles C_1, C_2, C_3, …, C_n form a multiply connected region and $f(z)$ is analytic in this region.

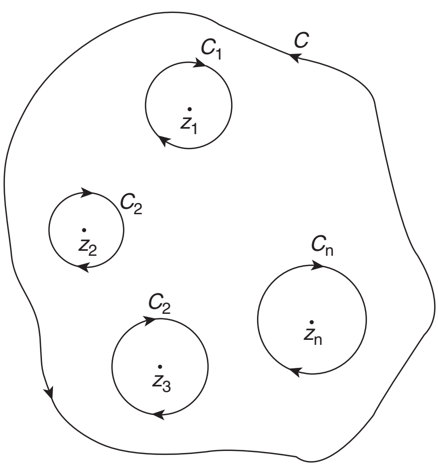

Fig. 4.36

By Cauchy's integral theorem for a multiply connected region,

$$\oint_C f(z)\,\mathrm{d}z = \oint_{C_1} f(z)\,\mathrm{d}z + \oint_{C_2} f(z)\,\mathrm{d}z + \cdots + \oint_{C_n} f(z)\,\mathrm{d}z$$

$$= 2\pi i\left[\text{Res. at } z_1\right] + 2\pi i\left[\text{Res. at } z_2\right] + \cdots + 2\pi i\left[\text{Res. at } z_n\right]$$

$$= 2\pi i \text{ (sum of residues)}$$

Example 1

Evaluate $\oint_C \frac{z^2-4z+4}{z+i}\,dz$ *where C is* $|z| = 2$. **[Winter 2013]**

Solution

(i) Let $f(z)=\frac{z^2-4z+4}{z+i}$

The poles are given by $z=-i$.

(ii) C is the circle $|z| = 2$ with the centre at (0, 0) and a radius of 2 (Fig. 4.37).

(iii) For $z=-i$, $|z| = |-i| = 1 < 2$

Hence, $z=-i$ lies inside C.

(iv) $z=-i$ is a simple pole.

$$\text{Res}\left[f(z);\, z=-i\right] = \lim_{z\to -i}(z+i)f(z)$$
$$= \lim_{z\to -i}(z^2-4z+4)$$
$$= i^2+4i+4$$
$$= 3+4i$$

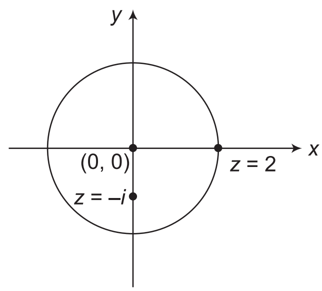

Fig. 4.37

(v) By Cauchy's residue theorem,

$$\int_C f(z)\,dz = 2\pi i\text{ (sum of residues)}$$

$$\int_C \frac{z^2-4z+4}{z+i} = 2\pi i\,(3+4i)$$
$$= 2\pi(3\text{i}-4)$$

Example 2

Evaluate $\oint_C \frac{1}{(z-1)^2(z-3)}\,dz$ *where C is* $|z| = 2$.

Solution

(i) $f(z)=\frac{1}{(z-1)^2(z-3)}$

The poles are given by $z = 1$, $z = 3$.

(ii) C is a circle $|z| = 2$ with the centre at (0, 0) and a radius of 2 (Fig. 4.38).

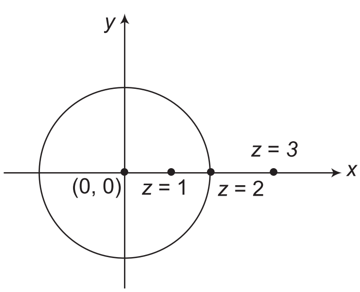

Fig. 4.38

(iii) For $z = 1$, $|z| = |1| = 1 < 2$

Hence, $z = 1$ lies inside C.

For $z = 3$, $|z| = |3| = 3 > 2$

Hence, $z = 3$ lies outside C.

(iv) $z = 1$ is a pole of order 2.

$$\text{Res}\left[f(z);\, z=1\right] = \frac{1}{(2-1)!} \lim_{z\to 1} \frac{\mathrm{d}}{\mathrm{d}z}\left[(z-1)^2\, f(z)\right]$$

$$= \lim_{z\to 1} \frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{1}{z-3}\right]$$

$$= \lim_{z\to 1}\left[-\frac{1}{(z-3)^2}\right]$$

$$= -\frac{1}{4}$$

(v) By Cauchy's residue theorem,

$$\oint_C f(z)\,\mathrm{d}z = 2\pi i \text{ (sum of residues)}$$

$$\oint_C \frac{1}{(z-1)^2(z-3)}\,\mathrm{d}z = 2\pi i\left(-\frac{1}{4}\right)$$

$$= -\frac{\pi i}{2}$$

Example 3

Evaluate $\displaystyle\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}\,\mathrm{d}z$, *where C is* $|z| = 3$.

Solution

(i) Let $f(z) = \dfrac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}$

The poles are given by $z = 1$, $z = 2$.

(ii) C is a circle $|z| = 3$ with the centre at (0, 0) and a radius of 3 (Fig. 4.39).

(iii) For $z = 1$, $|z| = |1| = 1 < 3$

Hence, $z = 1$ lies inside C.

For $z = 2$, $|z| = |2| = 2 < 3$

Hence, $z = 2$ lies inside C.

(iv) $z = 1$ is a simple pole.

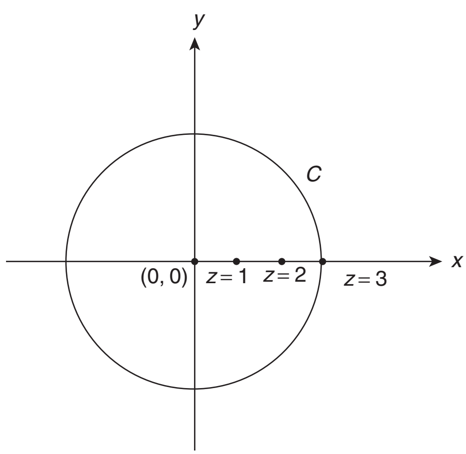

Fig. 4.39

$$\text{Res}\left[f(z);\, z=1\right] = \lim_{z\to 1}(z-1)\, f(z)$$

$$= \lim_{z\to 1}\frac{\sin \pi z^2 + \cos \pi z^2}{z-2}$$

$$= \frac{\sin \pi + \cos \pi}{-1}$$

$$= 1$$

$z = 2$ is a simple pole.

$$\text{Res}\left[f(z);\, z=2\right] = \lim_{z\to 2}(z-2)\, f(z)$$

$$= \lim_{z\to 2}\frac{\sin \pi z^2 + \cos \pi z^2}{z-1}$$

$$= \frac{\sin 4\pi + \cos 4\pi}{1}$$

$$= 1$$

(v) By Cauchy's residue theorem,

$$\int_C f(z)\, dz = 2\pi i \ \ \text{(sum of residues)}$$

$$\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}\, dz = 2\pi i(1+1)$$

$$= 4\pi i$$

Example 4

Evaluate $\oint_C \frac{5z+7}{z^2+2z-3}\, dz$, *where C is* $|z-2| = 2$. **[Summer 2013]**

Solution

(i) Let $f(z) = \frac{5z+7}{z^2+2z-3}$

$$= \frac{5z+7}{(z+3)(z-1)}$$

The poles are given by $z = -3$, $z = 1$.

(ii) C is a circle $|z-2| = 2$ with the centre at (2, 0) and a radius of 2 (Fig. 4.40).

(iii) For $z = -3$, $|z-2| = |-3-2| = 5 > 2$

Hence, $z = -3$ lies outside C.

For $z = 1$, $|z-2| = |1-2| = 1 < 2$

Hence, $z = 1$ lies inside C.

(iv) $z = 1$ is a simple pole.

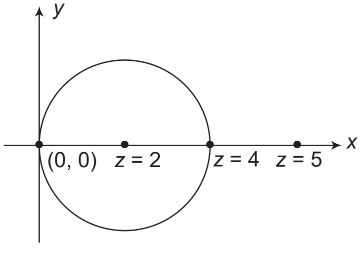

Fig. 4.40

$$\text{Res}\left[f(z);\, z=1\right] = \lim_{z\to 1}(z-1)\, f(z)$$

$$= \lim_{z\to 1}\frac{5z+7}{z+3}$$

$$= \frac{5+7}{1+3}$$

$$= 3$$

(v) By Cauchy's residue theorem,

$$\oint_C f(z)\,\mathrm{d}z = 2\pi i\text{ (sum of residues)}$$

$$\oint_C \frac{5z+7}{z^2+2z-3}\,\mathrm{d}z = 2\pi i\,(3)$$

$$= 6\pi\,\mathrm{i}$$

Example 5

Evaluate $\int_C \frac{3z^2+z-1}{(z^2-1)(z-3)}\,\mathrm{d}z$, *where C is the circle* $|z| = 2$.

Solution

(i) Let $f(z) = \frac{3z^2+z-1}{(z^2-1)(z-3)} = \frac{3z^2+z-1}{(z+1)(z-1)\,(z-3)}$

The poles are given by $z = -1$, $z = 1$, $z = 3$.

(ii) C is a circle $|z| = 2$ with the centre at (0, 0) and a radius of 2 (Fig. 4.41).

(iii) For $z = -1$, $|z| = |-1| = 1 < 2$

Hence, $z = -1$ lies inside C.

For $z = 1$, $|z| = |1| = 1 < 2$

Hence, $z = 1$ lies inside C.

For $z = 3$, $|z| = |3| = 3 > 2$

Hence, $z = 3$ lies outside C.

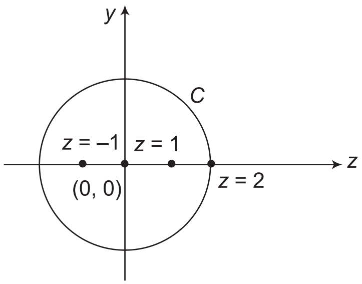

Fig. 4.41

(iv) $z = -1$ is a simple pole.

$$\text{Res}\left[f(z);\, z=-1\right] = \lim_{z\to -1}(z+1)\, f(z)$$

$$= \lim_{z\to -1}\frac{3z^2+z-1}{(z-1)(z-3)}$$

$$= \frac{3(-1)^2+(-1)-1}{(-1-1)\,(-1-3)}$$

$$= \frac{1}{8}$$

$z = 1$ is a simple pole.

$$\text{Res}\left[f(z); z=1\right] = \lim_{z\to 1}(z-1)\, f(z)$$
$$= \lim_{z\to 1}\frac{3z^2+z-1}{(z+1)(z-3)}$$
$$= \frac{3+1-1}{(1+1)\,(1-3)}$$
$$= -\frac{3}{4}$$

(v) By Cauchy's residue theorem,

$$\int_C f(z)\, dz = 2\pi i \text{ (sum of residues)}$$

$$\int_C \frac{3z^2+z-1}{(z^2-1)(z-3)}\, dz = 2\pi i\left(\frac{1}{8}-\frac{3}{4}\right)$$
$$= 2\pi i\left(-\frac{5}{8}\right)$$
$$= -\frac{5\pi i}{4}$$

Example 6

Using the residue theorem, evaluate $\int_C \frac{e^z+z}{z^3-z}\, dz$, *where* $C: |z| = \frac{\pi}{2}$.

[Summer 2015]

Solution

(i) Let $f(z) = \frac{e^z+z}{z^3-z} = \frac{e^z+z}{z(z+1)(z-1)}$

The poles are given by $z = 0$, $z = -1$, $z = 1$.

(ii) C is the circle $|z| = \frac{\pi}{2} = 1.57$ with the centre (0, 0) and a radius of $\frac{\pi}{2}$ (Fig. 4.42).

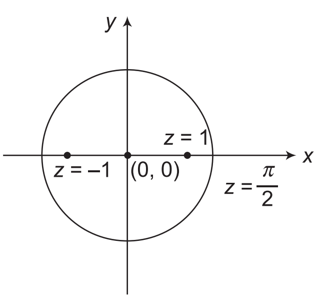

Fig. 4.42

(iii) For $z = 0$, $|z| = 0 < \frac{\pi}{2}$

Hence, $z = 0$ lies inside C.

For $z = -1$, $|z| = |-1| = 1 < \frac{\pi}{2}$

Hence, $z = -1$ lies inside C.

For $z = 1$, $|z| = |1| = 1 < \frac{\pi}{2}$

Hence, $z = 1$ lies inside C.

(iv) $z = 0$ is a simple pole.

$$\text{Res}\left[f(z); z = 0\right] = \lim_{z\to 0} z\, f(z)$$

$$= \lim_{z\to 0} \frac{e^z + z}{(z+1)(z-1)}$$

$$= \frac{e^0}{-1}$$

$$= -1$$

$z = -1$ is a simple pole.

$$\text{Res}\left[f(z); z = -1\right] = \lim_{z\to -1} (z+1) f(z)$$

$$= \lim_{z\to -1} \frac{e^z + z}{z(z-1)}$$

$$= \frac{e^{-1} - 1}{-1(-2)}$$

$$= \frac{1-e}{2e}$$

$z = 1$ is a simple pole.

$$\text{Res}\left[f(z); z = 1\right] = \lim_{z\to 1} (z-1) f(z)$$

$$= \lim_{z\to 1} \frac{e^z + z}{z(z+1)}$$

$$= \frac{e+1}{2}$$

(v) By Cauchy's residue theorem,

$$\int_C f(z)\,dz = 2\pi i \text{ (sum of residues)}$$

$$\int_C \frac{e^z + z}{z^3 - z}\,dz = 2\pi i\left(-1 + \frac{1-e}{2e} + \frac{e+1}{2}\right)$$

$$= 2\pi i\left(\frac{-2e + 1 - e + e^2 + e}{2e}\right)$$

$$= \frac{\pi i}{e}(e^2 - 2e + 1)$$

$$= \frac{\pi i}{e}(e-1)^2$$

Example 7

Use residues to evaluate the integrals of the function $\dfrac{\exp(-z)}{z^2}$ *around the circle* $|z| = 3$ *in the positive sense.*

[Winter 2012]

Solution

(i) Let $f(z) = \dfrac{\exp(-z)}{z^2} = \dfrac{e^{-z}}{z^2}$

The poles are given by $z = 0$.

(ii) C is the circle $|z| = 3$ with the centre at (0, 0) and a radius of 3 (Fig. 4.43).

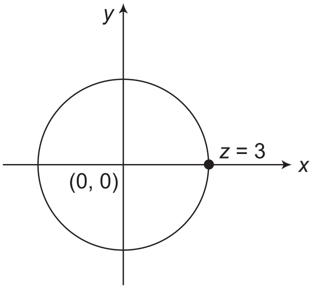

Fig. 4.43

(iii) For $z = 0$, $|z| = 0 < 3$

Hence, $z = 0$ lies inside C.

(iv) $z = 0$ is a pole of order 2.

$$\text{Res}\left[f(z); z = 0\right] = \frac{1}{(2-1)!}\lim_{z\to 0}\frac{\mathrm{d}}{\mathrm{d}z}\left[z^2 f(z)\right]$$

$$= \lim_{z\to 0} e^{-z}$$

$$= e^0$$

$$= 1$$

(v) By Cauchy's residue theorem,

$$\int_C f(z)\,\mathrm{d}z = 2\pi i \text{ (sum of residues)}$$

$$\int_C \frac{e^{-z}}{z^2}\,\mathrm{d}z = 2\pi i\,(1)$$

$$= 2\pi i$$

Example 8

Evaluate $\displaystyle\int_C \frac{e^{2z}}{(z-\pi i)^3}\,\mathrm{d}z$, *where C is* $|z - 2i| = 2$.

Solution

(i) Let $f(z) = \dfrac{e^{2z}}{(z-\pi i)^3}$

The poles are given by $z = \pi i$.

(ii) C is a circle $|z - 2i| = 2$ with the centre at (0, 2) and a radius of 2 (Fig. 4.44).

(iii) For $z = 2\pi$,

$$\left|z-2i\right| = \left|\pi i - 2i\right| = \pi - 2 = 1.14 < 2$$

Hence, $z = \pi i$ lies inside C.

(iv) $z = \pi i$ is a pole of order 3.

$$\begin{aligned}\text{Res}\left[f(z);\, z=\pi i\right] &= \frac{1}{(3-1)!}\lim_{z\to\pi i}\frac{d^2}{dz^2}\left[(z-\pi i)^3 f(z)\right]\\ &= \frac{1}{2!}\lim_{z\to\pi i}\frac{d^2}{dz^2}(e^{2z})\\ &= \frac{1}{2}\lim_{z\to\pi i}\frac{d}{dz}(2e^{2z})\\ &= \frac{1}{2}\lim_{z\to\pi i}4e^{2z}\\ &= 2e^{2\pi i}\\ &= 2(\cos 2\pi + i\sin 2\pi)\\ &= 2\end{aligned}$$

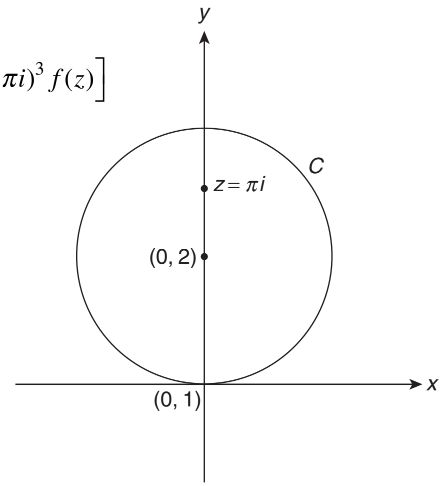

Fig. 4.44

(v) By Cauchy's residue theorem,

$$\int_C f(z)\,dz = 2\pi i \text{ (sum of residues)}$$

$$\begin{aligned}\int_C \frac{e^{2z}}{(z-\pi i)^3}\,dz &= 2\pi i(2)\\ &= 4\pi i\end{aligned}$$

Example 9

Evaluate $\displaystyle\int_C \frac{e^z\,dz}{(z^2+\pi^2)^2}$, *where C is the circle* $|z| = 4$.

Solution

(i) Let $f(z) = \dfrac{e^z}{(z^2+\pi^2)^2} = \dfrac{e^z}{(z+\pi i)^2(z-\pi i)^2}$

The poles are given by $z = -\pi i$, $z = \pi i$.

(ii) C is a circle $|z| = 4$ with the centre at (0, 0) and a radius of 4 (Fig. 4.45).

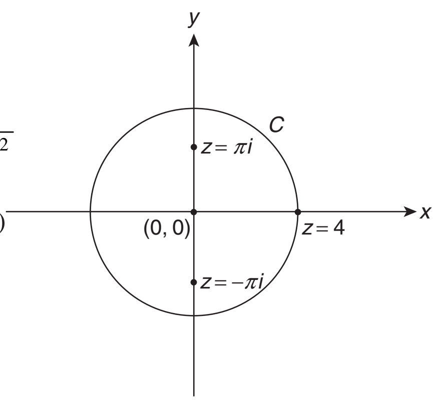

Fig. 4.45

(iii) For $z = -\pi i, |z| = |-\pi i| = \pi < 4$

Hence, $z = -\pi i$ lies inside C.

For $z = \pi i, |z| = |\pi i| = \pi < 4$

Hence, $z = \pi i$ lies inside C.

(iv) $z = -\pi i$ is a pole of order 2.

$$\text{Res}\left[f(z); z=-\pi i\right] = \frac{1}{(2-1)!}\lim_{z\to -\pi i}\frac{\mathrm{d}}{\mathrm{d}z}\left[(z+\pi i)^2 f(z)\right]$$

$$= \lim_{z\to -\pi i}\frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{e^z}{(z-\pi i)^2}\right]$$

$$= \lim_{z\to -\pi i}\left[\frac{(z-\pi i)^2 e^z - e^z\cdot 2(z-\pi i)(1)}{(z-\pi i)^4}\right]$$

$$= \lim_{z\to -\pi i}\left[\frac{e^z(z-\pi i)(z-\pi i-2)}{(z-\pi i)^4}\right]$$

$$= \lim_{z\to -\pi i}\left[\frac{e^z\left(z-\pi i-2\right)}{\left(z-\pi i\right)^3}\right]$$

$$= \frac{e^{-\pi i}\left(-\pi i-\pi i-2\right)}{\left(-\pi i-\pi i\right)^3}$$

$$= \frac{-\left(\cos\pi - i\sin\pi\right)\left(2\pi i+2\right)}{-8\pi^3 i^3}$$

$$= \frac{(\pi i+1)}{4\pi^3 i}$$

$$= \frac{-i(\pi i+1)}{4\pi^3}$$

$$= \frac{\pi - i}{4\pi^3}$$

Similarly, replacing i by $-i$ in $\text{Res}\left[f(z); z=-\pi i\right]$,

$$\text{Res}\left[f(z); z=\pi i\right] = \frac{\pi+i}{4\pi^3}$$

(v) By Cauchy's residue theorem,

$$\int_C f(z)\,\mathrm{d}z = 2\pi i\text{ (sum of residues)}$$

$$\int_C \frac{e^z}{(z^2+\pi^2)^2}\,\mathrm{d}z = 2\pi i\left(\frac{\pi-i}{4\pi^3}+\frac{\pi+i}{4\pi^3}\right)$$

$$= \frac{2\pi i(2\pi)}{4\pi^3}$$

$$= \frac{i}{\pi}$$

Example 10

Evaluate $\oint_C \frac{2z+6}{z^2+4}\,dz$ *where C is* $|z-i|=2$. **[Summer 2013]**

Solution

(i) Let $f(z)=\frac{2z+6}{z^2+4}$

$$=\frac{2z+6}{(z+2i)(z-2i)}$$

The poles are given by $z=-2i$, $z=2i$.

(ii) C is a circle $|z-i|=2$ with the centre at (0, 1) and a radius of 2 (Fig. 4.46).

(iii) For $z=2i$, $|z-i|=|2i-i|=|i|=1<2$

Hence, $z=2i$ lies inside C.

For $z=-2i$, $|z-i|=|-2i-i|=|-3i|=3>2$

Hence, $z=-2i$ lies outside C.

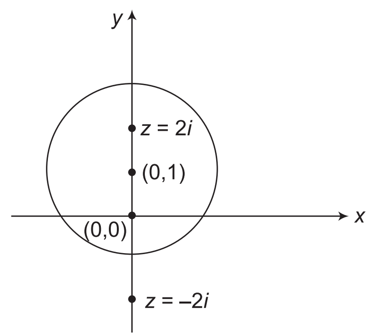

Fig. 4.46

(iv) $z=2i$ is a simple pole.

$$\text{Res}\left[f(z);\, z=2i\right]=\lim_{z\to 2i}\left[(z-2i)\,f(z)\right]$$

$$=\lim_{z\to 2i}\frac{2z+6}{z+2i}$$

$$=\frac{2(2i)+6}{2i+2i}$$

$$=\frac{2i+3}{2i}$$

(v) By Cauchy's residue theorem,

$$\oint_C f(z)\,dz=2\pi i\text{ (sum of residues)}$$

$$\oint_C \frac{2z+6}{z^2+4}\,dz=2\pi i\left(\frac{2i+3}{2i}\right)$$

$$=\pi(2i+3)$$

Example 11

Find the value of the integral $\int_C \frac{3z^2+2}{(z-1)(z^2+9)}\,dz$ *taken counterclockwise around the circle* $C: |z-2|=2$. **[Winter 2012]**

Solution

(i) Let $f(z)=\frac{3z^2+2}{(z-1)(z^2+9)}=\frac{3z^2+2}{(z-1)(z+3i)(z-3i)}$

The poles are given by $z = 1$, $z = -3i$, $z = 3i$.

(ii) C is the circle $|z-2| = 2$ with the centre at (2, 0) and a radius of 2 (Fig. 4.47).

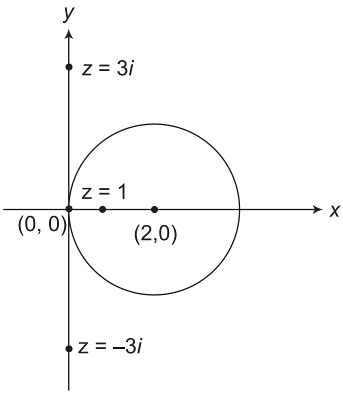

Fig. 4.47

(iii) For $z = 1$, $|z-2| = |1-2| = |-1| = 1 < 2$
Hence, $z = 1$ lies inside C.

For $z = 3i$,

$$|z-2| = |3i-2| = \sqrt{9+4} = \sqrt{13} > 2$$

Hence, $z = 3i$ lies outside C.

For $z = -3i$,

$$|z-2| = |-3i-2| = \sqrt{9+4} = \sqrt{13} > 2$$

Hence, $z = -3i$ lies outside C.

(iv) $z = 1$ is a simple pole.

$$\begin{aligned}\text{Res}\left[f(z); z=1\right] &= \lim_{z\to 1}(z-1)\, f(z) \\ &= \lim_{z\to 1}\frac{3z^2+2}{z^2+9} \\ &= \frac{3+2}{1+9} \\ &= \frac{1}{2}\end{aligned}$$

(v) By Cauchy's residue theorem,

$$\int_C f(z)\,dz = 2\pi i \text{ (sum of residues)}$$

$$\begin{aligned}\int_C \frac{3z^2+2}{(z-1)(z^2+9)}\,dz &= 2\pi i\left(\frac{1}{2}\right) \\ &= \pi i\end{aligned}$$

Example 12

Evaluate $\displaystyle\int_C \frac{z^2}{(z-1)^2(z+2)}\,dz$, *where C is the circle* $|z| = 2.5$.

[Summer 2014]

Solution

(i) Let $f(z) = \dfrac{z^2}{(z-1)^2(z+2)}$

The poles are given by $z = 1$, $z = -2$.

(ii) C is a circle $|z| = 2.5$ with the centre at (0, 0) and a radius of 2.5 (Fig. 4.48).

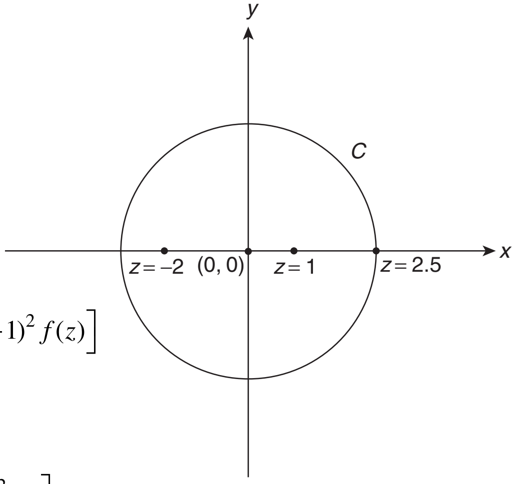

Fig. 4.48

(iii) For $z = 1, |z| = |1| = 1 < 2.5$

Hence, $z = 1$ lies inside C.

For $z = -2, |z| = |-2| = 2 < 2.5$

Hence, $z = -2$ lies inside C.

(iv) $z = 1$ is a pole of order 2.

$$\text{Res}\left[f(z); z = 1\right] = \frac{1}{(2-1)!}\lim_{z\to 1}\frac{\mathrm{d}}{\mathrm{d}z}\left[(z-1)^2 f(z)\right]$$

$$= \lim_{z\to 1}\frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{z^2}{z+2}\right]$$

$$= \lim_{z\to 1}\left[\frac{(z+2)(2z) - z^2(1)}{(z+2)^2}\right]$$

$$= \lim_{z\to 1}\frac{2z^2 + 4z - z^2}{(z+2)^2}$$

$$= \frac{2+4-1}{9}$$

$$= \frac{5}{9}$$

$z = -2$ is a simple pole.

$$\text{Res }[f(z);\, z = -2] = \lim_{z\to -2}(z+2)\, f(z)$$

$$= \lim_{z\to -2}\frac{z^2}{(z-1)^2}$$

$$= \frac{(-2)^2}{(-2-1)^2}$$

$$= \frac{4}{9}$$

(v) By Cauchy's residue theorem,

$$\int_C f(z)\,\mathrm{d}z = 2\pi i \text{ (sum of residues)}$$

$$\int_C \frac{z^2}{(z-1)^2(z+2)}\,\mathrm{d}z = 2\pi i\left(\frac{4}{9} + \frac{5}{9}\right)$$

$$= 2\pi i(1)$$

$$= 2\pi i$$

Example 13

Evaluate $\oint_C \frac{5z+7}{z^2+2z-3}\,dz$, *where C is* $|z-2| = 2$.

Solution

(i) Let $f(z) = \frac{5z+7}{z^2+2z-3} = \frac{5z+7}{(z+3)(z-1)}$

The poles are given by $z = 1$, $z = -3$.

(ii) C is the circle $|z-2| = 2$ with the centre at (2, 0) and a radius of 2 (Fig. 4.47).

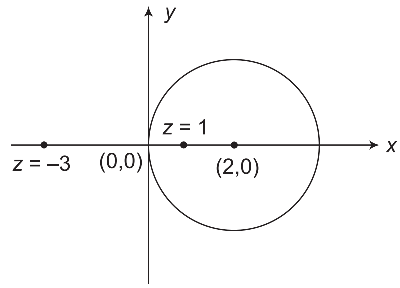

Fig. 4.49

(iii) For $z = 1$, $|z-2| = |-1| = 1 < 2$

Hence, $z = 1$ lies inside C.

For $z = -3$, $|z-2| = |-3-2| = |-5| = 5 > 2$

Hence, $z = -3$ lies outside C.

(iv) $z = 1$ is a simple pole

$$\begin{aligned}\text{Res}\left[f(z); z=1\right] &= \lim_{z\to 1}(z-1)f(z)\\ &= \lim_{z\to 1}\frac{5z+7}{z+3}\\ &= \frac{12}{4}\\ &= 3\end{aligned}$$

(v) By Cauchy's residue theorem,

$$\begin{aligned}\oint_C f(z)\,dz &= 2\pi i\,(\text{sum of residues})\\ &= 2\pi i(3)\\ &= 6\pi i\end{aligned}$$

Example 14

Evaluate $\int_C \frac{z-1}{(z+1)^2(z-2)}\,dz$, *where C is* $|z-i| = 2$.

Solution

(i) Let $f(z)=\dfrac{z-1}{(z+1)^2(z-2)}$

The poles are given by $z=-1$, $z=2$.

(ii) C is a circle $|z-i|=2$ with the centre at (0, 1) and a radius of 2 (Fig. 4.50).

(iii) For $z=-1, |z-i|=|-1-i|=\sqrt{2}<2$

Hence, $z=-1$ lies inside C.

For $z=2, |z-i|=|2-i|=\sqrt{5}>2$

Hence, $z=2$ lies outside C.

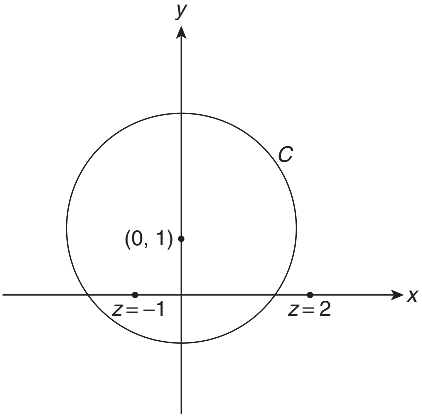

Fig. 4.50

(iv) $z=-1$ is a pole of order 2.

$$\begin{aligned}\text{Res}\left[f(z); z=-1\right] &= \frac{1}{(2-1)!}\lim_{z\to-1}\frac{\mathrm{d}}{\mathrm{d}z}\left[(z+1)^2 f(z)\right]\\ &= \lim_{z\to-1}\frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{z-1}{z-2}\right]\\ &= \lim_{z=-1}\frac{(z-2)(1)-(z-1)(1)}{(z-2)^2}\\ &= \lim_{z=-1}\frac{-1}{(z-2)^2}\\ &= -\frac{1}{(-1-2)^2}\\ &= -\frac{1}{9}\end{aligned}$$

(v) By Cauchy's residue theorem,

$$\int_C f(z)\,\mathrm{d}z = 2\pi i \text{ (sum of residues)}$$

$$\begin{aligned}\int_C \frac{z-1}{(z+1)^2(z-2)}\,\mathrm{d}z &= 2\pi i\left(-\frac{1}{9}\right)\\ &= -\frac{2\pi i}{9}\end{aligned}$$

Example 15

Evaluate $\displaystyle\int_C \frac{\mathrm{d}z}{(z^2+4)^2}$, *where C is the circle* $|z-i|=2$.

Solution

(i) Let $f(z) = \dfrac{1}{(z^2+4)^2} = \dfrac{1}{(z+2i)^2(z-2i)^2}$

The poles are given by $z = -2i$, $z = 2i$.

(ii) C is a circle $|z-i| = 2$ with the centre at (0, 1) and a radius of 2 (Fig. 4.51).

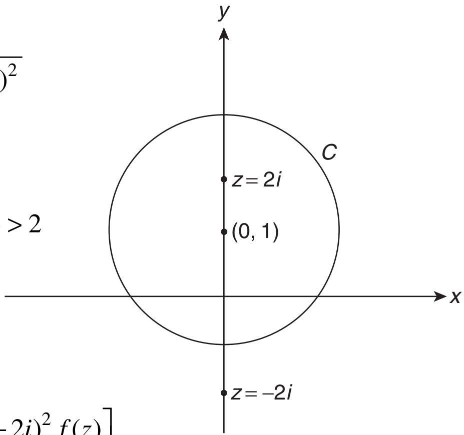

Fig. 4.51

(iii) For $z = -2i, |z-i| = |-2i-i| = |-3i| = 3 > 2$

Hence, $z = -2i$ lies outside C.

For $z = 2i, |z-i| = |2i-i| = |i| = 1 < 2$

Hence, $z = 2i$ lies inside C.

(iv) $z = 2i$ is a pole of order 2.

$$\text{Res}\left[f(z); z = 2i\right] = \frac{1}{(2-1)!}\lim_{z\to 2i}\frac{\mathrm{d}}{\mathrm{d}z}\left[(z-2i)^2 f(z)\right]$$

$$= \lim_{z\to 2i}\frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{1}{(z+2i)^2}\right]$$

$$= \lim_{z\to 2i}\left[-\frac{2}{(z+2i)^3}\right]$$

$$= -\frac{2}{(4i)^3}$$

$$= -\frac{2}{-64i}$$

$$= \frac{1}{32i}$$

(v) By Cauchy's residue theorem,

$$\int_C f(z)\,\mathrm{d}z = 2\pi i \text{ (sum of residues)}$$

$$\int_C \frac{\mathrm{d}z}{(z^2+4)^2} = 2\pi i\left(\frac{1}{32i}\right)$$

$$= \frac{\pi}{16}$$

Example 16

Evaluate $\displaystyle\int \frac{z^4}{(z+1)(z-i)^2}\,\mathrm{d}z$, *where C:* $9x^2 + 4y^2 = 36$ *by using the residue theorem.* **[Summer 2015]**

Solution

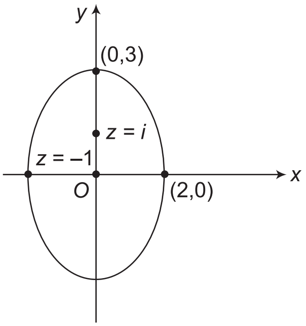

Fig. 4.52

(i) Let $f(z)=\dfrac{z^4}{(z+1)(z-i)^2}$

The poles are given by $z=-1$, $z=i$.

(ii) C is an ellipse $\dfrac{x^2}{4}+\dfrac{y^2}{9}=1$ (Fig. 4.52)

(iii) For $z=-1$, $x=-1$, $y=0$

$$\therefore \quad \frac{x^2}{4}+\frac{y^2}{9}=\frac{(-1)^2}{4}+0=\frac{1}{4}<1$$

Hence, $z=-1$ lies inside C.

For $z=i$, $x=0$, $y=1$

$$\therefore \quad \frac{x^2}{4}+\frac{y^2}{9}=0+\frac{1}{9}<1$$

Hence, $z=i$ lies inside C.

(iv) $z=-1$ is a simple pole.

$$\text{Res}\left[f(z);z=-1\right]=\lim_{z\to-1}(z+1)f(z)$$

$$=\lim_{z\to-1}\frac{z^4}{(z-i)^2}$$

$$=\frac{(-1)^4}{(-1-i)^2}$$

$$=\frac{1}{2i}$$

$z=i$ is a pole of order 2.

$$\text{Res}\left[f(z);z=i\right]=\frac{1}{(2-1)!}\lim_{z\to i}\frac{\mathrm{d}}{\mathrm{d}z}\left[(z-i)^2\ f(z)\right]$$

$$=\lim_{z\to i}\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{z^4}{z+1}\right)$$

$$=\lim_{z\to i}\frac{4z^3(z+1)-z^4}{(z+1)^2}$$

$$=\lim_{z\to i}\frac{3z^4+4z^3}{(z+1)^2}$$

$$=\frac{3i^4+4i^3}{(i+1)^2}$$

$$=\frac{3-4i}{2i}$$

(v) By Cauchy's residue theorem,

$$\int_C f(z)\,dz = 2\pi i \text{ (sum of residues)}$$

$$\int_C \frac{z^4}{(z+1)(z-i)^2}\,dz = 2\pi i\left(\frac{1}{2i} + \frac{3-4i}{2i}\right)$$

$$= 4\pi(1-i)$$

Example 17

Evaluate $\oint_C e^{\frac{3}{z}}dz$ *where C is* $|z| = 1$.

Solution

(i) Let $f(z) = e^{\frac{3}{z}}$

$$= 1 + \frac{3}{z} + \frac{1}{2!}\left(\frac{3}{z}\right)^2 + \cdots$$

Since the number of terms of negative powers of z are infinite, $z = 0$ is an essential singularity.

$$\text{Res}\left[f(z);\, z=0\right] = \text{coefficient of } \frac{1}{z}$$

$$= 3$$

By Cauchy's residue theorem,

$$\oint_C f(z)\,dz = 2\pi i \text{ (sum of residues)}$$

$$\oint_C e^{\frac{3}{z}}\,dz = 2\pi i(3)$$

$$= 6\pi i$$

Example 18

Evaluate $\int_C z^2 e^{\frac{1}{z}}\,dz$, *where C is* $|z| = 1$.

Solution

Let $$f(z) = z^2 e^{\frac{1}{z}}$$

$$= z^2\left(1 + \frac{1}{z} + \frac{1}{2!}\frac{1}{z^2} + \frac{1}{3!}\frac{1}{z^3} + \frac{1}{4!}\frac{1}{z^4} + \cdots\right)$$

$$= z^2 + z + \frac{1}{2} + \frac{1}{6z} + \frac{1}{24z^2} + \cdots$$

$$= z^2 + z + \frac{1}{2} + \frac{1}{6}z^{-1} + \frac{1}{24}z^{-2} + \cdots$$

Since the number of terms of negative powers of z are infinite, $z = 0$ is an essential singularity.

$$\text{Res}\,[f(z); z = 0] = \text{Coefficient of } \frac{1}{z}$$

$$= \frac{1}{6}$$

By Cauchy's residue theorem,

$$\int_C f(z)\,\mathrm{d}z = 2\pi i \text{ (sum of residues)}$$

$$\int_C z^2 e^{\frac{1}{z}}\,\mathrm{d}z = 2\pi i\left(\frac{1}{6}\right)$$

$$= \frac{\pi i}{3}$$

Example 19

Evaluate $\int_C e^{-\frac{1}{z}} \sin\left(\frac{1}{z}\right) \mathrm{d}z$, *where C is* $|z| = 1$.

Solution

Let $$f(z) = e^{-\frac{1}{z}} \sin\left(\frac{1}{z}\right)$$

$$= \left(1 - \frac{1}{z} + \frac{1}{2z^2} - \cdots\right)\left(\frac{1}{z} - \frac{1}{6z^3} + \cdots\right)$$

$$= \frac{1}{z} - \frac{1}{z^2} + \frac{1}{3z^3} - \cdots$$

$z = 0$ is an isolated essential singularity.

$$\text{Res}\left[f(z); z = 0\right] = \text{Coefficient of } \frac{1}{z}$$
$$= 1$$

By Cauchy's residue theorem,

$$\int_C f(z)\, dz = 2\pi i \text{ (sum of residues)}$$

$$\int_C e^{-\frac{1}{z}} \sin\left(\frac{1}{z}\right) dz = 2\pi i(1)$$
$$= 2\pi i$$

Example 20

Evaluate $\int_C \frac{dz}{\cos z}$ *where C is* $|z| = 2$.

Solution

(i) Let $f(z) = \frac{1}{\cos z}$

The poles are given by

$\cos z = 0$

$$z = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$$

(ii) C is a circle $|z| = 2$ with the centre at (0, 0) and a radius of 2 (Fig. 4.53).

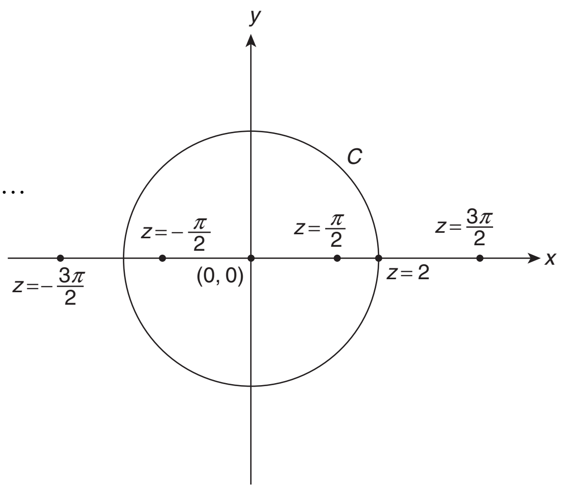

Fig. 4.53

(iii) For $z = \pm\frac{\pi}{2}$,

$$|z| = \left|\pm\frac{\pi}{2}\right| = 1.52 < 2$$

Hence, $z = \pm\frac{\pi}{2}$ lies inside C.

Other poles, i.e., $z = \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$ lies outside C.

(iv) $z = \frac{\pi}{2}$ is a simple pole.

$$\text{Res}\left[f(z); z = \frac{\pi}{2}\right] = \lim_{z \to \frac{\pi}{2}} \left(z - \frac{\pi}{2}\right) f(z)$$

$$= \lim_{z \to \frac{\pi}{2}} \left(z - \frac{\pi}{2}\right) \frac{1}{\cos z}$$

$$= \lim_{z \to \frac{\pi}{2}} \frac{\left(z - \frac{\pi}{2}\right)}{\cos z} \qquad \left[\frac{0}{0}\text{form}\right]$$

$$= \lim_{z \to \frac{\pi}{2}} \frac{1}{-\sin z} \qquad [\text{Using L' Hospital's rule}]$$

$$= -\frac{1}{\sin \frac{\pi}{2}}$$

$$= -1$$

Similarly,

$$\text{Res}\left[f(z); z = -\frac{\pi}{2}\right] = \lim_{z \to -\frac{\pi}{2}} \left(z + \frac{\pi}{2}\right) f(z)$$

$$= \lim_{z \to -\frac{\pi}{2}} \frac{\left(z + \frac{\pi}{2}\right)}{\cos z} \qquad \left[\frac{0}{0}\text{form}\right]$$

$$= \lim_{z \to -\frac{\pi}{2}} \frac{1}{-\sin z} \qquad [\text{Using L'Hospital's rule}]$$

$$= \frac{1}{-\sin\left(-\frac{\pi}{2}\right)}$$

$$= 1$$

By Cauchy's residue theorem,

$$\int_C f(z)\,dz = 2\pi i\,(\text{sum of residues})$$

$$\int_C \frac{dz}{\cos z} = 2\pi i(-1+1)$$

$$= 0$$

Example 21

Evaluate $\int_C \operatorname{cosec} z\, dz$, *where C is* $|z| = 1$.

Solution

(i) Let $f(z) = \text{cosec } z$

$$= \frac{1}{\sin z}$$

The poles are given by $\sin z = 0$

$$z = 0$$

$$f(z) = \frac{1}{z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots}$$

$$= \frac{1}{z\left(1 - \frac{z^2}{6} + \frac{z^4}{120} - \cdots\right)}$$

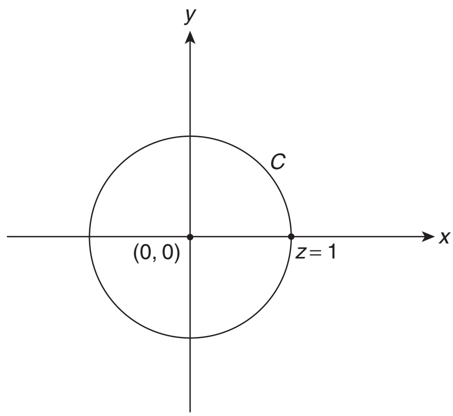

Fig. 4.54

(ii) C is a circle $|z| = 1$ with the centre at (0, 0) and a radius of 1 (Fig. 4.54).

(iii) For $z = 0$, $|z| = 0 < 1$

Hence, $z = 0$ lies inside C.

(iv) $z = 0$ is a simple pole.

$$\text{Res}\left[f(z); z = 0\right] = \lim_{z \to 0} z\, f(z)$$

$$= \lim_{z \to 0} \frac{1}{1 - \frac{z^2}{6} + \frac{z^4}{120} - \cdots}$$

$$= 1$$

By Cauchy's residue theorem,

$$\int_C f(z)\, dz = 2\pi i \text{ (sum of residues)}$$

$$\int_C \text{cosec } z\, dz = 2\pi i(1)$$

$$= 2\pi i$$

Example 22

Evaluate $\displaystyle\int_C \frac{dz}{\sinh 2z}$, *where C is* $|z| = 2$.

Solution

(i) Let $f(z) = \dfrac{1}{\sinh 2z}$

The poles are given by

$$\sinh 2z = 0$$
$$z = 0$$

$$f(z) = \frac{1}{2z + \frac{(2z)^3}{3!} + \frac{(2z)^5}{5!} + \cdots}$$

$$= \frac{1}{2z\left[1 + \frac{(2z)^2}{3!} + \frac{(2z)^4}{5!} + \cdots\right]}$$

(ii) C is a circle $|z| = 2$ with the centre at (0, 0) and a radius of 2 (Fig. 4.55).

(iii) For $z = 0$, $|z| = 0 < 2$

Hence, $z = 0$ lies inside C.

(iv) $z = 0$ is a simple pole.

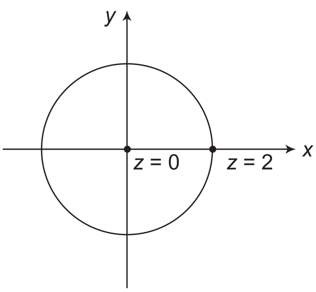

Fig. 4.55

$$\text{Res}\left[f(z); z = 0\right] = \lim_{z \to 0} z\, f(z)$$

$$= \lim_{z \to 0} \frac{1}{2\left[1 + \frac{(2z)^2}{3!} + \frac{(2z)^4}{5!} + \cdots\right]}$$

$$= \frac{1}{2}$$

By Cauchy's residue theorem,

$$\int_C f(z)\,dz = 2\pi i\,(\text{sum of residues})$$

$$\int_C \frac{dz}{\sinh 2z} = 2\pi i\left(\frac{1}{2}\right)$$

$$= \pi i$$

Example 23

Evaluate $\int_C \tan z\, dz$, *where C is the circle* $|z| = 2$. **[Summer 2013]**

Solution

(i) Let $f(z) = \tan z = \dfrac{\sin z}{\cos z}$

The poles are given by

$$\cos z = 0$$

$$z = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2} + \cdots$$

(ii) C is a circle $|z| = 2$ with the centre at (0, 0) and a radius of 2 (Fig. 4.56).

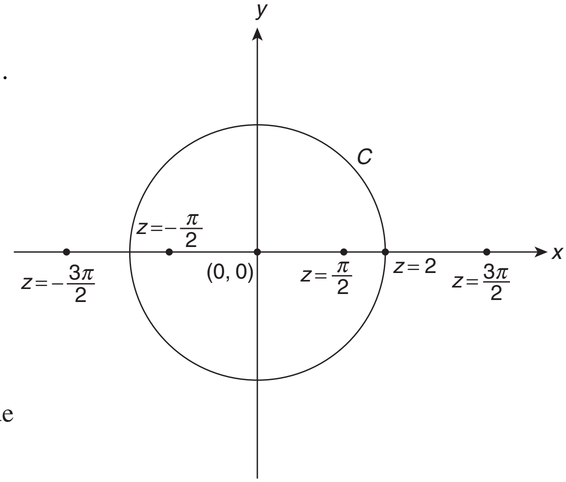

Fig. 4.56

(iii) For $z = \pm\frac{\pi}{2}$,

$$|z| = \left|\pm\frac{\pi}{2}\right| = 1.57 < 2$$

Hence, $z = \pm\frac{\pi}{2}$, lies inside C.

(iv) $z = \frac{\pi}{2}$ is a simple pole.

$$\begin{aligned}
\text{Res}\left[f(z); z = \frac{\pi}{2}\right] &= \lim_{z\to\frac{\pi}{2}}\left(z - \frac{\pi}{2}\right) f(z) \\
&= \lim_{z\to\frac{\pi}{2}}\left(z - \frac{\pi}{2}\right)\tan z \\
&= \lim_{z\to\frac{\pi}{2}}\frac{\left(z - \frac{\pi}{2}\right)}{\cot z} \qquad \left[\frac{0}{0}\text{ form}\right] \\
&= \lim_{z\to\frac{\pi}{2}}\frac{1}{-\text{cosec}^2 z} \qquad [\text{Using L'Hospital's rule}] \\
&= -\frac{1}{\text{cosec}^2\frac{\pi}{2}} \\
&= -1
\end{aligned}$$

$z = -\frac{\pi}{2}$ is a simple pole.

$$\begin{aligned}
\text{Res}\left[f(z); z = -\frac{\pi}{2}\right] &= \lim_{z\to-\frac{\pi}{2}}\left(z + \frac{\pi}{2}\right) f(z) \\
&= \lim_{z\to-\frac{\pi}{2}}\left(z + \frac{\pi}{2}\right)\tan z
\end{aligned}$$

$$= \lim_{z \to -\frac{\pi}{2}} \frac{z + \frac{\pi}{2}}{\cot z} \quad \left[\frac{0}{0} \text{form}\right]$$

$$= \lim_{z \to -\frac{\pi}{2}} \frac{1}{-\operatorname{cosec}^2 z} \quad \left[\text{Using L'Hospital's rule}\right]$$

$$= -\frac{1}{\operatorname{cosec}^2\left(-\frac{\pi}{2}\right)}$$

$$= -1$$

By Cauchy's residue theorem,

$$\int_C f(z)\, dz = 2\pi i \text{ (sum of residues)}$$

$$\int_C \tan z\, dz = 2\pi i(-1-1)$$

$$= -4\pi i$$

Example 24

Evaluate $\int_C \frac{dz}{z \sin z}$, *where C is* $|z| = 1$.

Solution

(i) Let $f(z) = \frac{1}{z \sin z}$

The poles are given by $z \sin z = 0$

$z = 0, \quad \sin z = 0$

$z = 0, \pm\pi, \pm 2\pi, \ldots$

(ii) C is a circle $|z| = 1$ with the centre at (0, 0) and a radius of 1 (Fig. 4.57).

(iii) For $z = 0$, $|z| = 0 < 1$

Hence, $z = 0$ lies inside C.

Other poles $z = \pm\pi, \pm 2\pi, \ldots$ lies outside C.

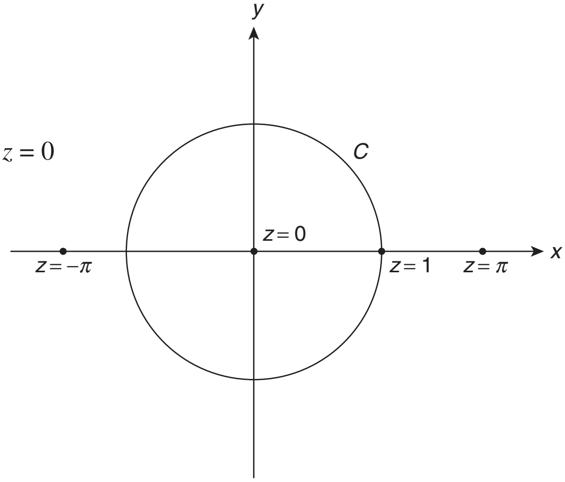

Fig. 4.57

(iv) $z = 0$ is a pole of order 2.

$$\begin{aligned}
\text{Res}\left[f(z); z=0\right] &= \frac{1}{(2-1)!}\lim_{z\to 0}\frac{\mathrm{d}}{\mathrm{d}z}\left[z^2\ f(z)\right] \\
&= \lim_{z\to 0}\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{z}{\sin z}\right) \\
&= \lim_{z\to 0}\frac{(1)\sin z - z\,(\cos z)}{(\sin z)^2} && \left[\frac{0}{0}\text{form}\right] \\
&= \lim_{z\to 0}\frac{\cos z - (\cos z - z\sin z)}{2\sin z\cos z} && \left[\text{Using L'Hospital's rule}\right] \\
&= \lim_{z\to 0}\frac{z\sin z}{2\sin z\cos z} \\
&= \lim_{z\to 0}\frac{z}{2\cos z} \\
&= 0
\end{aligned}$$

(v) By Cauchy's residue theorem,

$$\int_C f(z)\,\mathrm{d}z = 2\pi i\ (\text{sum of residues})$$

$$\begin{aligned}
\int_C \frac{\mathrm{d}z}{z\sin z} &= 2\pi i(0) \\
&= 0
\end{aligned}$$

Example 25

Evaluate $\int \frac{z\sec z}{1-z^2}\,\mathrm{d}z$, *where C is circle* $|z| = 2$.

Solution

(i) Let $f(z) = \frac{z\sec z}{1-z^2} = \frac{z}{(1-z^2)\cos z}$

$$= \frac{z}{(1+z)(1-z)\cos z}$$

The poles are given by

$$(1-z^2)\cos z = 0$$

$$(1-z^2) = 0,\ \cos z = 0$$

$$z = \pm 1,\ z = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$$

(ii) C is a circle $|z| = 2$ with the centre at (0, 0) and a radius of 2 (Fig. 4.58).

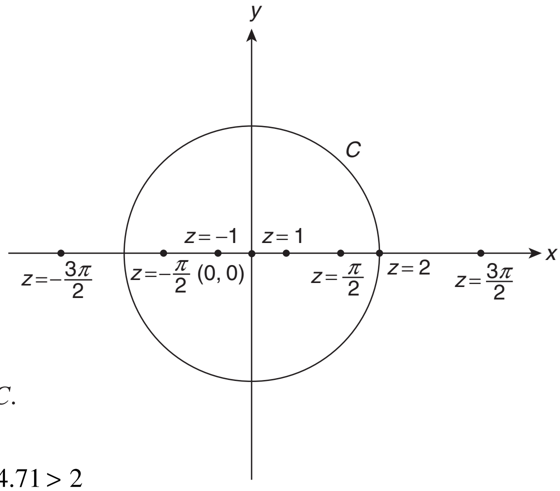

Fig. 4.58

(iii) For $z = \pm 1, |z| = |\pm 1| = 1 < 2$

Hence, $z = \pm 1$ lies inside C.

For $z = \pm\frac{\pi}{2}$,

$$|z| = \left|\pm\frac{\pi}{2}\right| = 1.57 < 2$$

Hence, $z = \pm\frac{\pi}{2}$ lies inside C.

For $z = \pm\frac{3\pi}{2}, |z| = \left|\pm\frac{3\pi}{2}\right| = 4.71 > 2$

Hence, $z = \pm\frac{3\pi}{2}$ lies outside C.

Only poles at $z = \pm 1,\ \pm\frac{\pi}{2}$ lie inside C.

(iv) $z = 1$ is a simple pole.

$$\begin{aligned}\text{Res}\left[f(z); z = 1\right] &= \lim_{z\to 1}(z-1)\, f(z)\\ &= \lim_{z\to 1}\left[-\frac{z}{(1+z)\cos z}\right]\\ &= -\frac{1}{2\cos 1}\end{aligned}$$

$z = -1$ is a simple pole.

$$\begin{aligned}\text{Res}\left[f(z); z = -1\right] &= \lim_{z\to -1}(z+1)\, f(z)\\ &= \lim_{z\to -1}\frac{z}{(1-z)\cos z}\\ &= -\frac{1}{2\cos 1}\end{aligned}$$

$z = \frac{\pi}{2}$ is a simple pole.

$$\begin{aligned}\text{Res}\left[f(z); z = \frac{\pi}{2}\right] &= \lim_{z\to\frac{\pi}{2}}\left(z-\frac{\pi}{2}\right) f(z)\\ &= \lim_{z\to\frac{\pi}{2}}\left(z-\frac{\pi}{2}\right)\frac{z}{\left(1-z^2\right)\cos z}\end{aligned}$$

$$= \lim_{z \to \frac{\pi}{2}} \left(\frac{z}{1-z^2} \right) \cdot \lim_{z \to \frac{\pi}{2}} \frac{z - \frac{\pi}{2}}{\cos z} \qquad \left[\frac{0}{0} \text{ form} \right]$$

$$= \frac{\frac{\pi}{2}}{1 - \frac{\pi^2}{4}} \cdot \lim_{z \to \frac{\pi}{2}} \frac{1}{(-\sin z)} \qquad [\text{Using L' Hospital's rule}]$$

$$= \frac{2\pi}{4-\pi^2} \cdot \frac{1}{\left(-\sin \frac{\pi}{2} \right)}$$

$$= -\frac{2\pi}{4-\pi^2}$$

$$= \frac{2\pi}{\pi^2 - 4}$$

$z = -\frac{\pi}{2}$ is a simple pole.

$$\text{Res}\left[f(z); z = -\frac{\pi}{2} \right] = \lim_{z \to -\frac{\pi}{2}} \left(z + \frac{\pi}{2} \right) f(z)$$

$$= \lim_{z \to -\frac{\pi}{2}} \left(z + \frac{\pi}{2} \right) \frac{z}{(1-z^2) \cos z}$$

$$= \lim_{z \to -\frac{\pi}{2}} \left(\frac{z}{1-z^2} \right) \cdot \lim_{z \to -\frac{\pi}{2}} \frac{z + \frac{\pi}{2}}{\cos z} \qquad \left[\frac{0}{0} \text{ form} \right]$$

$$= \frac{-\frac{\pi}{2}}{1 - \frac{\pi^2}{4}} \lim_{z \to -\frac{\pi}{2}} \frac{1}{(-\sin z)} \qquad [\text{Using L' Hospital's rule}]$$

$$= -\frac{2\pi}{4-\pi^2} \cdot \frac{1}{\left[-\sin\left(-\frac{\pi}{2} \right) \right]}$$

$$= \frac{2\pi}{\pi^2 - 4}$$

By Cauchy's residue theorem,

$$\int_C f(z)\,\mathrm{d}z = 2\pi i \text{ (sum of residues)}$$

$$\int_C \frac{z\sec z}{1-z^2}\,\mathrm{d}z = 2\pi i\left[-\frac{1}{2\cos 1}-\frac{1}{2\cos 1}+\frac{2\pi}{(\pi^2-4)}+\frac{2\pi}{(\pi^2-4)}\right]$$

$$= 2\pi i\left[-\frac{1}{\cos 1}+\frac{4\pi}{(\pi^2-4)}\right]$$

$$= -\frac{2\pi i}{\cos 1}+\frac{8\pi^2 i}{\pi^2-4}$$

Example 26

Evaluate $\int \frac{\cos \pi z}{z^2-1}\,\mathrm{d}z$, *where C is the rectangle whose vertices are* $2 \pm i, -2 \pm i$.

Solution

(i) Let $f(z) = \frac{\cos \pi z}{z^2-1} = \frac{\cos \pi z}{(z+1)(z-1)}$

The poles are given by $z = -1, z = 1$.

(ii) C is a rectangle with vertices $2 \pm i$ and $-2 \pm i$. (Fig. 4.59).

(iii) The poles $z = \pm 1$ lie inside C.

(iv) $z = 1$ is a simple pole.

$$\text{Res}\left[f(z); z=1\right] = \lim_{z\to 1}(z-1)\,f(z)$$

$$= \lim_{z\to 1}\frac{\cos \pi z}{z+1}$$

$$= -\frac{1}{2}$$

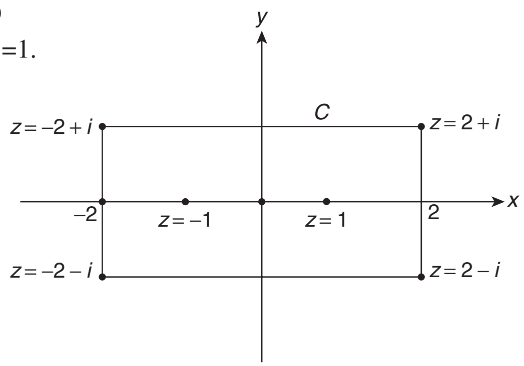

Fig. 4.59

$z = -1$ is a simple pole.

$$\text{Res}\left[f(z); z=-1\right] = \lim_{z\to -1}(z+1)\,f(z)$$

$$= \lim_{z\to -1}\frac{\cos \pi z}{z-1}$$

$$= \frac{-1}{-2}$$

$$= \frac{1}{2}$$

By Cauchy's residue theorem,

$$\int_C f(z)\,dz = 2\pi i\,(\text{sum of residues})$$

$$\int_C \frac{\cos \pi z}{z^2-1}\,dz = 2\pi i\left(-\frac{1}{2}+\frac{1}{2}\right)$$

$$= 0$$

EXERCISE 4.5

Evaluate the following integrals using Cauchy's residue theorem:

1. $\int_C \frac{\sin z}{z^6}\,dz$, where C is $|z| = 1$

$\left[\textbf{Ans.:} \frac{\pi i}{60}\right]$

2. $\int_C z\,e^{\frac{1}{z}}\,dz$, where C is $|z| = 1$

[**Ans.:** πi]

3. $\int_C \frac{z}{(z-1)^2(z+1)}\,dz$, where C is $|z| = \frac{3}{4}$

[**Ans.:** 0]

4. $\int_C \frac{z}{(z+1)^2(z-2)}\,dz$, where C is $|z - i| = 2$

$\left[\textbf{Ans.:} -\frac{4\pi i}{9}\right]$

5. $\int_C \frac{dz}{(z^2+1)^2}$, where C is $|z - i| = 1$

$\left[\textbf{Ans.:} \frac{\pi}{2}\right]$

6. $\int_C \frac{(z+4)^2}{z^4+5z^3+6z^2}$, where C is $|z| = 1$

$\left[\textbf{Ans.:} -\frac{16\pi i}{9}\right]$

7. $\int_C \frac{3z^2+2z-4}{z^3-4z}$, where C is $|z - i| = 3$

[**Ans.:** $6\pi i$]