

## 2.14 ROW SPACE, COLUMN SPACE AND NULL SPACE

Let

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

be an  $m \times n$  matrix. The vectors

$$\begin{aligned} \mathbf{r}_1 &= [a_{11} \ a_{12} \ \cdots \ a_{1n}] \\ \mathbf{r}_2 &= [a_{21} \ a_{22} \ \cdots \ a_{2n}] \\ &\vdots \\ \mathbf{r}_m &= [a_{m1} \ a_{m2} \ \cdots \ a_{mn}] \end{aligned}$$

in  $R^n$  are called row vectors of  $A$ , and the vectors

$$\mathbf{c}_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \mathbf{c}_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \dots, \mathbf{c}_n = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

in  $R^m$  are called column vectors of  $A$ .

### (1) Row Space

The subspace of  $R^n$  spanned by the row vectors of  $A$  is called the row space of  $A$ .

### (2) Column Space

The subspace of  $R^m$  spanned by the column vectors of  $A$  is called the column space of  $A$ .

### (3) Null Space

The solution space of the homogeneous system of equations  $A\mathbf{x} = \mathbf{0}$  is called the null space of  $A$ .

#### 2.14.1 Basis for Row Space

**Theorem 2.18:** Elementary row transformations do not change the row space and null space of a matrix.

**Note:** If a matrix  $A$  is reduced to row echelon form  $B$  then the row spaces of  $A$  and  $B$  are same.

**Theorem 2.19:** If  $B$  is the row echelon form of  $A$  then the row vectors of  $B$  with leading 1's (i.e. non-zero row vectors) form a basis for the row space of  $B$ , and hence form a basis for the row space of  $A$ .

**Note:**

- (i) A basis for the row space of a matrix  $A$  may not consist entirely of row vectors.
- (ii) A basis for the row space of  $A$  consisting entirely of row vectors of  $A$  can be obtained by finding the basis for column space of  $A^T$ .

### 2.14.2 Basis for Column Space

**Theorem 2.20:** If  $A$  and  $B$  are row equivalent matrices then

- (i) A set of column vectors of matrix  $A$  is linearly independent if and only if the corresponding column vectors of  $B$  are linearly independent
- (ii) A set of column vectors of matrix  $A$  forms a basis for the column space of  $A$  if and only if the corresponding column vectors of  $B$  form a basis for the column space of  $B$

**Theorem 2.21:** If  $B$  is the row echelon form of a matrix  $A$  then

- (i) The column vectors containing the leading 1's of row vectors form a basis for the column space of  $B$
- (ii) The column vectors of  $A$  corresponding to the column vectors of  $B$  containing the leading 1's form a basis for the column space of  $A$

### 2.14.3 Basis for Null Space

The basis for the null space of  $A$  is the basis for the solution space of the homogeneous system  $Ax = \mathbf{0}$ . This method has been discussed in 2.10.

**Example 1:** Find a basis for the row and column spaces of  $A$ .

$$A = \begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{bmatrix}$$

**Solution:**

$$A = \begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{bmatrix}$$

Reducing the matrix  $A$  to row echelon form,

$$\begin{array}{l} R_2 - 2R_1, \quad R_3 + R_1 \\ \sim \begin{bmatrix} 1 & 4 & 5 & 2 \\ 0 & -7 & -7 & -4 \\ 0 & 7 & 7 & 4 \end{bmatrix} \end{array}$$

$$\begin{array}{l} R_3 + R_2 \\ \sim \begin{bmatrix} 1 & 4 & 5 & 2 \\ 0 & -7 & -7 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$\begin{aligned} & \left(-\frac{1}{7}\right)R_2 \\ & \sim \begin{bmatrix} 1 & 4 & 5 & 2 \\ 0 & 1 & 1 & \frac{4}{7} \\ 0 & 0 & 0 & 0 \end{bmatrix} = B \end{aligned}$$

Basis for the row space of  $A =$  Non-zero rows of  $B = \left\{ (1, 4, 5, 2), \left(0, 1, 1, \frac{4}{7}\right) \right\}$

The leading 1's appear in columns 1 and 2.

Hence, basis for the column space of  $A = \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix} \right\}$

**Example 2:** Find a basis for the row and column spaces of

$$A = \begin{bmatrix} 1 & 4 & 5 & 4 \\ 2 & 9 & 8 & 2 \\ 2 & 9 & 9 & 7 \\ -1 & -4 & -5 & -4 \end{bmatrix}$$

**Solution:**

$$A = \begin{bmatrix} 1 & 4 & 5 & 4 \\ 2 & 9 & 8 & 2 \\ 2 & 9 & 9 & 7 \\ -1 & -4 & -5 & -4 \end{bmatrix}$$

Reducing the matrix  $A$  to row echelon form,

$$\begin{aligned} & R_2 - 2R_1, R_3 - 2R_1, R_4 + R_1 \\ & \sim \begin{bmatrix} 1 & 4 & 5 & 4 \\ 0 & 1 & -2 & -6 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} & R_3 - R_2 \\ & \sim \begin{bmatrix} 1 & 4 & 5 & 4 \\ 0 & 1 & -2 & -6 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} = B \end{aligned}$$

Basis for the row space of  $A$  = non-zero rows of  $B = \{(1, 4, 5, 4), (0, 1, -2, -6), (0, 0, 1, 5)\}$

The leading 1's appear in columns 1, 2 and 3.

$$\text{Hence, basis for the column space of } A = \left\{ \begin{bmatrix} 1 \\ 2 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ 9 \\ 9 \\ -4 \end{bmatrix}, \begin{bmatrix} 5 \\ 8 \\ 9 \\ -5 \end{bmatrix} \right\}$$

**Example 3:** Find a basis for the row space of

$$A = \begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 3 & -2 & 1 & 4 & -1 \\ -1 & 0 & -1 & -2 & -1 \\ 2 & 3 & 5 & 7 & 8 \end{bmatrix}$$

consisting entirely the row vectors of  $A$ .

**Solution:** We know that

$$\text{Row space of } A = \text{Column space of } A^T$$

$\therefore$  Basis for the row space of  $A$  = Transpose of the basis for the column space of  $A^T$ .

$$A^T = \begin{bmatrix} 1 & 3 & -1 & 2 \\ 4 & -2 & 0 & 3 \\ 5 & 1 & -1 & 5 \\ 6 & 4 & -2 & 7 \\ 9 & -1 & -1 & 8 \end{bmatrix}$$

Reducing the matrix  $A^T$  to row echelon form,

$$\begin{aligned} & R_2 - 4R_1, R_3 - 5R_1, R_4 - 6R_1, R_5 - 9R_1 \\ A & \sim \begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & -14 & 4 & -5 \\ 0 & -14 & 4 & -5 \\ 0 & -14 & 4 & -5 \\ 0 & -28 & 8 & -10 \end{bmatrix} \\ & R_3 - R_2, R_4 - R_2, R_5 - 2R_2 \\ & \sim \begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & -14 & 4 & -5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

$$\left(-\frac{1}{14}\right)R_2$$

$$\sim \begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & 1 & -\frac{4}{14} & \frac{5}{14} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = B$$

The leading 1's appear in columns 1 and 2.

$$\text{Basis for the column space of } A^T = \left\{ \begin{bmatrix} 1 \\ 4 \\ 5 \\ 6 \\ 9 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 1 \\ 4 \\ -1 \end{bmatrix} \right\}$$

Hence, basis for the row space of  $A = \{(1, 4, 5, 6, 9), (3, -2, 1, 4, -1)\}$

**Example 4:** Find a basis for the column space of

$$A = \begin{bmatrix} 1 & -2 & 7 & 0 \\ 1 & -1 & 4 & 0 \\ 3 & 2 & -3 & 5 \\ 2 & 1 & -1 & 3 \end{bmatrix}$$

consisting of vectors that are not entirely the column vectors of  $A$ .

**Solution:** We know that

$$\text{Column space of } A = \text{Row space of } A^T$$

$\therefore$  Basis for the column space of  $A =$  Transpose of the basis for the row space of  $A^T$

$$A^T = \begin{bmatrix} 1 & 1 & 3 & 2 \\ -2 & -1 & 2 & 1 \\ 7 & 4 & -3 & -1 \\ 0 & 0 & 5 & 3 \end{bmatrix}$$

Reducing the matrix  $A^T$  to row echelon form,

$$R_2 + 2R_1, R_3 - 7R_1$$

$$\sim \begin{bmatrix} 1 & 1 & 3 & 2 \\ 0 & 1 & 8 & 5 \\ 0 & -3 & -24 & -15 \\ 0 & 0 & 5 & 3 \end{bmatrix}$$

$$\begin{aligned}
 & R_3 + 3R_2 \\
 & \sim \begin{bmatrix} 1 & 1 & 3 & 2 \\ 0 & 1 & 8 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 3 \end{bmatrix} \\
 & \left(\frac{1}{5}\right)R_4 \\
 & \sim \begin{bmatrix} 1 & 1 & 3 & 2 \\ 0 & 1 & 8 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \frac{3}{5} \end{bmatrix} = B
 \end{aligned}$$

Basis for the row space of  $A^T =$  Non-zero rows of  $B$

$$= \left\{ (1, 1, 3, 2), (0, 1, 8, 5), \left(0, 0, 1, \frac{3}{5}\right) \right\}$$

Hence, basis for the column space of  $A = \left\{ \begin{bmatrix} 1 \\ 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 8 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ \frac{3}{5} \end{bmatrix} \right\}$

**Example 5:** Find a basis for the space spanned by the vectors  $\mathbf{v}_1 = (1, 1, 0, 0)$ ,  $\mathbf{v}_2 = (0, 0, 1, 1)$ ,  $\mathbf{v}_3 = (-2, 0, 2, 2)$ ,  $\mathbf{v}_4 = (0, -3, 0, 3)$ .

**Solution:** The space spanned by these vectors is the row space of the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ -2 & 0 & 2 & 2 \\ 0 & -3 & 0 & 3 \end{bmatrix}$$

Reducing the matrix  $A$  to row echelon form,

$$\begin{aligned}
 & R_3 + 2R_1 \\
 & \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & -3 & 0 & 3 \end{bmatrix}
 \end{aligned}$$

$$\begin{aligned}
 & \left(\frac{1}{2}\right)R_3, \left(\frac{1}{3}\right)R_4 \\
 & \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & -1 & 0 & 1 \end{bmatrix} \\
 & R_2 + R_3, R_4 + R_3 \\
 & \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \\
 & R_3 - R_2 \\
 & \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \\
 & (-1)R_3 \\
 & \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \\
 & R_4 - R_3 \\
 & \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = B
 \end{aligned}$$

Basis for the given space = Basis for the row space of  $A$  = Non-zero rows of  $B$   
 $= \{(1, 1, 0, 0), (0, 1, 2, 2), (0, 0, 1, 1), (0, 0, 0, 1)\}$

**Example 6:** Find a basis for the space spanned by the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 3 \\ 2 \\ 3 \\ 2 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix}, \mathbf{v}_5 = \begin{bmatrix} 5 \\ 3 \\ 5 \\ 3 \end{bmatrix}$$

**Solution:** The space spanned by these vectors is the column space of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 3 & 5 \\ 2 & 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 3 & 5 \\ 2 & 1 & 2 & 3 & 3 \end{bmatrix}$$

Reducing the matrix  $A$  to row echelon form,

$$\begin{aligned} & R_2 - 2R_1, R_3 - R_1, R_4 - 2R_1 \\ & \sim \begin{bmatrix} 1 & 2 & 3 & 3 & 5 \\ 0 & -3 & -4 & -3 & -7 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & -4 & -3 & -7 \end{bmatrix} \\ & R_4 - R_2 \\ & \sim \begin{bmatrix} 1 & 2 & 3 & 3 & 5 \\ 0 & -3 & -4 & -3 & -7 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ & \left(-\frac{1}{3}\right)R_2 \\ & \sim \begin{bmatrix} 1 & 2 & 3 & 3 & 5 \\ 0 & 1 & \frac{4}{3} & 1 & \frac{7}{3} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = B \end{aligned}$$

The leading 1's appear in columns 1 and 2.

$$\text{Basis for the column space} = \{\mathbf{v}_1, \mathbf{v}_2\} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix} \right\} \text{ which is also the basis for the}$$

space spanned by the given vectors.

**Theorem 2.22:** A system of linear equations  $A\mathbf{x} = \mathbf{b}$  is consistent if and only if  $\mathbf{b}$  is in the column space of  $A$ .