**Department of Mathematics**
**School of Advanced Sciences**
**BMAT101P – Calculus - Laboratory (MATLAB)**
**Experiment 2–B**

# Evaluating maxima and minima of functions of two variables

**Aim:** To find Maximum and Minimum values (Extreme values) of a function $f(x,y)$ using MATLAB.

## Mathematical form:

Let $z = f(x,y)$ be the given function. Critical points are points in the $xy-$plane where the tangent plane is horizontal. The tangent plane is horizontal, if its normal vector points in the $z-$direction. Hence, critical points are solutions of the equations: $f_x(x,y)=0$ and $f_y(a,b)=0$.

Procedure for finding the maximum or minimum values of *f*(*x*,*y*):

(1) For the given function *f*(*x*,*y*) find $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ and equate it to zero and solve them to find the roots $(x_1, y_1)$, $(x_2, y_2)$,.... These points may be maximum or minimum points.

(2) Find the values $r = \frac{\partial^2 f}{\partial x^2}$, $s = \frac{\partial^2 f}{\partial x \partial y}$, $t = \frac{\partial^2 f}{\partial y^2}$ at these points.

(3) (a) If $rt - s^2 > 0$ and $r < 0$ at a certain point, then the function is maximum at that point.

(b)If $rt - s^2 > 0$ and $r > 0$ at a certain point , then the function is minimum at that point.

(c) If $rt - s^2 < 0$ for a certain point, then the function is neither maximum nor minimum at that point. This point is known as saddle point.

(d) If $rt - s^2 = 0$ at a certain point, then nothing can be said whether the function is maximum or minimum at that point. In this case further investigation are required.

## MATLAB Syntax used:

| | |
|---|---|
| `diff` | `diff(expr)` differentiates a symbolic expression expr with respect to its free variable as determined by symvar. |
| `solve` | Symbolic solution of algebraic equations, The input to solve can be either symbolic expressions or strings |
| `size` | Dimensions of data and model objects and to access a specific size output. |
| `figure` | Create figure graphics object, Figure objects are the individual windows on the screen in which the MATLAB software displays graphical output |
| `double` | Convert to double precision, `double(x)` returns the double-precision value for `X`. If `X` is already a double-precision array, double has no effect. |
| `sprintf` | Format data into string. It applies the *format* to all elements of array *A* and any additional array arguments in column order, and returns the results to string *str*. `sprintf('%f',var)` is used to format the floating-point number `var` into string. |
| `solve` | Symbolic solution of algebraic equations, The input to solve can be either symbolic expressions or strings. |
| `fsurf` | `fsurf(f)` creates a surface plot of the function z = f(x,y) over the default interval `[-5, 5]` for x and y. |
| `plot3` | The `plot3` function displays a three-dimensional plot of a set of data points. |

**MATLAB code:**

```
clc
clear
syms x y
f(x,y)=input('Enter the function f(x,y):');
p=diff(f,x); q=diff(f,y);
[ax,ay]=solve(p,q);
ax=double(ax);ay=double(ay);
r=diff(p,x);  s=diff(p,y); t=diff(q,y);D=r*t-s^2;
figure
fsurf(f);
legstr={'Function Plot'};% for Legend
for i=1:size(ax)
T1=D(ax(i),ay(i));
T2=r(ax(i),ay(i));
T3=f(ax(i),ay(i));
if(double(T1)==0)
sprintf('At (%f,%f) further investigation is required',ax(i),ay(i))
legstr=[legstr,{'Case of Further investigation'}];
mkr='ko';
elseif (double(T1)<0)
sprintf('The point (%f,%f) is a saddle point', ax(i),ay(i))
legstr=[legstr,{'Saddle Point'}]; % updating Legend
mkr='bv'; % marker
else
if (double(T2) < 0)
sprintf('The maximum value of the function is f(%f,%f)=%f', ax(i),ay(i), T3)
legstr=[legstr,{'Maximum value of the function'}];% updating Legend
mkr='g+';% marker
else
sprintf('The minimum value of the function is f(%f,%f)=%f', ax(i),ay(i), T3)
legstr=[legstr,{'Minimum value of the function'}];% updating Legend
mkr='r*'; % marker
end
end
hold on
plot3(ax(i),ay(i),T3,mkr,'Linewidth',4);
end
legend(legstr,'Location','Best');
```

**Example 1.** Obtain the maximum and minimum values of $f(x,y) = 2(x^2 - y^2) - x^4 + y^4$

**Solution:**

| **S.No.** | **Critical Points** | ***r*** | ***D=rt–s²*** | **Remarks** |
|---|---|---|---|---|
| 1 | (0,0) | 4 | $-16 < 0$ | Saddle Point |
| 2 | (0,1) | 4 | 32 | Minimum |
| 3 | (0, –1) | 4 | 32 | Minimum |
| 4 | (1,0) | -8 | 32 | Maximum |
| 5 | (1,1) | -8 | $-64 < 0$ | Saddle Point |
| 6 | (1, –1) | -8 | $-64 < 0$ | Saddle Point |
| 7 | (–1,0) | -8 | 32 | Maximum |
| 8 | (–1,1) | -8 | $-64 < 0$ | Saddle Point |
| 9 | (–1, –1) | -8 | $-64 < 0$ | Saddle Point |

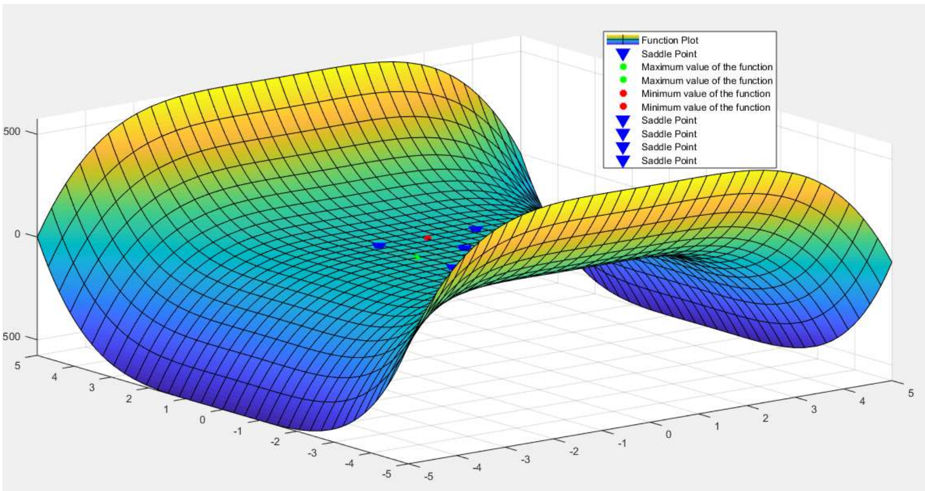
The Minimum value of $f(x,y)$ is $-1$ at $(0,1)$ & $(0,-1)$ and the Maximum value for $f(x,y)$ is $+1$ at $(1,0)$ & $(-1,0)$

**Input:**

```
Enter the function f(x,y):2*(x^2-y^2)-x^4+y^4
```

**Out Put:**

```
 ans =
'The point (0.000000,0.000000) is a saddle point'
ans =
'The maximum value of the function is f(-1.000000,0.000000)=1.000000'
ans =
 'The maximum value of the function is f(1.000000,0.000000)=1.000000'
ans =
'The minimum value of the function is f(0.000000,-1.000000)=-1.000000'
ans =
 'The minimum value of the function is f(0.000000,1.000000)=-1.000000'
ans =
    'The point (-1.000000,-1.000000) is a saddle point'
ans =
    'The point (1.000000,-1.000000) is a saddle point'
ans =
    'The point (-1.000000,1.000000) is a saddle point'
ans =
    'The point (1.000000,1.000000) is a saddle point'
```




**Example. 2** Four small towns in a rural area wish to pool their resources to build a television station. If the towns are located at the points $(-5,0)$, $(1,7)$, $(9,0)$ and $(0,-8)$ on a rectangular map grid, where units are in miles, at what point $S(x,\ y)$ should the station be located to minimize the sum of the distances from the towns?

Solution: Let $S(x,y)$ be the location where the television station is to be set up.

The location of the towns are $A(-5,0)$, $B(1,7)$, $C(9,0)$ and $D(0,-8)$.

The point $S(x,y)$ where the sum of the distances from the above points is to be minimized is the same point that minimizes the sum of the squares of the distances; namely,

$$S(x,y)=\left[(x+5)^2+y^2\right]+\left[(x-1)^2+(y-7)^2\right]+\left[(x-9)^2+y^2\right]+\left[x^2+(y+8)^2\right]$$

$$S_x=2(x+5)+2(x-1)+2(x-9)+2x.$$

$$S_y=2y+2(y-7)+2y+2(y+8).$$

Then $S_x = 0 \Rightarrow x = 5/4$ and $S_y = 0 \Rightarrow y = -1/4$.

$r = S_{xx} = 8 > 0$, $s = S_{xy} = 0$ and $t = S_{yy} = 8 > 0$.

Hence $rt - s^2 > 0$, $r > 0$ at $(5/4, -1/4)$.

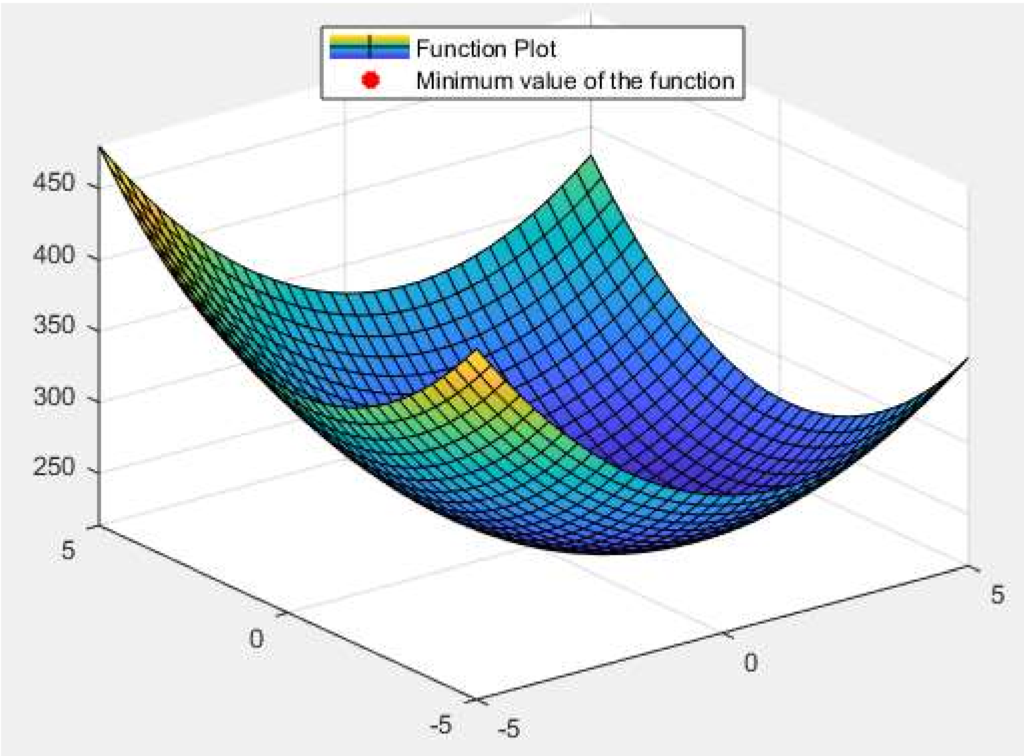
Therefore, the television station can be set up at the location $S(5/4, -1/4)$ on the rectangular map grid such that the distance from $S$ to each of the towns is a minimum.

**Input**

```
Enter the function f(x,y):
(x+5)^2+y^2+(x-1)^2+(y-7)^2+(x-9)^2+y^2+x^2+(y+8)^2
```

**Output**

```
ans =
    'The minimum value of the function is f(1.250000,-0.250000)=
                                                           213.500000'
```




**Exercise**

Find the maxima and minima for the following functions

1. $f(x, y) = x^4 + y^4 - x^2 - y^2 + 1$.
2. $f(x, y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$.