7.4: Difference Equations:

Definition: A difference equation is a relation between the differences of an unknown function at one or more general values of the argument.

Examples

1. $y_{n+2} + 3y_{n+1} + 2y_n = 0$

2. $y_{n+2} - y_{n+1} + y_n = 2^n$

Oder of a difference equation is the difference between the largest and the smallest arguments occuring in the difference equation divided by the unit of the increment.

order of the example ① above is

$$\frac{n+2-n}{1} = 2$$

Solution of a difference equation is an expression for y_n which satisfies the given difference equation

The general solution of a difference equation is that in which the number of arbitrary constants is equal to the order of the difference equation.

Formation of Difference Equations:

1. Form the difference equation for $y_n = A\,2^n + B\,5^n$, where A and B are arbitrary constants.

<u>Sol</u>: Given $y_n = A\,2^n + B\,5^n$.

So, $y_{n+1} = 2A\,2^n + 5B\,5^n$

$$= 2\underbrace{(A\,2^n + B\,5^n)} + 3B\,5^n$$

$$= 2y_n + 3B\,5^n \longrightarrow (1)$$

and $y_{n+2} = 2y_{n+1} + \underbrace{15B\,5^n}_{5(3B5^n)}$

$$= 2y_{n+1} + 5(y_{n+1} - 2y_n)$$

(from (1))

Hence, the required difference equation is

$$\boxed{y_{n+2} - 7y_{n+1} + 10y_n = 0}$$

(2) Form the difference equation for

$y_n = A(-2)^n + B\,3^n$, where A and B are arbitrary constants.

Sol: Given $y_n = A(-2)^n + B\,3^n$

So, $y_{n+1} = -2\,A(-2)^n + 3\,B\,3^n$

$= -2\left(A(-2)^n + B\,3^n\right) + 5\,B\,3^n$

$= -2\,y_n + 5\,B\,3^n \longrightarrow (1)$

and $y_{n+2} = -2\,y_{n+1} + 3(5\,B\,3^n)$

$= -2\,y_{n+1} + 3(y_{n+1} + 2y_n)$ (from (1))

Hence, the required difference equation is

$$\boxed{y_{n+2} - y_{n+1} - 6\,y_n = 0}$$

3. Form the difference equation for

$y_n = (A + Bn)2^n$, where A and B are arbitrary constants.

<u>Sol:</u> Given $y_n = (A + Bn)2^n$

or, $y_n = A2^n + Bn.2^n$

So, $y_{n+1} = 2A2^n + 2B(n+1)2^n$

$= 2(A2^n + Bn2^n) + 2B2^n$

$= 2y_n + 2B2^n \longrightarrow (1)$

and $y_{n+2} = 2y_{n+1} + \underbrace{4B2^n}_{2(2B2^n)}$

$= 2y_{n+1} + 2(y_{n+1} - 2y_n)$

(from (1))

Hence, the ~~required~~ required difference equation is

$$\boxed{y_{n+2} - 4y_{n+1} + 4y_n = 0}$$

Solution of Difference Equations Using Z-Transform:

1. Solve $u_{n+2} + 5u_{n+1} + 6u_n = 5^n$ with $u_0 = 0$, $u_1 = 0$ using Z-Transform.

Sol: Given $u_{n+2} + 5u_{n+1} + 6u_n = 5^n$

Taking Z-Transform, we get

$$Z[u_{n+2}] + 5Z[u_{n+1}] + 6Z[u_n] = Z[5^n]$$

$$\Rightarrow z^2\left[Z[u_n] - u_0 - \frac{u_1}{z}\right] + 5.z\left[Z[u_n] - u_0\right]$$

$$+ 6Z[u_n] = \frac{z}{z-5}$$

$$\Rightarrow (z^2 + 5z + 6)\,Z[u_n] = \frac{z}{z-5} \quad \left(\text{since } u_0 = 0 \text{ and } u_1 = 0\right)$$

$$\Rightarrow Z[u_n] = \frac{z}{(z-5)(z+2)(z+3)}$$

$$\Rightarrow \quad \frac{Z[u_n]}{z} = \frac{1}{(z-5)(z+2)(z+6)} \longrightarrow \circledast$$

Now,

$$\frac{1}{(z-5)(z+2)(z+3)} = \frac{A}{z-5} + \frac{B}{z+2} + \frac{C}{z+3}$$

$$\Rightarrow \quad 1 = A(z+2)(z+3) + B(z-5)(z+3) + C(z-5)(z+2)$$

which gives $A = \frac{1}{56}$, $B = \frac{-1}{7}$

and $C = \frac{1}{8}$.

Therefore, from $\circledast$, we have

$$\frac{Z[u_n]}{z} = \frac{1}{56}\left(\frac{1}{z-5}\right) - \frac{1}{7}\left(\frac{1}{z+2}\right) + \frac{1}{8}\left(\frac{1}{z+3}\right)$$

$$\Rightarrow Z[u_n] = \frac{1}{56}\left(\frac{z}{z-5}\right) - \frac{1}{7}\left(\frac{z}{z+2}\right) + \frac{1}{8}\left(\frac{z}{z+3}\right)$$

$$\Rightarrow u_n = \frac{1}{56} Z^{-1}\left[\frac{z}{z-5}\right] - \frac{1}{7} Z^{-1}\left[\frac{z}{z+2}\right] + \frac{1}{8} Z^{-1}\left[\frac{z}{z+3}\right]$$

Hence, $u_n = \frac{1}{56}(5^n) - \frac{1}{7}(-2)^n + \frac{1}{8}(-3)^n$.

② Solve $u_{n+2} + 2u_{n+1} + u_n = 2^n$

with $u_0 = 0$ and $u_1 = 0$ using

Z-Transform.

<u>Sol:</u> Given $u_{n+2} + 2u_{n+1} + u_n = 2^n$

Taking Z-Transform, we get

$$Z[u_{n+2}] + 2Z[u_{n+1}] + Z[u_n] = Z[2^n]$$

$$\Rightarrow z^2\left[Z[u_n] - u_0 - \frac{u_1}{z}\right] - 2z\left[Z[u_n] - u_0\right]$$

$$+ Z[u_n] = \frac{z}{z-2}$$

$$\Rightarrow (z^2 + 2z + 1)\, Z[u_n] = \frac{z}{z-2} \quad (\text{Since } u_0 = 0,\ u_1 = 0)$$

$$\Rightarrow \frac{Z(u_n)}{z} = \frac{1}{(z-2)(z+1)^2} \longrightarrow (*)$$

Now,

$$\frac{1}{(z-2)(z+1)^2} = \frac{A}{z-2} + \frac{B}{z+1} + \frac{C}{(z+1)^2}$$

$$\Rightarrow \quad 1 = A(z+1)^2 + B(z-2)(z+1) + C(z-2)$$

which gives $A = \frac{1}{9}$, $B = -\frac{1}{9}$ and $C = -\frac{1}{3}$

Therefore, from (*), we have

$$\frac{Z[u_n]}{z} = \frac{1}{9}\left(\frac{1}{z-2} - \frac{1}{z+1}\right) - \frac{1}{3}\,\frac{1}{(z+1)^2}$$

$$\Rightarrow Z[u_n] = \frac{1}{9}\left(\frac{z}{z-2} - \frac{z}{z+1}\right) - \frac{1}{3}\left(\frac{z}{(z+1)^2}\right)$$

$$\Rightarrow \quad u_n = \frac{1}{9}\left(Z^{-1}\left[\frac{z}{z-2}\right] - Z^{-1}\left[\frac{z}{z+1}\right]\right) - \frac{1}{3}\,Z^{-1}\left[\frac{z}{(z+1)^2}\right]$$

$$= \frac{1}{9}\left(2^n - (-1)^n\right) + \frac{1}{3}\,Z^{-1}\left[\frac{-z}{(z+1)^2}\right]$$

$$\boxed{\begin{aligned} &Z[n(-1)^n] \\ &= -z\frac{d}{dz}\left(\frac{z}{z+1}\right) \\ &= (-z)\frac{1}{(z+1)^2} = \frac{-z}{(z+1)^2} \end{aligned}}$$

Hence, $u_n = \frac{1}{9}\left(2^n - (-1)^n\right) + \frac{1}{3}\left(n(-1)^n\right)$

③ Form the difference equation for the Fibonacci sequence and hence solve it using Z-Transform.

Sol: The Fibonacci sequence is the series of numbers

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, ...

The difference equation for the above series of numbers is

$$y_{n+2} = y_n + y_{n+1} \quad \text{with } y_0 = 0,\ y_1 = 1$$

or, $y_{n+2} - y_{n+1} - y_n = 0$ with $y_0 = 0,\ y_1 = 1$

Taking Z-Transform, we get

$$Z[y_{n+2}] - Z[y_{n+1}] - Z[y_n] = Z[0]$$

$$\Rightarrow z^2\left[Z[y_n] - y_0 - \frac{y_1}{z}\right] - z\left[Z[y_n] - y_0\right] - Z[y_n] = 0$$

$$\Rightarrow (z^2 - z - 1)\, Z[y_n] = z \quad \left(\text{Since } y_0 = 0 \text{ and } y_1 = 1\right)$$

$$\Rightarrow \frac{Z[y_n]}{z} = \frac{1}{z^2 - z - 1}$$

$$\Rightarrow \frac{Z[y_n]}{z} = \frac{1}{\left[z-\left(\frac{1}{2}+\frac{\sqrt{5}}{2}\right)\right]\left[z-\left(\frac{1}{2}-\frac{\sqrt{5}}{2}\right)\right]} \quad \rightarrow (*)$$

Now,

$$\frac{1}{\left(z-\left(\frac{1}{2}+\frac{\sqrt{5}}{2}\right)\right)\left(z-\left(\frac{1}{2}-\frac{\sqrt{5}}{2}\right)\right)} = \frac{A}{z-\left(\frac{1}{2}+\frac{\sqrt{5}}{2}\right)} + \frac{B}{z-\left(\frac{1}{2}-\frac{\sqrt{5}}{2}\right)}$$

which gives, $A = \frac{1}{\sqrt{5}}$ and $B = \frac{-1}{\sqrt{5}}$

Therefore, from $(*)$, we have

$$\frac{Z[y_n]}{z} = \frac{1}{\sqrt{5}}\left(\frac{1}{z-\left(\frac{1}{2}+\frac{\sqrt{5}}{2}\right)} - \frac{1}{z-\left(\frac{1}{2}-\frac{\sqrt{5}}{2}\right)}\right)$$

$$\Rightarrow y_n = \frac{1}{\sqrt{5}}\left(Z^{-1}\left[\frac{z}{z-\left(\frac{1}{2}+\frac{\sqrt{5}}{2}\right)}\right] - Z^{-1}\left[\frac{z}{z-\left(\frac{1}{2}-\frac{\sqrt{5}}{2}\right)}\right]\right)$$

Hence, $y_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1}{2}+\frac{\sqrt{5}}{2}\right)^n - \left(\frac{1}{2}-\frac{\sqrt{5}}{2}\right)^n\right]$

4. Solve $y_{n+2} + 4y_{n+1} + 3y_n = 3^n$ with $y_0 = 0$, $y_1 = 1$. using Z-Transform.

<u>Answer:</u>

$$y_n = \frac{3}{8}(-1)^n + \frac{1}{24}3^n - \frac{5}{12}(-3)^n.$$