Module 2.3: Lagrange's Linear Equation

A linear partial differential equation of order one, involving a dependent variable z and two independent variables x and y, of the form $Pp + Qq = R$, where P, Q, R are functions of x, y, z is called Lagrange's linear equation.

Working Rule to solve $Pp + Qq = R$

step 1: write down the subsidiary equations (or auxiliary equations)

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

step 2: Find any two independent solutions of the subsidiary equations. Let the two solutions be $u = a$ and $v = b$, where 'a' and 'b' are arbitrary constants.

step 3: Now the general of $Pp + Qq = R$ is given by $f(u, v) = 0$ or, $u = f(v)$.

Problems:

1. Find the general solution of $px + qy = z$.

Sol: The given p.d.e. is $px + qy = z \longrightarrow (1)$

The subsidiary equations are

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$$

From the first two fractions, we have

$$\frac{dx}{x} = \frac{dy}{y}$$

Integrating, $\int \frac{dx}{x} = \int \frac{dy}{y}$

$\Rightarrow \log x = \log y + \log c_1$

$\Rightarrow \log\left(\frac{x}{y}\right) = \log c_1$

$\Rightarrow \frac{x}{y} = c_1$

From the last two fractions, we have

$$\frac{dy}{y} = \frac{dz}{z}$$

Integrating, we get $\log y = \log z + \log c_2$

$\Rightarrow \log\left(\frac{y}{z}\right) = \log c_2 \Rightarrow \frac{y}{z} = c_2$.

Hence the general solution of (1) is $\phi(c_1, c_2) = 0$

i.e., $\phi\left(\frac{x}{y}, \frac{y}{z}\right) = 0$, where ϕ is arbitrary function.

2: Solve $(y-z)p + (x-y)q = z - x$.

Sol: The subsidiary equations are

$$\frac{dx}{y-z} = \frac{dy}{x-y} = \frac{dz}{z-x} \longrightarrow (1)$$

Now, $$\frac{dx}{y-z} = \frac{dy}{x-y} = \frac{dz}{z-x} = \frac{dx+dy+dz}{(y-z)+(x-y)+z-x}$$

$$= \frac{dx+dy+dz}{0}$$

Therefore, $dx+dy+dz = 0$

Integrating, we get $x+y+z=c_1$

Again from ①, we get

$$\frac{x\,dx}{x(y-z)} = \frac{z\,dy}{z(x-y)} = \frac{y\,dz}{y(z-x)}$$

$$= \frac{x\,dx + z\,dy + y\,dz}{0}$$

Therefore, $x\,dx + z\,dy + y\,dz = 0$

Integrating, we get $\frac{x^2}{2} + yz = c_2$

Hence the general solution is $\boxed{\phi\left(x+y+z, \frac{x^2}{2}+yz\right)}$ i.e, $\boxed{\phi(c_1, c_2)=0}$

3. Solve $px^2+qy^2=z(x+y)$

<u>Sol:</u> Subsidiary equations are

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{z(x+y)} \longrightarrow ①$$

From the first two fractions of ①, we have

$$\frac{dx}{x^2} = \frac{dy}{y^2}.$$

Integrating, we get $-\frac{1}{x} = -\frac{1}{y} + c_1$

or, $\frac{1}{y} - \frac{1}{x} = c_1$

Again, from ①, we have $\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dx-dy}{x^2-y^2}$

Now, $\frac{dz}{z(x+y)} = \frac{dx-dy}{(x-y)(x+y)} \Rightarrow \frac{dz}{z} = \frac{dx-dy}{x-y}$

$$\Rightarrow \frac{dz}{z} = \frac{d(x-y)}{x-y}$$

Integrating, we get $\log z = \log(x-y) + \log c_2$

or, $\log\left(\frac{z}{x-y}\right) = \log c_2 \Rightarrow \frac{z}{x-y} = c_2$.

Hence the general solution is $\phi(c_1, c_2) = 0$.

i.e., $\phi\left(\frac{x-y}{xy}, \frac{z}{x-y}\right) = 0$.

4. solve $x(y^2-z^2)p - y(z^2+x^2)q = z(x^2+y^2)$

Sol: Hint: $\frac{dx}{x(y^2-z^2)} = \frac{dy}{-y(z^2+x^2)} = \frac{dz}{z(x^2+y^2)} \longrightarrow (1)$

Each fraction $= \frac{x\,dx + y\,dy + z\,dz}{0}$. Therefore, $x\,dx + y\,dy + z\,dz = 0$

Integrating, we get $x^2+y^2+z^2 = c_1$

Again, Each fraction $= \frac{\frac{1}{x}dx - \frac{1}{y}dy - \frac{1}{z}dz}{0}$

$\therefore \frac{1}{x}dx - \frac{1}{y}dy - \frac{1}{z}dz = 0$

Integrating, we get $\log x - \log y - \log z = \log c_2$

$\Rightarrow \log\left(\frac{x}{yz}\right) = \log c_2 \Rightarrow \frac{x}{yz} = c_2$

Hence the general solution is $\phi(c_1, c_2) = 0$

i.e., $\phi\left(x^2+y^2+z^2, \frac{x}{yz}\right) = 0$

5. Solve $(z^2-2yz-y^2)p + (xy+zx)q = xy - zx$.

Ans: $\phi(x^2+y^2+z^2, y^2-2yz-z^2) = 0$