

Module 6.2:

Fourier Sine Transform:

we have

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \sin(px) \left(\int_0^{\infty} \sin(pt) f(t) dt \right) dp$$

which is the Fourier sine integral.

Taking $F_s(p) = \int_0^{\infty} \sin(pt) f(t) dt$, then

we get
$$f(x) = \frac{2}{\pi} \int_0^{\infty} \sin(px) F_s(p) dp$$

The function $F_s(p)$ is the Fourier sine transform (or, infinite Fourier sine transform) of $f(x)$ and $f(x)$ is the inverse Fourier sine transform of $F_s(p)$.

Fourier Cosine Transform:

we have

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \cos(px) \left(\int_0^{\infty} \cos(pt) f(t) dt \right) dp$$

which is the Fourier cosine integral.

Taking

$$F_c(p) = \int_0^{\infty} \cos(pt) f(t) dt$$

then we

get

$$f(x) = \frac{1}{2\pi} \int_0^{\infty} \cos(px) F_c(p) dp$$

The function $F_c(p)$ is the Fourier cosine transform (or, infinite Fourier cosine transform) of $f(x)$ and $f(x)$ is the inverse Fourier cosine transform of $F_c(p)$.

Problems:

1. Find the Fourier sine and cosine transforms of $f(x) = x$.

Sol: (i) Fourier sine transform of $f(x)$ is

$$F_s(p) = \int_0^{\infty} \sin(px) f(x) dx$$

$$= \int_0^{\infty} x \sin px dx$$

and Fourier cosine transform of $f(x)$ is

$$F_c(p) = \int_0^{\infty} \cos(px) f(x) dx = \int_0^{\infty} x \cos(px) dx$$

Now,
$$\int_0^{\infty} x e^{-ipx} dx = \left[x \cdot \frac{e^{-ipx}}{-ip} \right]_0^{\infty} - \left[1 \cdot \frac{e^{-ipx}}{(-ip)^2} \right]_0^{\infty}$$

$$\Rightarrow \int_0^{\infty} x \cos(px) - i \sin(px) dx = 0 + \frac{1}{p^2} (0 - 1) \quad (\because i^2 = -1)$$

$$\Rightarrow \int_0^{\infty} x \cos px dx - i \int_0^{\infty} x \sin px dx = -\frac{1}{p^2}$$

Equating, real and imaginary parts,

we get

$$\int_0^{\infty} x \sin px dx = 0 \text{ and } \int_0^{\infty} x \cos px dx = -\frac{1}{p^2}$$

Hence $F_s(p) = 0$ and $F_c(p) = -\frac{1}{p^2}$.

2. Find the Fourier sine and cosine

transforms of $f(x) = \frac{e^{-ax}}{x}$ ($a > 0$).

Sol: (i) Fourier sine transform of $f(x)$ is

$$F_s\{f(x)\} = F_s(p) = \int_0^{\infty} f(x) \sin(px) dx$$

$$\Rightarrow F_s\{f(x)\} = \int_0^{\infty} \frac{e^{-ax}}{x} \sin(px) dx$$

Differentiating both sides w.r.t. p , we get

$$\frac{d}{dp} (F_s(p)) = \int_0^{\infty} \frac{e^{-ax}}{x} \times \cos(px) dx$$

$$= \int_0^{\infty} e^{-ax} \cos(px) dx$$

$$= \frac{a}{a^2 + p^2}$$

Integrating both sides w.r.t. p , we get

$$F_s(p) = \int \frac{a}{a^2 + p^2} dp \Rightarrow F(p) = \tan^{-1}\left(\frac{p}{a}\right) + C_1$$

Similarly, Fourier cosine transform of

$$f(x) \text{ is } F_c(p) = -\frac{1}{2} \cdot \log(p^2 + a^2) + C_2$$

3. Find the Fourier sine and cosine transform of $f(x) = e^{-ax}$ ($a > 0$) and hence evaluate

(i) $\int_0^{\infty} \frac{\cos px}{a^2 + p^2} dp$ and (ii) $\int_0^{\infty} \frac{p \sin px}{a^2 + p^2} dp$

Sol: (i) $F_c\{f(x)\} = \int_0^{\infty} f(x) \cos px \, dx$

$$= \int_0^{\infty} e^{-ax} \cos px \, dx$$

$$= \frac{e^{-ax}}{a^2 + p^2} \left[-a \cos px + p \sin px \right]_0^{\infty}$$

$$= 0 - \frac{1}{a^2 + p^2} (-a \cdot 1 + p \cdot 0)$$

$$\therefore F_c(p) = \frac{a}{a^2 + p^2}$$

By the inverse Fourier cosine transform, we get

$$f(x) = \frac{2}{\pi} \int_0^{\infty} F_c(p) \cos px \, dp$$

$$\Rightarrow e^{-ax} = \frac{2}{\pi} \int_0^{\infty} \frac{a}{a^2+p^2} \cos px \, dp$$

$$\text{or, } \int_0^{\infty} \frac{\cos px}{a^2+p^2} \, dp = \frac{\pi}{2a} e^{-ax}$$

(ii) Fourier sine transform of $f(x)$ is

$$F_s(p) = \int_0^{\infty} f(x) \sin(px) \, dx$$

$$= \int_0^{\infty} e^{-ax} \sin px \, dx$$

$$= \frac{e^{-ax}}{a^2+p^2} \left[-a \sin px - p \cos px \right]_0^{\infty}$$

$$= 0 - \frac{1}{a^2+p^2} [0 - p]$$

$$= \frac{p}{a^2+p^2}$$

By the inverse Fourier sine transform of $F_s(p)$,

we have

$$f(x) = \frac{2}{\pi} \int_0^{\infty} F_s(p) \sin px \, dp$$

$$\Rightarrow e^{-ax} = \frac{2}{\pi} \int_0^{\infty} \frac{p}{a^2+p^2} \sin px \, dp$$

$$\text{Hence } \int_0^{\infty} \frac{p}{a^2+p^2} \sin px \, dp = \frac{\pi}{2} e^{-ax}$$