

## 2.15 RANK AND NULLITY

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In the previous section we observed that in the row echelon form of a matrix, the number of non-zero rows (i.e. rows containing the leading 1's) form a basis for the row space of  $A$  and vectors corresponding to the columns containing the leading 1's form a basis for the column space of  $A$ .

Thus, dimension of row space = number of rows containing the leading 1's  
and dimension of column space = number of columns containing the leading 1's.

This concludes that for any matrix  $A$

$$\text{Dimension of row space} = \text{Dimension of column space}$$

### 2.15.1 Rank

The dimension of row/column space of a matrix  $A$  (or the number of non-zero rows in the row echelon form of  $A$ ) is called the rank of  $A$  and is denoted by  $\rho(A)$ .

**Note:** If  $A$  is an  $m \times n$  matrix then

$$\text{rank}(A) \leq \min(m, n)$$

Thus, the largest possible value of  $\text{rank}(A) = \min(m, n)$  where  $\min(m, n)$  means the smaller of the  $m$  and  $n$ .

e.g. if  $A$  is of order  $5 \times 3$  then

$$\begin{aligned} \text{The largest possible value of rank}(A) &= \min(5, 3) \\ &= 3 \end{aligned}$$

### 2.15.2 Nullity

The dimension of the null space of a matrix  $A$  is called the nullity of  $A$  and is denoted by nullity( $A$ ).

### 2.15.3 Dimension Theorem

**Theorem 2.23:** If  $A$  is an  $m \times n$  matrix then

$$\text{rank}(A) + \text{nullity}(A) = n \text{ (number of columns)}$$

**Theorem 2.24:** If  $A$  is an  $m \times n$  matrix then nullity( $A$ ) represents the number of parameters in the general solution of  $A\mathbf{x} = \mathbf{0}$

**Example 1:** Find the number of parameters in the general solution of  $A\mathbf{x} = \mathbf{0}$  if  $A$  is a  $5 \times 7$  matrix of rank 3.

**Solution:** The number of parameters = nullity( $A$ )  
 $= n - \text{rank}(A)$   
 $= 7 - 3$   
 $= 4$

**Example 2:** Find the rank and nullity of the matrix

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 4 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

**Solution:**

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 4 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

Reducing matrix  $A$  to row echelon form,

$$\left(\frac{1}{2}\right)R_1$$

$$\sim \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 4 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$R_2 - 4R_1$$

$$\sim \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Rank ( $A$ ) = Number of non-zero rows = 1

nullity ( $A$ ) =  $n - \text{rank } (A)$

$$= 3 - 1 = 2$$

### Exercise 2.7

1. Find a basis for the null space of

$$(i) A = \begin{bmatrix} 2 & -1 & -2 \\ -4 & 2 & 4 \\ -8 & 4 & 8 \end{bmatrix}$$

$$(ii) A = \begin{bmatrix} 1 & 2 & 2 & -1 & 1 \\ 0 & 2 & 2 & -2 & -1 \\ 2 & 6 & 2 & -4 & 1 \\ 1 & 4 & 0 & -3 & 0 \end{bmatrix}$$

$$\left[ \begin{array}{l} \text{Ans. : (i) } \left\{ \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\} \\ \text{(ii) } \left\{ \begin{bmatrix} -2 \\ \frac{1}{2} \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\} \end{array} \right]$$

2. Find a basis for the row space of

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 9 & -1 \\ -3 & 8 & 3 \\ -2 & 3 & 2 \end{bmatrix} \text{ consisting of}$$

vectors that are

(i) row vectors of  $A$

(ii) not entirely row vectors of  $A$

$$\left[ \begin{array}{l} \text{Ans. : (i) } \{(1, 2, -1), (1, 9, -1)\} \\ \text{(ii) } \{(1, 0, -1), (0, 1, 0)\} \end{array} \right]$$

3. Find a basis for the column space of

$$A = \begin{bmatrix} 1 & -2 & 5 \\ 2 & 3 & 2 \\ 0 & -7 & 8 \end{bmatrix} \text{ consisting of vectors}$$

that are

(i) column vectors of  $A$

(ii) not entirely column vectors of  $A$

$$\left[ \text{Ans. : (i) } \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ -7 \end{bmatrix} \right\} \text{ (ii) } \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\} \right]$$