1.3 Differential Equations with varible coefficents:

Cauchy's (Cauchy-Euler) Linear Differential Equation:

An equation of the form

$$x^n \frac{d^n y}{dx^n} + k_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + k_2 x^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \dots + k_n y = Q(x), \longrightarrow (1)$$

where $k_1, k_2, \dots, k_n$ are arbitrary constants, is called a Cauchy's linear differential equation in y of order n.

Operator form of eq^n (1) is

$$(x^n D^n + k_1 x^{n-1} D^{n-1} + \dots + k_n) y = Q(x), \text{ where } D \equiv \frac{d}{dx} \longrightarrow (2)$$

Eq^n (2) can be reduced to a linear differential equation with constant co-efficients by taking

$$x = e^t \text{ or } t = \log_e x \quad (x>0) \text{ and } \theta \equiv \frac{d}{dt}$$

Then, we get $xDy = \theta y$, $x^2 D^2 y = \theta(\theta-1) y, \dots$

$$\left[\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} \Rightarrow \frac{dy}{dt} = \frac{dy}{dx} \cdot e^t \right.$$

$$\Rightarrow \theta y = x \cdot Dy \quad (\because e^t = x)$$

$$\therefore xDy = \theta y \Big]$$

Solved Problems :

1. Solve $x^2y'' - xy' + y = 2\log_e x \; (x>0)$.

Sol: Operator form of the given diff. eqn is

$$[x^2D^2 - xD + 1]y = 2\log_e x, \quad \text{where } D \equiv \frac{d}{dx}. \longrightarrow (1)$$

This is a Cauchy's linear diff. eqn.

Let $x = e^t$ (or) $t = \log_e x$ $(x>0)$ and $\theta \equiv \frac{d}{dt}$.

Then, we get $xDy = \theta y$ and $x^2D^2y = \theta(\theta-1)y$.

Therefore, eqn (1) becomes

$$[\theta(\theta-1) - \theta + 1]y = 2t$$

or, $(\theta^2 - 2\theta + 1)y = 2t \longrightarrow (2)$

Let $f(\theta) \equiv \theta^2 - 2\theta + 1$. Then A.E. is $f(m) = 0$

i.e., $m^2 - 2m + 1 = 0 \Rightarrow m = 1, 1$

$\therefore y_c = (c_1 + c_2t)e^t$.

Let $y^* = A + Bt$ be the trail solution of y_p.

Then $\theta^2y^* - 2\theta y^* + y^* = 2t$

$\Rightarrow \frac{d^2}{dt^2}(A+Bt) - 2\frac{d}{dt}(A+Bt) + (A+Bt) = 2t$

$\Rightarrow 0 - 2B + A + Bt = 2t$

$\Rightarrow A - 2B = 0$ and $B = 2$

which gives $A = 4$

$\therefore y_p = 4 + 2t$ and hence $y = y_c + y_p$

i.e., $y = (c_1 + c_2 t)e^t + (4 + 2t)$

or, $y = (c_1 + c_2 \log_e x)x + 2(2 + \log_e x)$

This is the general solution of (1).

2. Solve $(x^2 D^2 - xD + 2)y = x \log_e x$, $D \equiv \frac{d}{dx}$, $(x>0)$

3. Solve $(x^2 D^2 + xD + 1)y = \log_e x \sin(\log_e x)$, $(x>0)$.

2) Sol: Given differential equation is

$$(x^2 D^2 - xD + 2)y = x \log_e x \quad \left(D \equiv \frac{d}{dx}\right) \longrightarrow (1)$$

Let $x = e^t$ or $t = \log_e x$ $(x>0)$ and $\theta \equiv \frac{d}{dt}$

Then, eqn (1) becomes

$$(\theta(\theta-1) - \theta + 2)y = e^t . t$$

or, $(\theta^2 - 2\theta + 2)y = te^t \longrightarrow (2)$

Clearly $y_c = e^t(c_1 \cos t + c_2 \sin t)$

Let $y^* = (A+Bt)e^t$ be the trail solution for y_p. Then $\theta^2 y^* - 2\theta y^* + 2y^* = te^t$

$$\Rightarrow [(A+Bt)e^t + 2\cancel{B}e^t] - 2[(A+\cancel{B}t)e^t + \cancel{B}e^t] + 2[(A+\cancel{B}t)e^t] = te^t$$

$$\Rightarrow (A+Bt)e^t = te^t$$

$$\Rightarrow A = 0 \text{ and } B = 1$$

$\therefore y_p = te^t$ and hence $y = y_c + y_p$

i.e., $y = e^t(c_1 \cos t + c_2 \sin t) + te^t$

- Therefore, the general solution of the given differential equation is

$$y = x(c_1 \cos(\log_e x) + c_2 \sin(\log_e x)) + x \log_e x.$$

— x —

Legendre's Linear differential equation:

An equation of the form

$$(a+bx)^n \frac{d^n y}{dx^n} + k_1 (a+bx)^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + k_n y = Q(x), \text{ where} \quad \longrightarrow (1)$$

$k_1, k_2, \ldots, k_n$ are arbitrary constants, is called a ~~(linear)~~ Legendre's linear equation in y of order n.

Operator form of eqn (1) is

$$\left[(a+bx)^n D^n + k_1 (a+bx)^{n-1} D^{n-1} + \cdots + k_n\right] y = Q(x) \quad \left(D \equiv \frac{d}{dx}\right) \quad \longrightarrow (2)$$

Eqn (2) can be reduced to a linear diffn eqn with constant coefficients by taking

$$a+bx = e^t \text{ (or) } t = \log_e(a+bx) \quad (a+bx>0).$$

and $\theta \equiv \frac{d}{dt}$, then we get $(a+bx)Dy = b\theta y$,

$(a+bx)^2 D^2 y = b^2\theta(\theta-1)y, \ldots$

1. Solve $\left((1+2x)^2 D^2 - 6(1+2x)D + 16\right) y = 8(1+2x)^2$, $\left(D \equiv \frac{d}{dx}\right)$.

Sol: Given differential equation is

$$\left((1+2x)^2 D^2 - 6(1+2x)D + 16\right) y = 8(1+2x)^2 \quad \longrightarrow (1)$$

let $1+2x = e^t$ or $t = \log_e(1+2x)$ $(1+2x>0)$.

and $\theta \equiv \frac{d}{dt}$, then we get $(1+2x)Dy = 2\theta y$

and $(1+2x)^2 D^2 y = 4\theta(\theta-1)y$.

Therefore, equation ① becomes

$$\left(4\theta(\theta-1)-6(2\theta)+16\right)y=8e^{2t}$$

or, $(\theta^2-4\theta+4)y=2e^{2t} \longrightarrow ②$

This is a linear differential equation with constant coefficients.

Clearly $y_c=(C_1+C_2t)e^{2t}$

and $y_p=t^2e^{2t}$. Therefore, $y=y_c+y_p$

i.e, $y=(C_1+C_2t)e^{2t}+t^2e^{2t}$

or, $y=\left[\left(C_1+C_2\log_e(1+2x)\right)+\log_e(1+2x)\right](1+2x)^2$

2. Solve $(1+x)^2y''-3(1+x)y'+4y=\sin(1+x)$.