Calculus
(BMAT101L)
Module 1

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Contents

Chapter 1

Differentiation and its Applications

1.1 Continuity and Differentiability

Definition 1.1.1 (Continuity)**.** Let $f : \mathscr{D} \to \mathbb{R}$, where $\mathscr{D} \subset \mathbb{R}$, and $c \in \mathscr{D}$. Then f is said to be continuous at c, if $\lim_{x\to c} f(x) = f(c)$. If f is continuous at every point of $\mathscr{D}$, then f is continuous on $\mathscr{D}$.

Definition 1.1.2 (Differentiability)**.** Let $f : \mathscr{D} \to \mathbb{R}$, where $\mathscr{D} \subset \mathbb{R}$, and $c \in \mathscr{D}$. Then f is said to be differentiable at c, if $l = \lim_{x\to c} \frac{f(x)-f(c)}{x-c}$ exists, and the limit l is called the derivative $f'(c)$ of f at c. If f is differentiable at every point of the domain $\mathscr{D}$, then f is differentiable on $\mathscr{D}$.

Definition 1.1.3 (Critical and Stationary Points)**.** Let f be defined on an interval $\mathscr{I}$. An interior point $c \in \mathscr{I}$ is called a critical point of f, if $f'(c)$ is undefined or $f'(c) = 0$ is zero. If f is differentiable at $c \in \mathscr{I}$ and $f'(c) = 0$. Then c is called a stationary point of f. The tangent to the plane curve $y = f(x)$ at c is either the x-axis or parallel to the x-axis.

1.2 Absolute Extrema on a Finite Closed Interval

Let $f : [a, b] \to \mathbb{R}$ and $c \in [a, b]$. We say that

- $f(c)$ is the absolute maximum value on $[a, b]$, if $f(x) \le f(c)$ for all $x \in [a, b]$, c is a point of absolute maximum of f and $f(c)$ is the absolute maximum value of f.
- $f(d)$ is the absolute minimum value on $[a, b]$, if $f(d) \le f(x)$ for all $x \in [a, b]$, d is a point of absolute minimum of f and $f(d)$ is the absolute minimum value of f.

Theorem 1.2.1 (Absolute Extrema Theorem)**.** If a function f is continuous on a finite closed interval $[a, b]$, then f has both an absolute maximum and an absolute minimum on $[a, b]$ which occur at the at critical points of f.

Finding Absolute Extrema on a Finite Closed Interval: To find absolute extrema of a continuous function f on a finite closed interval $[a, b]$: evaluate f at all its critical points in the open interval (a, b) and at the end points a and b. The largest of the computed values is the absolute maximum and the smallest is the absolute minimum for f on $[a, b]$.

Example 1.2.1. Find the absolute extrema of $f(x) = \sqrt{5 - x^2}$ on $[-2, 1]$.

Solution. Let $x \in (-2, 1)$. Then $f'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left[\sqrt{5 - x^2}\right] = -\frac{2x}{\sqrt{5-x^2}}\cdot$ Therefore, $f'(x) = 0$ for $x = 0$, which is a stationary point of f, , and $f(0) = \sqrt{5 - 0^2} = \sqrt{5}$.

At the left end point $x = -2$, $f(-2) = \sqrt{5 - (-2)^2} = 1$ and at the left end point $x = 1$, $f(1) = \sqrt{5 - (1)^2} = 2$.

Conclusion. The largest of the three functional values occurs at $x = 0$ and the smallest at $x = -2$. Therefore, f has the absolute maximum value of $\sqrt{5}$ at $x = 0$ and the absolute minimum value of 1 at $x = -2$.

Exercise 1.2.1. Find the absolute maxima and absolute minima of the following functions on the given intervals:

(a) $f(x) = (2x/3) - 5$ for all $x \in [-2, 3]$

(b) $f(x) = x^a(1-x)^b$ on $[0,1]$ where a and b are positive real numbers

(c) $f(x) = 2x^3 + 3x^2 - 12x$ for all $x \in [-3,2]$

(d) $f(x) = 3x/\sqrt{4x^2+1}$ for all $x \in [-1,1]$

(e) $f(x) = 1 + |9 - x^2|$ for all $x \in [-5,1]$.

Answers.

(a) $f'(x) = 2/3 \neq 0$ for all $x \in \mathbb{R}$ so that f has no critical points. While, $f(-2) = -19/3$ and $f(3) = -3$. Thus f has the absolute maximum -3 at $x = 3$ and the absolute minimum $-19/3$ at $x = -2$.

(b) $f'(x) = x^{a-1}(1-x)^{b-1}[a-(a+b)x]$ so that $x = 0$, $a/(a+b)$ are critical points, $f(0) = f(1) = 0$, while $f(a/(a+b)) = a^a b^b/(a+b)^{a+b}$. Thus f has the absolute maximum $a^a b^b/(a+b)^{a+b}$ at $x = a/(a+b)$ and an absolute minimum 0 at the end points $x = 0, 1$.

(c) $x = -2$, 1 are the critical points of f, the absolute maximum $f_{\max} = 20$ occurs at $x = 2$, the absolute minimum $f_{\min} = -7$ occurs at $x = 1$.

(d) No critical points of f, the absolute maximum $f_{\max} = 3/\sqrt{5}$ occurs at $x = 1$, the absolute minimum $f_{\min} = -3/\sqrt{5}$ occurs at $x = -1$.

(e) $x = 0$, -3 are the critical points of f, the absolute maximum $f_{\max} = 17$ occurs at $x = -5$, the absolute minimum $f_{\min} = 1$ occurs at $x = -3$.

1.3 Mean Value Theorems

Theorem 1.3.1 (Rolle's Theorem). Consider $f : [a,b] \to \mathbb{R}$. Suppose that

(a) f is continuous at every point of the closed interval $[a,b]$,
(b) f is differentiable on (a,b), and
(c) $f(a) = f(b)$.

Then there is *at least one* number c in (a,b) such that $f'(c) = 0$.

Example 1.3.1. Explain why Rolle's theorem is not applicable to $f(x) = \frac{x^2-4x}{x-2}$ on $[0,4]$.

Solution. Note that $f(x)$ is not defined at $x = 2 \in [0,4]$. Thus f is not continuous on $[0,4]$. Therefore, Rolle's theorem is not applicable to f on $[0,4]$.

Exercise 1.3.1. Give the reason why Rolle's theorem is not applicable to each of the following functions $f(x)$:

(a) $\sec x$ on $[0,2\pi]$
(b) $|x|$ on $[-1,1]$
(c) $1-(x-1)^{2/3}$ on $[0,2]$
(d) $\tan x$ on $[0,\pi/4]$

Answers.

(a) Since $f(x) = \pm\infty$ at $x = \pi/2$ and $3\pi/2$, f is not continuous on $[0,2\pi]$.
(b) f is not differentiable at $x = 0 \in (-1,1)$.
(c) $f'(x) = -2/3(x-1)^{1/3} = \infty$ at $x = 1$. Thus f is not differentiable on $(0,2)$.

Example 1.3.2. Verify whether Rolle's theorem is applicable to $f(x) = \sqrt{1-x^2}$ on $[-1,1]$. If it is so, find an appropriate constant c in $(-1,1)$.

Solution. Note that f is a polynomial function of rational degree which is known to be continuous on $\mathbb{R}^1$ and hence on $[-1,1]$. Also, $f(-1) = f(1) = 0$. Further,

$$f'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left\{\sqrt{1-x^2}\right\} = -\frac{2x}{2\sqrt{1-x^2}} = -\frac{x}{\sqrt{1-x^2}} \tag{1.3.1}$$

which exists on the open interval $(-1,1)$. Thus Rolle's theorem is applicable to f on $[-1,1]$. Therefore, there exists at least one point $c \in (-1,1)$ such that $f'(c) = 0$. To find such a c, we solve the equation $f'(x) = 0$, that is $-\frac{x}{\sqrt{1-x^2}} = 0$. This gives $x = 0$. Thus taking $c = 0$, we infer that $f'(c) = 0$, and c is the constant of Rolle's theorem we need.

Example 1.3.3. Verify Rolle's theorem for $f(x) = \log\left[\frac{x^2+ab}{x(a+b)}\right]$ on $[a,b]$, where $a > 0$, and find an appropriate constant c in (a,b).

Solution. The function $f(x)$ is known to be continuous on $(0,\infty)$ and hence on $[a,b]$. Also, $f(a) = f(b) = \log 1 = 0$. Further, f is differentiable on (a,b) with

$$f'(x) = \frac{x(a+b)}{x^2+ab}\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{x^2+ab}{x(a+b)}\right\} = \frac{x}{x^2+ab}\frac{\mathrm{d}}{\mathrm{d}x}\left\{x+\frac{ab}{x}\right\} = \frac{x}{x^2+ab}\left(1-\frac{ab}{x^2}\right).$$

for all $x \in (a,b)$. Thus Rolle's theorem is applicable to f on $[a,b]$. Therefore, there exists at least one point $c \in (a,b)$ such that $f'(c) = 0$. But $f'(x) = 0$ only if $1 - ab/x^2 = 0$, that is when $x = \sqrt{ab}$. Thus taking $c = \sqrt{ab}$, we see that $f'(c) = 0$, and c is the constant of Rolle's theorem.

Exercise 1.3.2. Verify whether Rolle's theorem and find an appropriate constant c of it for each of the following functions $f(x)$:

(a) $\sqrt{x} - x/3$ on $[0,9]$
(b) $(x-a)^m(x-b)^n$ on $[a,b]$
(c) $(\sin x)/e^x$ on $[0,\pi]$
(d) $\log(4+2x-x^2)$ on $[-1,3]$
(e) $\frac{x}{2} - \sqrt{x}$ on $[0,4]$

Answers.
(a) $c = 9/4$ (b) $c = \frac{an+bm}{m+n}$ (c) $c = \pi/4 \in (0,\pi)$ (d) $c = 1$ (e) $c = 1$

Theorem 1.3.2 (The Mean Value Theorem)**.** Suppose that

(a) $f(x)$ is continuous on the closed interval $[a,b]$,
(b) f is differentiable on (a,b).

Then there is at least one number c in (a,b) such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Example 1.3.4. Explain why the mean value theorem is not applicable to $f(x) = x^{2/3}$ in $[-1,8]$.

Solution. Note that $f'(x) = (2/3)x^{-1+2/3} = 2/3x^{1/3} = \infty$ at $x = 0$. Thus f is not differentiable at $x = 0 \in (-1,8)$. Hence the mean value theorem is not applicable to f in $[-1,8]$.

Exercise 1.3.3. Explain why the mean value theorem is not applicable to the following functions:

(a) $f(x) = \begin{cases} \frac{\sin x}{x}, & (-\pi \le x < 0) \\ 0, & (x = 0) \end{cases}$

(b) $f(x) = \begin{cases} x^2 - x, & (-2 \le x \le -1) \\ 2x^2 - 3x - 3, & (-1 < x \le 0) \end{cases}$

(c) $f(x) = \sqrt{2x - 1}$ in $[0, 1]$

Answers.

(a) f is not continuous at $x = 0 \in [-\pi, 0]$

(b) f is not differentiable at $x = -1 \in (-2, 0)$

(c) f is not differentiable at $x = 1/2 \in (0, 1)$

Example 1.3.5. Verify the mean value theorem for $f(x) = x + 1/x$ on $[1/2, 1]$, and find the constant $c \in (1/2, 1)$.

Solution. Note that f is continuous and differentiable on $\mathbb{R} - \{0\}$. In particular, it is continuous on $[1/2, 1]$, and differentiable on $(1/2, 1)$.

Hence, by Lagrange's mean-value theorem, there exist at least one point $c \in (1/2, 1)$ such that

$$f'(c) = \frac{f(1) - f(1/2)}{1 - 1/2} \text{ or } 1 - \frac{1}{c^2} = \frac{(1+1) - [(1/2) + 2]}{1/2}. \tag{1.3.2}$$

This holds good only if $c = 1/\sqrt{2} \in (1/2, 1)$ which is the constant of the mean value theorem for f.

Exercise 1.3.4. Verify the mean value theorem for each of the following functions $f(x)$ in the given interval and find an appropriate constant c in each case:

(a) $\log x$ on $x \in [1, e]$

(b) $f(x) = lx^2 + mx + n$ on $x \in [a, b]$

(c) $f(x) = x^3 + x - 4$; on $x \in [-1, 2]$

(d) $f(x) = \sqrt{25 - x^2}$ on $x \in [-5, 3]$

(e) $f(x) = \frac{x+1}{x-1}$ on $x \in [2, 3]$

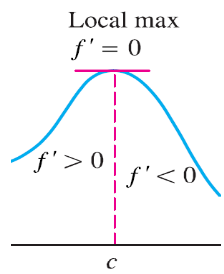
Answers.
(a) $c = e - 1$ (b) $c = (a + b)/2$ (c) $c = 1$ (d) $c = -\sqrt{5}$ (e) $c = 1 + \sqrt{2}$

1.4 First Derivative Test - Local Extrema

Theorem 1.4.1 (First Derivative Test for Local Maxima). If $f'(x)$ changes its sign from positive to negative on passing through the critical point c from left to the right, that is

$$f'(x) \begin{cases} > 0 & \text{for } x < c \\ < 0 & \text{for } x > c \end{cases}$$

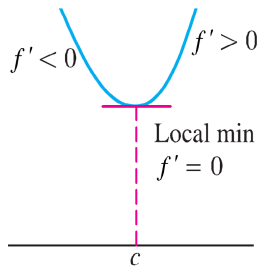
then f has a local maximum at c.



Theorem 1.4.2 (First Derivative Test for Local Minima)**.** If $f'(x)$ changes its sign from negative to positive on passing through the critical point c from left to the right, that is

$$f'(x) \begin{cases} < 0 & \text{for } x < c \\ > 0 & \text{for } x > c \end{cases}$$

then f has a local minimum at c.



Example 1.4.1. Use the first derivative test to identify the local extrema of $f(x) = x^2 - 4x + 4$.

Solution. Note that the first derivative $f'(x) = 2x - 4$ is negative for $x < 2$ and positive for $x > 2$. Thus the graph of f descends towards $x = 2$ on the left side and then ascends away from $x = 2$ on the right side. Therefore, the critical point $x = 2$ is a local minimum of f with the local minimum value $f(2) = 0$.

Example 1.4.2. Use the first derivative test to find its local extrema of $f(x) = x^4 - 8x^2 + 16$.

Solution. Since $f'(x) = 2(x^2 - 4)(2x) = 4x(x - 2)(x + 2)$, the stationary points are $x = 0, \pm 2$.

x-range	$4x$	$x - 2$	$x + 2$	$f'(x)$	Interval
$x < -2$	−	−	−	−	$(-\infty, -2)$
$-2 < x < 0$	−	−	+	+	$(-2, 0)$
$0 < x < 2$	+	−	+	−	$(0, 2)$
$x > 2$	+	+	+	+	$(2, \infty)$

From the table, we see that

- $f'(x)$ changes its sign from negative to positive on passing through ± 2 from left to the right, and hence ± 2 are the points of local minimum for f.
- $f'(x)$ changes its sign from positive to negative on passing through 0 from left to the right, and hence 0 is a point of local maximum for f.

Example 1.4.3. Use the first derivative test to find the local extremum of $f(x) = \frac{x^2}{4-x^2}$, $x \neq \pm 2$.

Solution. By the quotient rule, $f'(x) = \frac{8x}{(4-x^2)^2} = 0$ for $x = 0$. That is $x = 0$ is a stationary point of f. Since $f'(x)$ changes its sign from negative to positive on passing through 0 from left to the right, 0 is a point of local minimum for f with minimum value $f(0) = 0$.

Exercise 1.4.1. Use the first derivative test to find the local extrema of the following functions:

(a) $x\sqrt{8 - x^2}$, where $-2\sqrt{2} < x < 2\sqrt{2}$

(b) $\frac{x^2-3}{x-2}$ for $x \neq 2$

(c) $3x^4 + 4x^3 - 12x^2 + 2$

(d) $12x - x^3$

(e) $(x - 3)e^x$

Answers.

(a) -2 is a point of local minimum, while 2 is a point of local maximum for f

(b) 1 is a point of local maximum, while 3 is a point of local minimum for f

(c) -2 is a point of local minimum, while 1, 2 are points of local minimum for f.

(d) -2 is a point of local minimum, while 2 is a point of local minimum for f

(e) 2 is a point of local maximum for f.

1.5 Second Derivative Test - Concavity

Definition 1.5.1 (Concavity)**.** If the graph of a function $f(x)$ lies above the tangents at its points on an interval I, we say that C is *concave up* in I. While, If the graph of $f(x)$ lies below the tangents at its points on an interval I, we say that C is *concave down* in I.

Theorem 1.5.1 (Test for Concavity)**.** Let $y = f(x)$ be a plane curve C. Then the graph of $f(x)$ is concave up or down in I according as $f''(x) > 0$ or $f''(x) < 0$ for all $x \in I$ respectively.

Definition 1.5.2 (Point of Inflection)**.** A point P on a curve $y = f(x)$ is called a *point of inflection*, if f is continuous at P and the concavity of the curve reverses on passing through P. Thus $P(c, f(c))$ is a point of inflection on the curve $y = f(x)$, if the sign of $f''(x)$ is different on either side of the ordinate $x = c$.

Example 1.5.1. If $f(x) = \frac{x^3}{3} - \frac{x^2}{2} - 2x + \frac{1}{3}$, use the second derivative test to separate the intervals on which f is concave up and the intervals on which f is concave down. Also, find the points of inflection of f.

Solution. We have

- $f'(x) = x^2 - x - 2 = (x+1)(x-2)$; the critical points are -1, 2; $f''(x) = 2x - 1$.
- Since $f''(x) > 0$ for $x > 1/2$, f is concave up on $(1/2, \infty)$, and $f''(x) < 0$ for $x < 1/2$ so that f is concave down on $(\infty, 1/2)$. Also, f is continuous and its concavity reverses on passing through $1/2$, we see that $1/2$ is point of inflection of f.

Example 1.5.2. Use the first derivative test to find the points of local extrema for $f(x) = \frac{x}{\sqrt{x^2+1}}$, and use the second derivative test to identify the intervals of concavity of f. Also, find the points of inflection of f.

Solution.

- The first derivative of f is $f'(x) = 1/(x^2+1)^{3/2}$, which is neither zero nor infinity. Thus f has no critical points, and hence no local maxima and no local minima.
- Since $f''(x) = -\frac{3x}{(x^2+1)^{5/2}} > 0$ for $x < 0$, and < 0 for $x > 0$, f is concave up on $(-\infty, 0)$ and concave down on $(0, \infty)$.
- Since f is continuous and its concavity reverses on passing through the origin, $x = 0$ is a point of inflection.

Exercise 1.5.1. Use the second derivative test to separate the intervals on which f is concave up and the intervals on which f is concave down. Also, find the points of inflection of f in each case:

(a) $\frac{3}{4}(x^2 - 1)^{2/3}$

(b) $x^3 - 3x^2 + 1$

(c) xe^{-x}

(d) $5 + 12x - x^3$

(e) $x/(x^2 + 2)$

Answers.

(a) f is concave up on $(-\infty, -\sqrt{3}) \cup (\sqrt{3}, \infty)$, concave down on $(-\sqrt{3}, \sqrt{3})$; $x = \pm\sqrt{3}$ are points of inflection.

(b) f is concave down on $(-\infty, 1)$, concave up on $(1, \infty)$; $x = 1$ is a point of inflection.

(c) f is concave down on $(-\infty, 2)$, concave up on $(2, \infty)$; $x = 2$ is a point of inflection.

(c) f is concave up on $(-\infty, 0)$, concave down on $(0, \infty)$; $x = 0$ is a point of inflection.

(d) f is concave up on $(-\sqrt{6}, 0) \bigcup (\sqrt{6}, \infty)$, concave down on $(-\infty, -\sqrt{6}) \bigcup (0, \sqrt{6})$; $x = 0, \pm\sqrt{6}$ are points of inflection.

(e) f is concave up on $(-2, 2)$ and concave down on $(-\infty, -2) \bigcup (2, \infty)$; $x = \pm 2$ are points of inflection.

Reference

1. Hass, J., Heil, C., Weir, M. D., Thomas' Calculus, 14th Ed., Pearson Edu. Inc., **(2018)**: Pages 183-214 link text

Chapter 2

Areas and Volumes of solids of Revolution by a Definite Integral

2.1 Area of the Region bounded by two Plane Curves

The area A of a region $\mathscr{D}$ enclosed by the graphs of $f(x)$ and $g(x)$ between the ordinates $x = a$ and $x = b$ is given by the definite integral of $|f(x) - g(x)|$ from $x = a$ and $x = b$. That is

$$A = \int_{x=a}^{b} |f(x) - g(x)| \, dx.$$

If $f(x) \geq g(x)$ for $x \in [a, c]$ and $f(x) \leq g(x)$ for $x \in [c, b]$, then

$$A = \int_{x=a}^{c} [f(x) - g(x)] \, dx + \int_{x=c}^{b} [g(x) - f(x)] \, dx.$$

Example 2.1.1. Find the area of the region enclosed by the parabolas $y = 2x - x^2$ and $y = x^2$.

Solution. Let $f(x) = 2x - x^2$ and $g(x) = x^2$. The the parabolas $y = 2x - x^2$ and $y = x^2$ intersect in the points $(0, 0)$ and $(1, 1)$. Also, $f(x) - g(x) = 2x - x^2 - x^2 = 2(x - x^2) \geq 0$ for all $0 \leq x \leq 1$. Therefore,

$$A = \int_{x=0}^{1} |f(x) - g(x)| \, dx = \int_{x=0}^{1} 2(x - x^2) \, dx = 2\left|\frac{x^2}{2} - \frac{x^3}{3}\right|_{x=0}^{1} = \frac{1}{3}.$$

Example 2.1.2. Find the area of the region enclosed by the the curves $y = \sin x$ and $y = \cos x$ between the ordinates $x = 0$ and $x = \pi/2$.

Solution. The curves $y = \sin x$ and $y = \cos x$ intersect at $x = \pi/4$ between the ordinates $x = 0$ and $x = \pi/2$. Also, $\cos x \geq \sin x$ for $0 \leq x \leq \pi/4$, while $\sin x \geq \cos x$ for $\pi/4 \leq x \leq \pi/2$. Therefore,

$$\begin{aligned} A &= \int_{x=0}^{\pi/4} [\cos x - \sin x] \, dx + \int_{x=\pi/4}^{\pi/2} [\sin x - \cos x] \, dx \\ &= \Big|\sin x + \cos x\Big|_{x=0}^{\pi/4} + \Big| - \cos x - \sin x\Big|_{x=\pi/4}^{\pi/2} = 2(\sqrt{2} - 1). \end{aligned}$$

Example 2.1.3. Find the area of the region enclosed by the graphs of $y = |x|$ and $y = 1 - |x|$.

Solution. The graphs intersect in the points, given by $1 - |x| = |x|$, that is $x = \pm 1/2$. Also, $1 - |x| \geq |x|$ for all $-\frac{1}{2} \leq x \leq \frac{1}{2}$. The required area is $A = 1/2$.

Exercise 2.1.1. Find the area of the region enclosed by

(a) the cubical parabola $y = x^3$ and the straight line $y = x$ in the first quadrant;
(b) the sinusoidal curves $y = \sin x$ and $y = \sin 2x$ between the ordinates $x = 0$ and $x = \pi$;
(c) the parabolas $y^2 = 4ax$ and $x^2 = 4ay$ in the first quadrant;
(d) the graphs of $y = 2x$ and $y = x^2 - 8$.

Answers.

(a) The curve $y = x^3$ and the straight line $y = x$ intersect in the points $(0,0)$ and $(1,1)$. So the x-limits are $x = 0$ to $x = 1$. Also, $x \geq x^3$ for all $0 \leq x \leq 1$; $A = 1/4$.

(b) The curves $y = \sin x$ and $y = \sin 2x$ intersect in the point where $x = \frac{\pi}{3}$. Also, $\sin 2x \geq \sin x$ for $0 \leq x \leq \frac{\pi}{3}$, while $\sin x \geq \sin 2x$ for $\frac{\pi}{3} \leq x \leq \pi$; $A = 5/2$.

(c) The parabolas intersect in the points, given by $4ax = (x^2/4a)^2$, that is at $x = 0$ and $x = 4a$. Also, $2\sqrt{ax} \geq x^2/4$ for all $0 \leq x \leq 4a$; $A = 16a^2/3$.

(d) The graphs intersect in the points, given by $2x = x^2 - 8$, that is or $x = -2$ and $x = 4$. Also $2x \geq x^2 - 8$ for all $-2 \leq x \leq 4$; $A = 56$.

2.2 Volumes of Solids of Revolution - Disk Method

Let $f(x)$ be a continuous function on $[a,b]$. The volume of solid of revolution obtained by revolving the arc of the plane curve $y = f(x)$ from $x = a$ to $x = b$ about the x-axis, is

$$V = \int_a^b \pi y^2 \, dx = \int_a^b \pi [f(x)]^2 \, dx. \tag{2.2.1}$$

Let $g(y)$ be a continuous function on $[c,d]$. The volume of solid of revolution obtained by revolving the arc of the plane curve $x = g(y)$ from $y = c$ to $y = d$ about the y-axis, is

$$V = \int_c^d \pi x^2 \, dy = \int_c^d \pi [g(y)]^2 \, dy. \tag{2.2.2}$$

Example 2.2.1. Find the volume of the solid of revolution of the arc of the parabola $y = \sqrt{x}$ from $x = 0$ to $x = 1$ about the x-axis.

Solution. The solid of revolution of the arc of the parabola is a paraboloid with x-axis as its axis, and its volume is $V = \int_{x=0}^{1} \pi y^2 \, dx = \int_{x=0}^{1} \pi x \, dx = \pi/2$.

Exercise 2.2.1. Find the volume of the solid of revolution of

(a) the semi-circular arc $x^2 + y^2 = a^2$ from $x = -a$ to $x = a$ about the x-axis;

(b) the arc of the curve $y = x^3$ from $y = 0$ to $y = 8$ about the y-axis;

(c) the hyperbola $xy = 2$ about the y-axis from $y = 1$ to $y = 8$;

(d) the hyperbola $y^2 - x^2 = 1$ from $x = -a$ to $x = a$ about the x-axis.

Answers.

(a) $V = 4\pi a^3/3$

(b) $V = 96\pi/5$

(c) $V = 7\pi/2$.

Exercise 2.2.2.

(a) Regarding a cone of height h and radius a, as a solid of revolution of the straight line segment joining the vertex $(0,0)$ to the point (a,h) from $y = 0$ to $y = h$ about the y-axis, find its volume. **Answer.** $V = \pi a^2 h/3$

(b) Regarding a cylinder of height h and radius a, as a solid of revolution of the rectangle with edges $x = 0$, $x = a$, $y = 0$ and $y = h$ about the y-axis, find its volume. **Answer.** $V = \pi a^2 h$

2.3 Volumes of Solids of Revolution - Washer Method

The volume of solid of revolution of the region enclosed by the plane curves $y = f(x)$ and $y = g(x)$ with $f(x) \geq g(x)$ from $x = a$ to $x = b$ about the x-axis, is

$$V = \int_a^b \pi[f(x)^2 - g(x)^2]\,\mathrm{d}x. \tag{2.3.1}$$

The volume of solid of revolution of the region enclosed by the plane curves $x = f(y)$ and $x = g(y)$ with $f(y) \geq g(y)$ from $y = c$ to $y = d$ about the y-axis, is

$$V = \int_c^d \pi[f(y)^2 - g(y)^2]\,\mathrm{d}y. \tag{2.3.2}$$

Example 2.3.1. Find the volume of the solid of revolution of the region enclosed by the parabola $y = x^2$ and the line $y = x$ about the x-axis.

Solution. The two curves intersect in the points $(0, 0)$ and $(1, 1)$, and $x^2 \leq x$ for all $0 \leq x \leq 1$. Therefore, the volume of the solid of revolution is

$$V = \int_{x=0}^1 \pi[x^2 - (x^2)^2]\,\mathrm{d}x = \int_{x=0}^1 \pi(x^2 - x^4)\,\mathrm{d}x = \pi\left|\frac{x^3}{3} - \frac{x^5}{5}\right|_{x=0}^1 = 2\pi/5.$$

Example 2.3.2. Find the volume of the solid of revolution of the region enclosed by the parabola $y = x^2 + 1$ and the line $x + y = 3$ about the x-axis.

Solution. The two curves intersect in the points, where $x = -2$ and $x = 1$. Also, $x^2 + 1 \leq 3 - x$ for all $-2 \leq x \leq 1$. Hence

$$V = \int_{x=-2}^1 \pi[(3 - x)^2 - (x^2 + 1)^2]\,\mathrm{d}x = \int_{x=-2}^1 \pi(8 - 6x - x^2 - x^4)\,\mathrm{d}x = 117\pi/5.$$

Exercise 2.3.1 (Self-check)**.** Find the volume of the solid of revolution of each of the following regions enclosed by the given curves about the x-axis (between the given limits):

(a) $y = x^3$ and $y = x^2$
(b) $y^2 = 4(x - 1)$ and $y = x - 1$
(c) $y = x^2 + 2$ and $y = 10 - x^2$
(d) $y = 1/x$ and $2y = 5 - 2x$

Exercise 2.3.2 (Self-check)**.** Find the volume of the solid of revolution of each of the following regions enclosed by the given curves about the y-axis:

(a) $y = x^{1/3}$ and $x = 4y$, $x, y \geq 0$
(b) $x^2 - 2x$ and $y = x$
(c) $y = 16 - x$ and $y = 3x + 2$
(d) $y = x^3$ and $y = x^{1/3}$

2.4 Other Axes of Revolution

It is possible to use the method of disks and the method of washers to find the volume of a solid of revolution whose axis of revolution is a line other than one of the coordinate axes. We integrate an appropriate cross-sectional area to find the volume.

Example 2.4.1. Find the volume of the solid generated when the region under the curve $y = x^2$ over the interval $[0, 2]$ is revolved about the line $y = -1$.

Solution. At each x in the interval $0 \le x \le 2$, the cross-section of the solid perpendicular to the axis $y = -1$ is a washer with outer radius $x^2 + 1$ and inner radius 1. Thus the area of the typical washer is

$$A(x) = \pi[(x^2 + 1)^2 - 1^2] = \pi(x^4 + 2x^2).$$

Therefore, the volume of the solid of revolution is

$$V = \int_{x=0}^{2} \pi A(x)\ \mathrm{d}x = \int_{x=0}^{2} \pi[x^4 + 2x^2]\ \mathrm{d}x = \pi \left| \frac{x^5}{5} + \frac{2x^3}{3} \right|_{x=0}^{2} = 176\pi/15.$$

Exercise 2.4.1 (Self-check)**.** Find the volume of the solid of revolution of each of the following regions enclosed by the given curves about the given axis:

(a) $y = x^{1/2}$, $y = 0$ and $x = 9$, about $x = 9$
(b) $y = x^{1/2}$, $y = 0$ and $x = 9$, about $y = 3$
(c) $x = y^2$ and $x = y$, about $y = -1$
(d) $x = y^2$ and $x = y$, about $x = -1$
(e) $y = x^2$ aand $y = x^3$, about $x = 1$.

Reference

1. Hass, J., Heil, C., Weir, M. D., Thomas' Calculus, 14th Ed., Pearson Edu. Inc., **(2018)**: Pages 296-306, 316-325 link text