$$\frac{\partial T}{\partial z} = 4y - 16, \quad \text{and} \quad \frac{\partial \phi}{\partial z} = 8z$$

By Lagrange's method

$$\frac{\partial T}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 0 \quad \Rightarrow \quad 16x + 8\lambda x = 0 \qquad ...(3)$$

$$\frac{\partial T}{\partial y} + \lambda \frac{\partial \phi}{\partial y} = 0 \quad \Rightarrow \quad 4z + 2\lambda y = 0 \qquad ...(4)$$

$$\frac{\partial T}{\partial z} + \lambda \frac{\partial \phi}{\partial z} = 0 \quad \Rightarrow \quad 4y - 16 + 8\lambda z = 0 \qquad ...(5)$$

From (3), $\lambda = -2$

Putting the value of λ in (4) and (5), we get

$$4z + 2(-2)y = 0 \quad \Rightarrow \quad z - y = 0 \qquad ...(6)$$

and $$4y - 16 + 8(-2)z = 0 \quad \Rightarrow \quad y - 4z = 4 \qquad ...(7)$$

Adding (6) and (7), we get

$$-3z = 4 \text{ and } z = -\frac{4}{3}$$

From (6), $y = z = -\frac{4}{3}$

On putting the values of y and z in (2), we get

$$4x^2 + \left(-\frac{4}{3}\right)^2 + 4\left(-\frac{4}{3}\right)^2 - 16 = 0 \quad \Rightarrow \quad x^2 = \frac{16}{9} \Rightarrow x = \pm\frac{4}{3}$$

Hence, the hottest points on the probe surface is $\left(\pm\frac{4}{3}, \frac{-4}{3}, \frac{-4}{3}\right)$ **Ans.**

Example 26. *A tent of a given volume has a square base of side 2a, has its four-side vertical of length b and is surmounted by a regular pyramid of height h. Find the values of a and b in terms of h such that the canvas required for its construction is minimum.*

Solution. Let V be the volume and S be the surface of the tent.

$$V = 4a^2 b + \frac{1}{3}(4a^2)h \qquad \left[\text{Volume of pyramid} = \frac{1}{3} \text{ Area of the base} \times \text{height}\right]$$

$$S = 8ab + 4a\sqrt{a^2 + h^2} \qquad \left[\text{Surface Area of pyramid} = \frac{1}{2} \text{ perimeter} \times \text{slant height}\right]$$

$$\frac{\partial S}{\partial a} + \lambda \frac{\partial V}{\partial a} = 0$$

$$\Rightarrow \quad 8b + 4\sqrt{a^2 + h^2} + \frac{4a^2}{\sqrt{a^2 + h^2}} + \lambda\left[8ab + \frac{8ah}{3}\right] = 0 \qquad ...(1)$$

$$\frac{\partial S}{\partial b} + \lambda \frac{\partial V}{\partial b} = 0 \quad \Rightarrow \quad 8a + 4\lambda a^2 = 0 \qquad ...(2)$$

$$\frac{\partial S}{\partial h} + \lambda \frac{\partial V}{\partial h} = 0 \quad \Rightarrow \quad \frac{4ah}{\sqrt{a^2 + h^2}} + \frac{4}{3}\lambda a^2 = 0 \qquad ...(3)$$

From (2) $\quad \lambda a + 2 = 0 \quad \Rightarrow \quad \lambda a = -2 \qquad ...(4)$

From (3) $\quad 12ah + 4\lambda a^2 \sqrt{a^2 + h^2} = 0$

$$\Rightarrow \quad 3h + \lambda a\sqrt{a^2 + h^2} = 0 \qquad ...(5)$$

Substituting the value of $\lambda\, a$ from (4) in (5), we get

$$3h - 2\sqrt{a^2 + h^2} = 0 \Rightarrow 9\,h^2 = 4a^2 + 4h^2 \Rightarrow 4a^2 = 5h^2$$

$$a = \frac{\sqrt{5}}{2}h$$

Substituting $\lambda\, a = -2$ and $a = \frac{\sqrt{5}}{2}h$ in (1) and simplifying, we get

$$8b + 4\sqrt{\frac{5h^2}{4} + h^2} + \frac{5h^2}{\sqrt{\frac{5h^2}{4} + h^2}} - 2\left[8b + \frac{8h}{3}\right] = 0$$

$$\Rightarrow \quad 8\,b + 6h + \frac{10\,h}{3} - 16\,b - \frac{16\,h}{3} = 0$$

$$\Rightarrow \quad -8b + 4h = 0 \Rightarrow b = \frac{h}{2}.$$

Thus, when $a = \frac{\sqrt{5}}{2}h$ and $b = \frac{h}{2}$ we get the stationary value of S. **Ans.**

Example 27. *Find the maximum and minimum distances of the point* (3, 4, 12) *from the sphere* $x^2 + y^2 + z^2 = 1$. (*AMIETE, June 2010*)

Solution. Let the co-ordinates of the given point be (x, y, z), then its distance (D) from (3, 4, 12).

$$D = \sqrt{(x-3)^2 + (y-4)^2 + (z-12)^2}$$

$$\Rightarrow \quad F(x, y, z) = (x-3)^2 + (y-4)^2 + (z-12)^2$$

$$x^2 + y^2 + z^2 = 1$$

$$\phi(x, y, z) = x^2 + y^2 + z^2 - 1$$

$$\frac{\partial F}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 2(x-3) + 2\lambda x = 0 \qquad ...(1)$$

$$\frac{\partial F}{\partial y} + \lambda \frac{\partial \phi}{\partial y} = 2(y-4) + 2\lambda y = 0 \qquad ...(2)$$

$$\frac{\partial F}{\partial z} + \lambda \frac{\partial \phi}{\partial z} = 2(z-12) + 2\lambda z = 0 \qquad ...(3)$$

Multiplying (1) by x, (2) by y and (3) by z and adding, we get

$$(x^2 + y^2 + z^2) - 3x - 4y - 12z + \lambda(x^2 + y^2 + z^2) = 0$$

$$1 - 3x - 4y - 12z + \lambda = 0 \qquad ...(4)$$

From (1) $\quad x = \frac{3}{1+\lambda} \qquad ...(5)$

From (2) $\quad y = \frac{4}{1+\lambda} \qquad ...(6)$

From (3) $\quad z = \frac{12}{1+\lambda} \qquad ...(7)$

Putting these values of x, y, z in (4),we have

$$1+\lambda - \frac{9}{1+\lambda} - \frac{16}{1+\lambda} - \frac{144}{1+\lambda} = 0 \quad \Rightarrow \quad (1+\lambda)^2 = 169$$

$$\Rightarrow \quad 1 + \lambda = \pm 13$$

Putting the value of $1 + \lambda$ in (5), (6) and (7) we have the points

$$\left(\frac{3}{13}, \frac{4}{13}, \frac{12}{13}\right) \text{ and } \left(\frac{-3}{13}, \frac{-4}{13}, \frac{-12}{13}\right).$$

The minimum distance $= \sqrt{\left(3-\frac{3}{13}\right)^2+\left(4-\frac{4}{13}\right)^2+\left(12-\frac{12}{13}\right)^2} = 12$

The maximum distance $= \sqrt{\left(3+\frac{3}{13}\right)^2+\left(4+\frac{4}{13}\right)^2+\left(12+\frac{12}{13}\right)^2} = 14$ **Ans.**

Example 28. *Use the method of Lagrange's multipliers to find the extreme values of $f(x, y, z) = 2x + 3y + z$ subject to $x^2 + y^2 = 5$ and $x + z = 1$.*

(*A.M.I.E.T.E, June 2010, Dec. 2007, Uttarakhand, I Semester, Dec. 2006*)

Solution. Let

$$f(x, y, z) = 2x + 3y + z \quad ...(1)$$

$$\phi(x, y) = x^2 + y^2 - 5 \quad ...(2)$$

$$\psi(x, z) = x + z - 1 \quad ...(3)$$

Lagranges Multipliers Equations are

$$\frac{\partial f}{\partial x}+\lambda\frac{\partial \phi}{\partial x}+\mu\frac{\partial \psi}{\partial x} = 0 \Rightarrow 2 + \lambda(2x) + \mu(1) = 0 \quad ...(4)$$

$$\frac{\partial f}{\partial y}+\lambda\frac{\partial \phi}{\partial y}+\mu\frac{\partial \psi}{\partial y} = 0 \Rightarrow 3 + \lambda(2y) + \mu(0) = 0 \quad ...(5)$$

$$\frac{\partial f}{\partial z}+\lambda\frac{\partial \phi}{\partial z}+\mu\frac{\partial \psi}{\partial z} = 0 \Rightarrow 1 + \lambda(0) + \mu(1) = 0 \Rightarrow \mu = -1 \quad ...(6)$$

Putting the value of μ in (4) and (5), we get

$$2 + 2\lambda x - 1 = 0 \Rightarrow 2\lambda x = -1, \Rightarrow x = -\frac{1}{2\lambda}$$

$$3 + 2\lambda y = 0 \Rightarrow 2\lambda y = -3, \Rightarrow y = -\frac{3}{2\lambda}$$

Putting the values of x, y in $x^2 + y^2 = 5$, we get

$$\frac{1}{4\lambda^2}+\frac{9}{4\lambda^2}=5 \Rightarrow \frac{10}{4\lambda^2} = 5 \Rightarrow 2\lambda^2 = 1$$

$$\Rightarrow \lambda^2 = \frac{1}{2} \Rightarrow \lambda = \pm\frac{1}{\sqrt{2}}$$

We know that $x = -\frac{1}{2\lambda} = \pm\frac{\sqrt{2}}{2} = \pm\frac{1}{\sqrt{2}}$

$$y = -\frac{3}{2\lambda} = \pm\frac{3\sqrt{2}}{2} = \pm\frac{3}{\sqrt{2}}$$

From (3), $x + z = 1 \Rightarrow z = 1 - x = 1 \mp \frac{1}{\sqrt{2}}$

Putting $x = \frac{1}{\sqrt{2}}$, $y = \frac{3}{\sqrt{2}}$ and $z = 1 - \frac{1}{\sqrt{2}}$ in (1), we get

$$f = \frac{2}{\sqrt{2}}+\frac{9}{\sqrt{2}}+1-\frac{1}{\sqrt{2}} = \frac{10}{\sqrt{2}}+1 = 5\sqrt{2}+1$$

Putting $x = -\frac{1}{\sqrt{2}}$, $y = -\frac{3}{\sqrt{2}}$ and $z = 1 + \frac{1}{\sqrt{2}}$ in (1), we get

$$f = 2\left(-\frac{1}{\sqrt{2}}\right)+3\left(-\frac{3}{\sqrt{2}}\right)+\left(1+\frac{1}{\sqrt{2}}\right) = -\frac{2}{\sqrt{2}}-\frac{9}{\sqrt{2}}+1+\frac{1}{\sqrt{2}}$$

$$= 1-5\sqrt{2}$$ **Ans.**