Module 4:

4.1:

(1) Solve $x''(t) + 3x'(t) + 2x(t) = H(t-2)$ with $x(0) = 0$ and $x'(0) = 0$ using Laplace transform.

Sol: Given differential equation is

$$x''(t) + 3x'(t) + 2x(t) = H(t-2)$$

Taking Laplace transform on both sides, we get

$$L\{x''(t)\} + 3L\{x'(t)\} + 2L\{x(t)\} = L\{H(t-2)\}$$

$$\Rightarrow \left(s^2 L\{x(t)\} - s\,x(0) - x'(0)\right) + 3\left(s L\{x(t)\} - x(0)\right)$$

$$+ 2L\{x(t)\} = \frac{e^{-2s}}{s} \quad \left(L\{H(t-a)\} = \frac{e^{-as}}{s}\right)$$

$$\Rightarrow (s^2 + 3s + 2) L\{x(t)\} = \frac{e^{-2s}}{s}$$

$$\Rightarrow L\{x(t)\} = e^{-2s} \cdot \frac{1}{s(s+1)(s+2)}$$

$$\Rightarrow x(t) = L^{-1}\left\{ e^{-2s} \frac{1}{s(s+1)(s+2)} \right\} \longrightarrow (*)$$

Now

$$\frac{1}{s(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}$$

$$\Rightarrow 1 = A(s+1)(s+2) + B\,s(s+2) + C\,s(s+1)$$

which gives $A = \frac{1}{2}$, $B = -1$, $C = \frac{1}{2}$

Therefore, from ⊛, we have

$$x(t) = L^{-1}\left\{ e^{-2s}\left(\frac{1}{2s} - \frac{1}{s+1} + \frac{1}{2(s+2)}\right)\right\}$$

$$= \frac{1}{2} L^{-1}\left\{ e^{-2s}\,\frac{1}{s}\right\} - L^{-1}\left\{ e^{-2s}\,\frac{1}{s+1}\right\}$$
$$+ \frac{1}{2} L^{-1}\left\{ e^{-2s}\,\frac{1}{s+2}\right\}$$

$$= \frac{1}{2} L^{-1}\left\{ e^{-2s}\, L\{1\}\right\} - L^{-1}\left\{ e^{-2s}\, L\{e^{-t}\}\right\}$$
$$+ \frac{1}{2} L^{-1}\left\{ e^{-2s}\, L\{e^{-2t}\}\right\}$$

$$= \frac{1}{2}\, u(t-2) - e^{-(t-2)}\, u(t-2)$$
$$+ \frac{1}{2} e^{-2(t-2)}\, u(t-2).$$

(by second shifting theorem)

2. Solve $x''(t) - x'(t) - 2x(t) = \delta(t-1)$ with $x(0)=0$, $x'(0)=0$ using Laplace transform.

Sol: Given differential equation is

$$x''(t) - x'(t) - 2x(t) = \delta(t-1)$$

Taking Laplace transform on both sides, we get

$$L\{x''(t)\} - L\{x'(t)\} - 2L\{x(t)\} = L\{\delta(t-1)\}$$

$$\Rightarrow (s^2 L\{x(t)\} - s\,x(0) - x'(0)) - (s L\{x(t) - x(0)) - 2L\{x(t)\} = L\{\delta(t-1)\}$$

$$\Rightarrow (s^2 - s - 2)\, L\{x(t)\} = e^{-s} \quad \left(L\{\delta(t-a)\} = e^{-as}\right)$$

$$\Rightarrow L\{x(t)\} = e^{-s} \cdot \frac{1}{(s+1)(s-2)}$$

$$\Rightarrow x(t) = L^{-1}\left\{e^{-s}\frac{1}{(s-2)(s+1)}\right\} \longrightarrow (*)$$

Now, $\dfrac{1}{(s-2)(s+1)} = \dfrac{A}{s-2} + \dfrac{B}{s+1}$

$$\Rightarrow 1 = A(s+1) + B(s-2)$$

which gives $A = \frac{1}{3}$ and $B = \frac{-1}{3}$.

Therefore, from ⊛, we have

$$x(t) = L^{-1}\left\{e^{-s}\left(\frac{1}{3(s-2)} - \frac{1}{3(s+1)}\right)\right\}$$

$$= \frac{1}{3}\left(L^{-1}\left\{e^{-s}\frac{1}{s-2}\right\} - L^{-1}\left\{e^{-s}\frac{1}{s+1}\right\}\right)$$

$$= \frac{1}{3}\left(L^{-1}\left\{e^{-s}.L\{e^{2t}\}\right\} - L^{-1}\left\{e^{-s}L\{e^{-t}\}\right\}\right)$$

$$= \frac{1}{3}\left(e^{2(t-1)}u(t-1) - e^{-(t-1)}u(t-1)\right)$$

Solve the following using Laplace Transform.

1. $x''(t) + 5x'(t) + 6x(t) = H(t-1)$

   with $x(0) = 0$, $x'(0) = 1$

2. $x''(t) + 2x'(t) + x(t) = \delta(t)$

   with $x(0) = 1$ and $x'(0) = 0$

3. $x''(t) + 6x'(t) + 8x(t) = \delta(t-1)$

   with $x(0) = 0$ and $x'(0) = 0$