

7.2 ■ ELECTROMAGNETIC INDUCTION

7.2.1 ■ Faraday's Law

In 1831 Michael Faraday reported on a series of experiments, including three that (with some violence to history) can be characterized as follows:

Experiment 1. He pulled a loop of wire to the right through a magnetic field (Fig. 7.21a). A current flowed in the loop.

Experiment 2. He moved the *magnet* to the *left*, holding the loop still (Fig. 7.21b). Again, a current flowed in the loop.

Experiment 3. With both the loop and the magnet at rest (Fig. 7.21c), he changed the *strength* of the field (he used an electromagnet, and varied the current in the coil). Once again, current flowed in the loop.

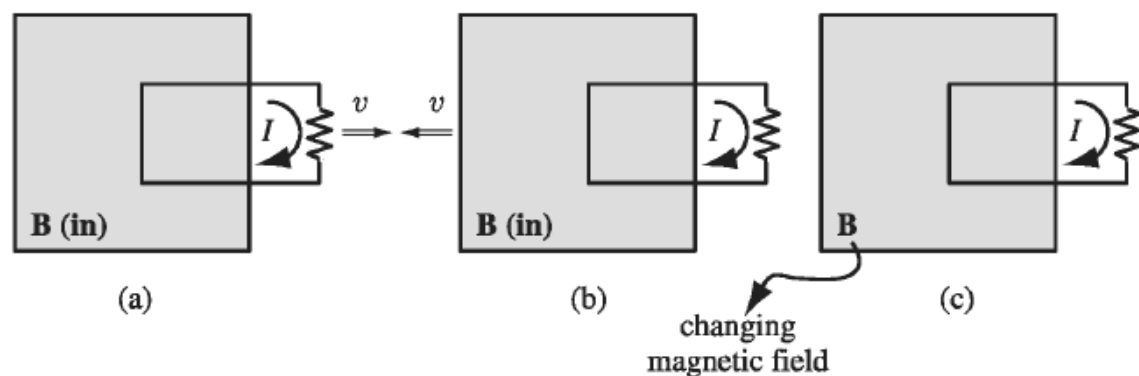


FIGURE 7.21

The first experiment, of course, is a straightforward case of motional emf; according to the flux rule:

$$\mathcal{E} = -\frac{d\Phi}{dt}.$$

I don't think it will surprise you to learn that exactly the same emf arises in Experiment 2—all that really matters is the *relative* motion of the magnet and the loop. Indeed, in the light of special relativity it *has* to be so. But Faraday knew nothing of relativity, and in classical electrodynamics this simple reciprocity is a remarkable coincidence. For if the *loop* moves, it's a *magnetic* force that sets up the emf, but if the loop is *stationary*, the force *cannot* be magnetic—stationary charges experience no magnetic forces. In that case, what *is* responsible? What sort of field exerts a force on charges at rest? Well, *electric* fields do, of course, but in this case there doesn't seem to be any electric field in sight.

Faraday had an ingenious inspiration:

A changing magnetic field induces an electric field.

It is this induced⁸ electric field that accounts for the emf in Experiment 2.⁹ Indeed, if (as Faraday found empirically) the emf is again equal to the rate of change of the flux,

$$\mathcal{E} = \oint \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi}{dt}, \quad (7.14)$$

then $\mathbf{E}$ is related to the change in $\mathbf{B}$ by the equation

$$\oint \mathbf{E} \cdot d\mathbf{l} = - \int \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{a}. \quad (7.15)$$

This is **Faraday's law**, in integral form. We can convert it to differential form by applying Stokes' theorem:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}.$$

(7.16)

Note that Faraday's law reduces to the old rule $\oint \mathbf{E} \cdot d\mathbf{l} = 0$ (or, in differential form, $\nabla \times \mathbf{E} = \mathbf{0}$) in the static case (constant $\mathbf{B}$) as, of course, it should.

In Experiment 3, the magnetic field changes for entirely different reasons, but according to Faraday's law an electric field will again be induced, giving rise to an emf $-d\Phi/dt$. Indeed, one can subsume all three cases (and for that matter any combination of them) into a kind of **universal flux rule**:

Whenever (and for whatever reason) the magnetic flux through a loop changes, an emf

$$\mathcal{E} = -\frac{d\Phi}{dt} \quad (7.17)$$

will appear in the loop.

Many people call *this* "Faraday's law." Maybe I'm overly fastidious, but I find this confusing. There are really *two* totally different mechanisms underlying Eq. 7.17, and to identify them both as "Faraday's law" is a little like saying that because identical twins look alike we ought to call them by the same name. In Faraday's first experiment it's the Lorentz force law at work; the emf is *magnetic*. But in the other two it's an *electric* field (induced by the changing magnetic field) that does the job. Viewed in this light, it is quite astonishing that all three processes yield the same formula for the emf. In fact, it was precisely this "coincidence" that led Einstein to the special theory of relativity—he sought a deeper understanding of what is, in classical electrodynamics, a peculiar accident. But that's a story for Chapter 12. In the meantime, I shall reserve the term "Faraday's law" for electric fields induced by changing magnetic fields, and I do *not* regard Experiment 1 as an instance of Faraday's law.

Keeping track of the *signs* in Faraday's law can be a real headache. For instance, in Ex. 7.5 we would like to know which *way* around the ring the induced current flows. In principle, the right-hand rule does the job (we called Φ positive to the left, in Fig. 7.22, so the positive direction for current in the ring is counterclockwise, as viewed from the left; since the first spike in Fig. 7.23b is *negative*, the first current pulse flows *clockwise*, and the second counterclockwise). But there's a handy rule, called **Lenz's law**, whose sole purpose is to help you get the directions right:¹⁰

Nature abhors a change in flux.

The induced current will flow in such a direction that the flux *it* produces tends to cancel the change. (As the front end of the magnet in Ex. 7.5 enters the ring, the flux increases, so the current in the ring must generate a field to the *right*—it therefore flows *clockwise*.) Notice that it is the *change* in flux, not the flux itself, that nature abhors (when the tail end of the magnet exits the ring, the flux *drops*, so the induced current flows *counterclockwise*, in an effort to restore it). Faraday induction is a kind of “inertial” phenomenon: A conducting loop “likes” to maintain a constant flux through it; if you try to *change* the flux, the loop responds by sending a current around in such a direction as to frustrate your efforts. (It doesn't *succeed* completely; the flux produced by the induced current is typically only a tiny fraction of the original. All Lenz's law tells you is the *direction* of the flow.)

Example 7.6. The “jumping ring” demonstration. If you wind a solenoidal coil around an iron core (the iron is there to beef up the magnetic field), place a metal ring on top, and plug it in, the ring will jump several feet in the air (Fig. 7.24). Why?

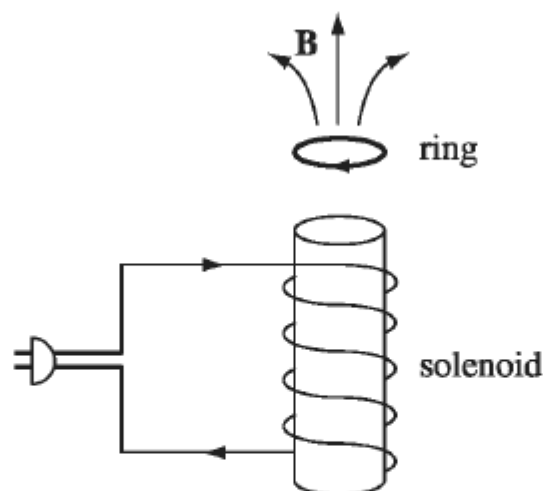


FIGURE 7.24

Solution

Before you turned on the current, the flux through the ring was *zero*. *Afterward* a flux appeared (upward, in the diagram), and the emf generated in the ring led to a current (in the ring) which, according to Lenz’s law, was in such a direction that *its* field tended to cancel this new flux. This means that the current in the loop is *opposite* to the current in the solenoid. And opposite currents repel, so the ring flies off.¹¹

Volume Current density

When the flow of charge is distributed throughout a three-dimensional region, we describe it by the **volume current density**, $\mathbf{J}$, defined as follows: Consider a “tube” of infinitesimal cross section $da_{\perp}$, running parallel to the flow (Fig. 5.14). If the current in this tube is $d\mathbf{I}$, the volume current density is

$$\mathbf{J} \equiv \frac{d\mathbf{I}}{da_{\perp}}. \quad (5.25)$$

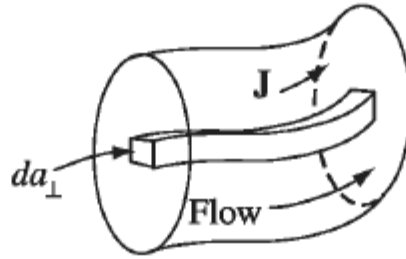


FIGURE 5.14

Equation of Continuity

According to Eq. 5.25, the total current crossing a surface S can be written as

$$I = \int_S J da_{\perp} = \int_S \mathbf{J} \cdot d\mathbf{a}. \quad (5.28)$$

(The dot product serves neatly to pick out the appropriate component of $d\mathbf{a}$.) In particular, the charge per unit time leaving a volume $\mathcal{V}$ is

$$\oint_S \mathbf{J} \cdot d\mathbf{a} = \int_{\mathcal{V}} (\nabla \cdot \mathbf{J}) d\tau.$$

Because charge is conserved, whatever flows out through the surface must come at the expense of what remains inside:

$$\int_{\mathcal{V}} (\nabla \cdot \mathbf{J}) d\tau = -\frac{d}{dt} \int_{\mathcal{V}} \rho d\tau = -\int_{\mathcal{V}} \left(\frac{\partial \rho}{\partial t} \right) d\tau.$$

(The minus sign reflects the fact that an *outward* flow *decreases* the charge left in $\mathcal{V}$.) Since this applies to *any* volume, we conclude that

$$\boxed{\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}}. \quad (5.29)$$

This is the precise mathematical statement of local charge conservation; it is called the **continuity equation**.

5.2.1 ■ Steady Currents

Stationary charges produce electric fields that are constant in time; hence the term **electrostatics**.⁸ *Steady currents* produce magnetic fields that are constant in time; the theory of steady currents is called **magnetostatics**.

Stationary charges	⇒	constant electric fields: electrostatics.
Steady currents	⇒	constant magnetic fields: magnetostatics.

By **steady current** I mean a continuous flow that has been going on forever, without change and without charge piling up anywhere. (Some people call them “stationary currents”; to my ear, that’s a contradiction in terms.) Formally, electro/magnetostatics is the régime

$$\frac{\partial \rho}{\partial t} = 0, \quad \frac{\partial \mathbf{J}}{\partial t} = \mathbf{0}, \quad (5.32)$$

at all places and all times. Of course, there’s no such thing in practice as a *truly* steady current, any more than there is a *truly* stationary charge. In this sense, both electrostatics and magnetostatics describe artificial worlds that exist only in textbooks. However, they represent suitable *approximations* as long as the actual fluctuations are remote, or gradual—in fact, for most purposes magnetostatics applies very well to household currents, which alternate 120 times a second!

Notice that a moving *point* charge *cannot possibly constitute a steady current*. If it’s here one instant, it’s gone the next. This may seem like a minor thing to *you*, but it’s a major headache for *me*. I developed each topic in electrostatics by starting out with the simple case of a point charge at rest; then I generalized to an arbitrary charge distribution by invoking the superposition principle. This approach is not open to us in magnetostatics because a moving point charge does not produce a static field in the first place. We are *forced* to deal with extended current distributions right from the start, and, as a result, the arguments are bound to be more cumbersome.

When a steady current flows in a wire, its magnitude I must be the same all along the line; otherwise, charge would be piling up somewhere, and it wouldn’t be a steady current. More generally, since $\partial \rho / \partial t = 0$ in magnetostatics, the continuity equation (5.29) becomes

$$\nabla \cdot \mathbf{J} = 0. \quad (5.33)$$

5.3.3 ■ Ampère's Law

The equation for the curl of $\mathbf{B}$,

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}, \quad (5.56)$$

is called **Ampère's law** (in differential form). It can be converted to integral form by the usual device of applying one of the fundamental theorems—in this case Stokes' theorem:

$$\int (\nabla \times \mathbf{B}) \cdot d\mathbf{a} = \oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 \int \mathbf{J} \cdot d\mathbf{a}.$$

Now, $\int \mathbf{J} \cdot d\mathbf{a}$ is the total current passing through the surface (Fig. 5.31), which we call I_{enc} (the **current enclosed** by the **Amperian loop**). Thus

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}}. \quad (5.57)$$

7.3 ■ MAXWELL'S EQUATIONS

7.3.1 ■ Electrodynamics Before Maxwell

So far, we have encountered the following laws, specifying the divergence and curl of electric and magnetic fields:

- (i) $\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0} \rho$ (Gauss's law),
- (ii) $\nabla \cdot \mathbf{B} = 0$ (no name),
- (iii) $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$ (Faraday's law),
- (iv) $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$ (Ampère's law).

These equations represent the state of electromagnetic theory in the mid-nineteenth century, when Maxwell began his work. They were not written in so compact a form, in those days, but their physical content was familiar. Now, it happens that there is a fatal inconsistency in these formulas. It has to do with the old rule that divergence of curl is always zero. If you apply the divergence to number (iii), everything works out:

$$\nabla \cdot (\nabla \times \mathbf{E}) = \nabla \cdot \left(-\frac{\partial \mathbf{B}}{\partial t} \right) = -\frac{\partial}{\partial t} (\nabla \cdot \mathbf{B}).$$

The left side is zero because divergence of curl is zero; the right side is zero by virtue of equation (ii). But when you do the same thing to number (iv), you get into trouble:

$$\nabla \cdot (\nabla \times \mathbf{B}) = \mu_0 (\nabla \cdot \mathbf{J}); \quad (7.36)$$

the left side must be zero, but the right side, in general, is *not*. For *steady* currents, the divergence of $\mathbf{J}$ is zero, but when we go beyond magnetostatics Ampère's law cannot be right.

There's another way to see that Ampère's law is bound to fail for nonsteady currents. Suppose we're in the process of charging up a capacitor (Fig. 7.43). In integral form, Ampère's law reads

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}}.$$

I want to apply it to the Amperian loop shown in the diagram. How do I determine I_{enc} ? Well, it's the total current passing through the loop, or, more precisely, the current piercing a surface that has the loop for its boundary. In this case, the *simplest* surface lies in the plane of the loop—the wire punctures this surface, so $I_{\text{enc}} = I$. Fine—but what if I draw instead the balloon-shaped surface in Fig. 7.43? *No* current passes through *this* surface, and I conclude that $I_{\text{enc}} = 0$! We never had this problem in magnetostatics because the conflict arises only when charge

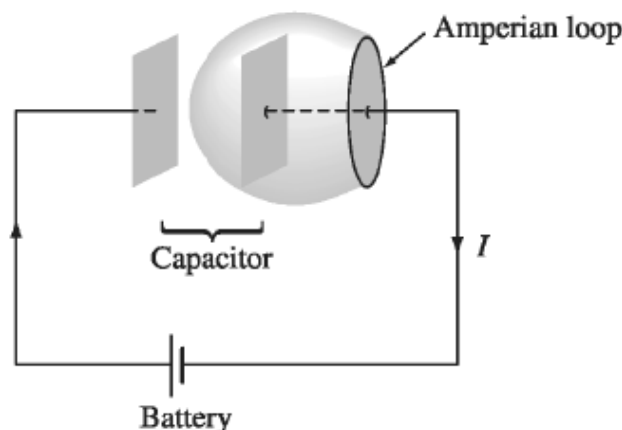


FIGURE 7.43

is piling up somewhere (in this case, on the capacitor plates). But for *nonsteady* currents (such as this one) “the current enclosed by the loop” is an ill-defined notion; it depends entirely on what surface you use. (If this seems pedantic to you—“obviously one should use the plane surface”—remember that the Amperian loop could be some contorted shape that doesn’t even lie in a plane.)

Of course, we had no right to *expect* Ampère’s law to hold outside of magnetostatics; after all, we derived it from the Biot-Savart law. However, in Maxwell’s time there was no *experimental* reason to doubt that Ampère’s law was of wider validity. The flaw was a purely theoretical one, and Maxwell fixed it by purely theoretical arguments.

7.3.2 ■ How Maxwell Fixed Ampère’s Law

The problem is on the right side of Eq. 7.36, which *should be* zero, but *isn’t*. Applying the continuity equation (5.29) and Gauss’s law, the offending term can be rewritten:

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial t}(\epsilon_0 \nabla \cdot \mathbf{E}) = -\nabla \cdot \left(\epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right).$$

If we were to combine $\epsilon_0(\partial \mathbf{E}/\partial t)$ with $\mathbf{J}$, in Ampère’s law, it would be just right to kill off the extra divergence:

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}. \quad (7.37)$$

(Maxwell himself had other reasons for wanting to add this quantity to Ampère’s law. To him, the rescue of the continuity equation was a happy dividend rather than a primary motive. But today we recognize this argument as a far more compelling one than Maxwell’s, which was based on a now-discredited model of the ether.)²⁰

Such a modification changes nothing, as far as *magnetostatics* is concerned: when $\mathbf{E}$ is constant, we still have $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$. In fact, Maxwell’s term is hard to detect in ordinary electromagnetic experiments, where it must compete for attention with $\mathbf{J}$ —that’s why Faraday and the others never discovered it in the laboratory. However, it plays a crucial role in the propagation of electromagnetic waves, as we’ll see in Chapter 9.

Apart from curing the defect in Ampère’s law, Maxwell’s term has a certain aesthetic appeal: Just as a changing *magnetic* field induces an *electric* field (Faraday’s law), so²¹

A changing electric field induces a magnetic field.

Of course, theoretical convenience and aesthetic consistency are only *suggestive*—there might, after all, be other ways to doctor up Ampère’s law. The real confirmation of Maxwell’s theory came in 1888 with Hertz’s experiments on electromagnetic waves.

Maxwell called his extra term the **displacement current**:

$$\mathbf{J}_d \equiv \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}. \quad (7.38)$$

(It’s a misleading name; $\epsilon_0(\partial \mathbf{E}/\partial t)$ has nothing to do with current, except that it adds to $\mathbf{J}$ in Ampère’s law.) Let’s see now how displacement current resolves the paradox of the charging capacitor (Fig. 7.43). If the capacitor plates are very close together (I didn’t *draw* them that way, but the calculation is simpler if you assume this), then the electric field between them is

$$E = \frac{1}{\epsilon_0} \sigma = \frac{1}{\epsilon_0} \frac{Q}{A},$$

where Q is the charge on the plate and A is its area. Thus, between the plates

$$\frac{\partial E}{\partial t} = \frac{1}{\epsilon_0 A} \frac{dQ}{dt} = \frac{1}{\epsilon_0 A} I.$$

Now, Eq. 7.37 reads, in integral form,

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}} + \mu_0 \epsilon_0 \int \left(\frac{\partial \mathbf{E}}{\partial t} \right) \cdot d\mathbf{a}. \quad (7.39)$$

If we choose the *flat* surface, then $E = 0$ and $I_{\text{enc}} = I$. If, on the other hand, we use the balloon-shaped surface, then $I_{\text{enc}} = 0$, but $\int (\partial \mathbf{E}/\partial t) \cdot d\mathbf{a} = I/\epsilon_0$. So we get the same answer for either surface, though in the first case it comes from the conduction current, and in the second from the displacement current.

7.3.3 ■ Maxwell’s Equations

In the last section we put the finishing touches on Maxwell’s equations:

(i) $\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0} \rho$	(Gauss’s law),	(7.40)
(ii) $\nabla \cdot \mathbf{B} = 0$	(no name),	
(iii) $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$	(Faraday’s law),	
(iv) $\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$	(Ampère’s law with Maxwell’s correction).	

9.2 ■ ELECTROMAGNETIC WAVES IN VACUUM

9.2.1 ■ The Wave Equation for $\mathbf{E}$ and $\mathbf{B}$

In regions of space where there is no charge or current, Maxwell's equations read

$$\left. \begin{array}{ll} \text{(i)} \quad \nabla \cdot \mathbf{E} = 0, & \text{(iii)} \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \\ \text{(ii)} \quad \nabla \cdot \mathbf{B} = 0, & \text{(iv)} \quad \nabla \times \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}. \end{array} \right\} \quad (9.40)$$

They constitute a set of coupled, first-order, partial differential equations for $\mathbf{E}$ and $\mathbf{B}$. They can be *decoupled* by applying the curl to (iii) and (iv):

$$\begin{aligned} \nabla \times (\nabla \times \mathbf{E}) &= \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = \nabla \times \left(-\frac{\partial \mathbf{B}}{\partial t} \right) \\ &= -\frac{\partial}{\partial t} (\nabla \times \mathbf{B}) = -\mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}, \end{aligned}$$

$$\begin{aligned} \nabla \times (\nabla \times \mathbf{B}) &= \nabla(\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B} = \nabla \times \left(\mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right) \\ &= \mu_0 \epsilon_0 \frac{\partial}{\partial t} (\nabla \times \mathbf{E}) = -\mu_0 \epsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2}. \end{aligned}$$

Or, since $\nabla \cdot \mathbf{E} = 0$ and $\nabla \cdot \mathbf{B} = 0$,

$$\nabla^2 \mathbf{E} = \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}, \quad \nabla^2 \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2}.$$

 (9.41)

We now have *separate* equations for $\mathbf{E}$ and $\mathbf{B}$, but they are of *second* order; that's the price you pay for decoupling them.

In vacuum, then, each Cartesian component of $\mathbf{E}$ and $\mathbf{B}$ satisfies the **three-dimensional wave equation**,

$$\nabla^2 f = \frac{1}{v^2} \frac{\partial^2 f}{\partial t^2}.$$

(This is the same as Eq. 9.2, except that $\partial^2 f / \partial z^2$ is replaced by its natural generalization, $\nabla^2 f$.) So Maxwell's equations imply that empty space supports the propagation of electromagnetic waves, traveling at a speed

$$v = \frac{1}{\sqrt{\epsilon_0 \mu_0}} = 3.00 \times 10^8 \text{ m/s}, \quad (9.42)$$

which happens to be precisely the velocity of light, c . The implication is astounding: Perhaps light *is* an electromagnetic wave.⁴ Of course, this conclusion does not surprise anyone today, but imagine what a revelation it was in Maxwell's time! Remember how ϵ_0 and μ_0 came into the theory in the first place: they were constants in Coulomb's law and the Biot-Savart law, respectively. You measure them in experiments involving charged pith balls, batteries, and wires—experiments having nothing whatever to do with light. And yet, according to Maxwell's theory, you can calculate c from these two numbers. Notice the crucial role played by Maxwell's contribution to Ampère's law ($\mu_0 \epsilon_0 \partial \mathbf{E} / \partial t$); without it, the wave equation would not emerge, and there would be no electromagnetic theory of light.

9.2.2 ■ Monochromatic Plane Waves

For reasons discussed in Sect. 9.1.2, we may confine our attention to sinusoidal waves of frequency ω . Since different frequencies in the visible range correspond to different *colors*, such waves are called **monochromatic** (Table 9.1). Suppose,

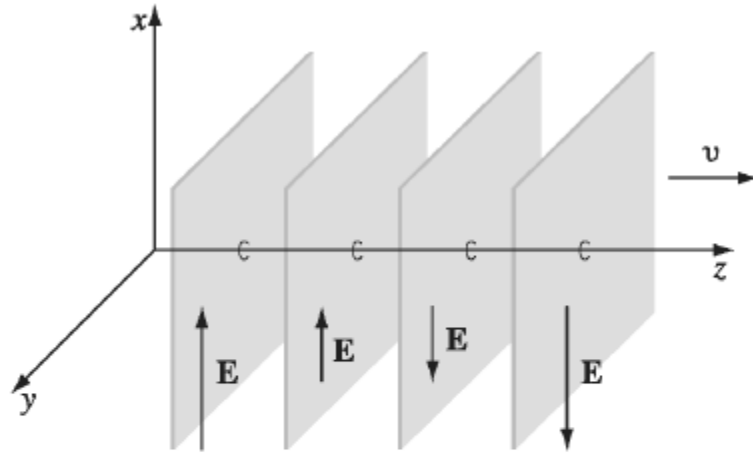


FIGURE 9.9

moreover, that the waves are traveling in the z direction and have no x or y dependence; these are called **plane waves**,⁵ because the fields are uniform over every plane perpendicular to the direction of propagation (Fig. 9.9). We are interested, then, in fields of the form

$$\tilde{\mathbf{E}}(z, t) = \tilde{\mathbf{E}}_0 e^{i(kz - \omega t)}, \quad \tilde{\mathbf{B}}(z, t) = \tilde{\mathbf{B}}_0 e^{i(kz - \omega t)}, \quad (9.43)$$

where $\tilde{\mathbf{E}}_0$ and $\tilde{\mathbf{B}}_0$ are the (complex) amplitudes (the *physical* fields, of course, are the real parts of $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{B}}$), and $\omega = ck$.

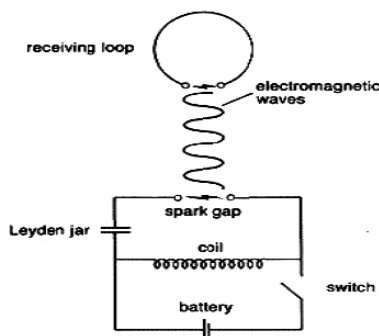
Heinrich Hertz's Wireless Experiment (1887)

See the excellent reference [Heinrich Rudolf Hertz](#).

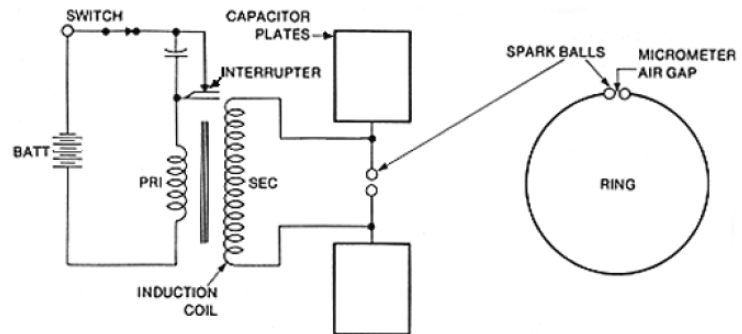
In the 1880s many were seeking experimental evidence to establish the equivalence of light and electromagnetic propagation. James Clerk Maxwell's mathematical theory of 1873 had predicted that electromagnetic disturbances should propagate through space at the speed of light and should exhibit the wave-like characteristics of light propagation.

In 1883 Hertz became a lecturer in theoretical physics at the University of Kiel and two years later he was appointed professor of physics at Karlsruhe Polytechnic. In 1887 Hertz designed a brilliant set of experiments tested Maxwell's hypothesis. He used an oscillator made of polished brass knobs, each connected to an induction coil and separated by a tiny gap over which sparks could leap. Hertz reasoned that, if Maxwell's predictions were correct, electromagnetic waves would be transmitted during each series of sparks. To confirm this, Hertz made a simple receiver of looped wire. At the ends of the loop were small knobs separated by a tiny gap. The receiver was placed several yards from the oscillator.

According to theory, if electromagnetic waves were spreading from the oscillator sparks, they would induce a current in the loop that would send sparks across the gap. This occurred when Hertz turned on the oscillator, producing the first transmission and reception of electromagnetic waves. Hertz also noted that electrical conductors reflect the waves and that they can be focused by concave reflectors. He found that nonconductors allow most of the waves to pass through. Another of his discoveries was the photoelectric effect.



Conceptual Schematic of Hertz's Experiment



More Detailed Schematic of Hertz's Experiment