

Bilinear transformation (Möbius transformation)

$$w = \frac{az+b}{cz+d}, \quad ad-bc \neq 0 \quad \text{--- (1)}$$

is known as bilinear transformation.

Note:- $\frac{dw}{dz} = \frac{ad-bc}{(cz+d)^2}$

If $ad-bc \neq 0$ then $f'(z) \neq 0$
and transformation is conformal.

from (1)

$$z = \frac{-dw+b}{cw-a}, \quad w \neq a/c \quad \text{--- (2)}$$

bilinear transformation.

Note:- from (1) it is clear that every point of z -plane mapped to unique point in w -plane except $z = -\frac{d}{c}$ (as $z = -\frac{d}{c}$ is singular)

* from (2) it is clear that every point of w -plane mapped to unique point in z -plane except $w = \frac{a}{c}$ (singular point)

Invariant point or fixed point of Bilinear transformation:-

$$w = \frac{az+b}{cz+d} \quad \text{--- (A)}$$

the points which coincide with the transformation ~~is~~ called invariant points.

ie the points for which $w = z$

$$z = \frac{az+b}{cz+d}$$

$$cz^2 + dz - az + b = 0$$

$$cz^2 + (d-a)z + b = 0 \rightarrow \text{quadratic eqn in } z \quad \text{--- (3)}$$

roots of (3) are called invariant points.

If two roots are equal ie $z_1 = z_2$ then transformation is parabolic.

Cross ratio:-

If z_1, z_2, z_3, z_4 are four points in z -plane then their cross ratio is defined as

$$\frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)}$$

Theorem:- Any bilinear transformation preserves the cross ratio of four points.

i.e. w_1, w_2, w_3, w_4 are four points in w -plane & corresponding points in z -plane are z_1, z_2, z_3, z_4 then.

$$\frac{(w_1 - w_2)(w_3 - w_4)}{(w_1 - w_4)(w_3 - w_2)} = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)}$$

Method to find bilinear transformation:-

(1) By finding a, b, c, d using given conditions

then $w = \frac{az+b}{cz+d}$

(2) with the help of cross ratio

$$\frac{(w - w_1)(w_2 - w_3)}{(w - w_3)(w_2 - w_1)} = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}$$

z	w
z_1	w_1
z_2	w_2
z_3	w_3

Q:- Find the bilinear transformation which maps $z_1=2, z_2=i, z_3=-2$ into points $w_1=1, w_2=i, w_3=-1$. discuss conformality also.

Solⁿ:-

$$w = \frac{az+b}{cz+d}$$

z	w
2	1
i	i
-2	-1

at $z=2, w=1$

$$1 = \frac{2a+b}{2c+d}$$

$$2c+d = 2a+b \quad \text{--- (1)}$$

at $z=i, w=i$

$$i = \frac{ia+b}{ic+d}$$

$$-c+id = ia+b \quad \text{--- (2)}$$

at $z=-2, w=-1$

$$-1 = \frac{-2a+b}{-2c+d}$$

$$2c-d = -2a+b \quad \text{--- (3)}$$

from (1) & (3) (Adding)

$$\boxed{b = 2c}$$

Subtracting ① & ③

$$\boxed{d = 2a}$$

from ②

$$-c + id = ia + b$$

$$-c + 2ai = ia - 2ai$$

$$\boxed{c = \frac{ai^3}{3}}$$

$$\boxed{d = 2a}$$

$$\boxed{b = \frac{2ai}{3}}$$

$$w = \frac{az + b}{cz + d}$$

$$w = \frac{az + \frac{2ai}{3}}{\frac{iaz}{3} + 2a}$$

$$\boxed{w = \frac{3z + 2i}{iz + 6}}$$

Invariant points: $w = z$

$$z = \frac{3z + 2i}{iz + 6}$$

$$iz^2 + 6z = 3z + 2i$$

$$iz^2 + 3z - 2i = 0$$

$$iz^2 - i^2 2z + z - 2i = 0$$

$$iz(z - 2i) + 1(z - 2i) = 0$$

$z = 2i, z = -i$ are invariant points.

* ~~Here~~ $w = \frac{az + b}{cz + d}$ is conformal

if $ad - bc \neq 0$

we have $a = 3, b = 2i, c = i, d = 6$

$ad - bc = 20 \neq 0$ i.e. mapping is

conformal.

Q:- Find bilinear transformation which maps points $z=0, -1, i$ onto $w=i, 0, \infty$. Also find image of $|z|=1$.

Solⁿ using
$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

z	w
0	i
-1	0
i	∞

$$\frac{(w-w_1)\left(\frac{w_2}{w_3}-1\right)}{\left(\frac{w}{w_3}-1\right)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-i)(-1)}{(-1)(-i)} = \frac{z(-1-i)}{(z-i)(-1)}$$

$$\frac{w-i}{-i} = \frac{z(1+i)}{z-i}$$

$$w-i = \frac{-i z(1+i)}{z-i}$$

$$w = \frac{(1-i)z}{z-i} + i$$

$$w = \frac{z+1}{z-i}$$

image of unit circle $|z|=1$

$$z = \frac{1+i\omega}{\omega-1}$$

we have $|z|=1$

$$\left| \frac{1+i\omega}{\omega-1} \right| = 1$$

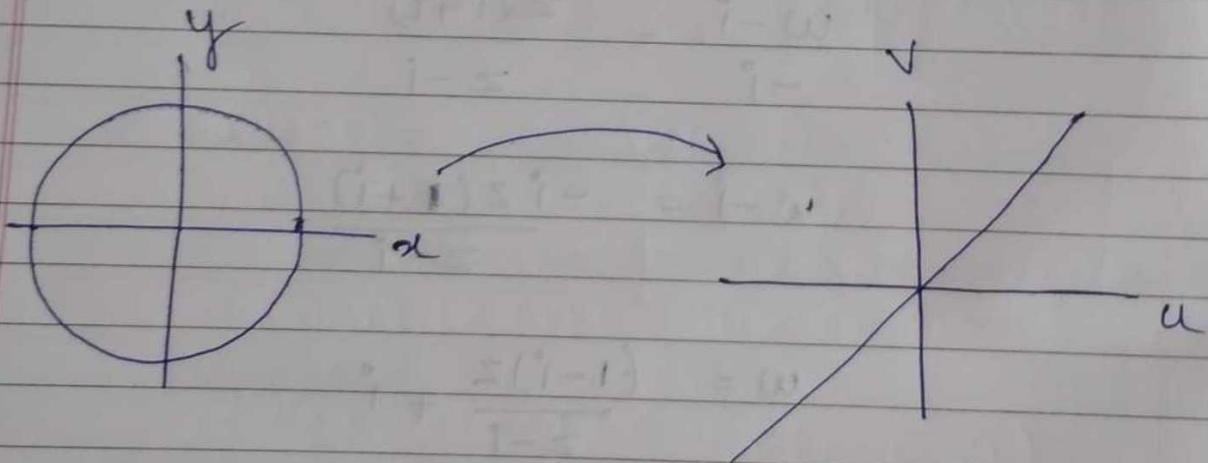
$$|1+i\omega| = |\omega-1|$$

$$|1+i(u+iv)| = |(u+iv)-1|$$

$$|(1-v)+iu| = |(u-1)+iv|$$

$$(1-v)^2 + u^2 = (u-1)^2 + v^2$$

$$\Rightarrow -2u + 2v = 0 \Rightarrow v = u$$



Q:- Find fixed point of transformation

$$(1) w = \frac{5z+4}{z+5} \quad (2) w = \frac{3z-4}{z-1}$$

$$(3) w = \frac{z-1}{z+1}$$

Q:- Find cross ratio of points $0, i, -i, 1$

Solⁿ:- cross ratio of four points is given by

$$\frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)}$$

$$(z_1 - z_4)(z_3 - z_2)$$

$$= \frac{-i(-i-1)}{(-1)(-2i)}$$

$$= \frac{i(i+1)}{2i} = \frac{i+1}{2} =$$

$$z_1 = 0$$

$$z_2 = i$$

$$z_3 = -i$$

$$z_4 = 1$$

Q:- Find cross ratio of points $0, i, -i, \infty$.

Q:- Find bilinear transformation which maps points $z = \infty, i, 0$ into points $w = 0, i, \infty$.

Q:- Find bilinear transformation which maps

$$Z = 0, -i, -1 \text{ onto } w = i, 1, 0$$

Solⁿ:-

z	w
0	i
$-i$	1
-1	0

We have

$$\frac{(w - w_1)(w_2 - w_3)}{(w - w_3)(w_2 - w_1)} = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}$$

$$\frac{(w - i)(1 - 0)}{(w - 0)(1 - i)} = \frac{(z - 0)(-i + 1)}{(z + 1)(-i - 0)}$$

$$\frac{(w - i)}{w(1 - i)} = \frac{z(1 - i)}{-i(z + 1)}$$

$$-i(w - i)(z + 1) = wz(1 - i)^2$$

$$-i(wz + w - iz - i) = wz(1 - 1 - 2i)$$

$$-i wz - i w + z + 1 = -2i wz$$

$$i wz - i w = z + 1$$

$$\boxed{w = \frac{z + 1}{iz - i}}$$

required mapping.

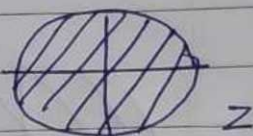
Q:- Find bilinear transformation which maps $z = \infty, i, 0$ into points $w = 0, i, \infty$.

Q:- Find bilinear transformation which maps the points $z = 1, i, -1$ into $w = i, 0, -i$. Hence find image of $|z| < 1$. ~~Also find invariant points.~~

Ans

$$w = \frac{i z + 1}{-i z + 1}$$

image of $|z| < 1$



$$-i w z + w = i z + 1$$

$$-i w z - i z = 1 - w$$

$$i z (w + 1) = w - 1$$

$$z = \frac{w - 1}{i(w + 1)}$$

$$|z| = \left| \frac{w - 1}{i(w + 1)} \right| < 1$$

$$|w - 1| < |i(w + 1)|$$

$$|w - 1|^2 < |i w + i|^2$$

$$|u + iv - 1|^2 < |i(u + iv) + i|^2$$

$$|(u - 1) + iv|^2 < |(-v) + i(u + 1)|^2$$

$$(u - 1)^2 + v^2 < (-v + i)^2 + u^2$$

$$u^2 + 1 - 2u + v^2 < v^2 + 1 - 2v + u^2$$

$$|u+i(v-1)|^2 < |-v+i(u+1)|^2$$

$$|(u-1)+iv|^2 < |-v+i(u+1)|^2$$

$$(u-1)^2 + v^2 < v^2 + (u+1)^2$$

$$u^2 + 1 - 2u + v^2 < v^2 + u^2 + 1 + 2u$$

$$4u > 0$$

$$\Rightarrow \boxed{u > 0}$$

i.e interior of circle i.e. $|z| < 1$ maps to right half of w plane.

