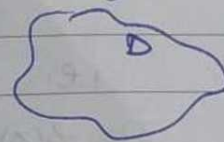


Analytic function:-

A function $f(z)$ is said to be analytic at point z_0 if f is differentiable not only at z_0 but every point of some neighbourhood of z_0 .



A function $f(z)$ is analytic in a domain D , if it is analytic at every point of domain D . An analytic function is also known as "holomorphic" or "regular" function.



Singular point:-

The point at which function is not differentiable is called a singular point of the function.

Entire function:-

A function which is analytic everywhere (for all points z in the complex plane) is known as entire function.

Ex:- Polynomial functions are entire.

Note:- Analytic function is always differentiable and continuous but converse is not true.

Cauchy-Riemann equations:- (C-R equations)

1) Necessary condition for $f(z)$ to be analytic :-
Theorem:- The necessary conditions for a function $f(z) = u + iv$ to be analytic at all points in a region R are

(i) u_x, u_y, v_x, v_y are continuous functions in region R .

(ii) $u_x = v_y$, $u_y = -v_x$
called C-R equations.

i.e. $f(z)$ must satisfy C-R equations

or If $f(z)$ is analytic in a domain $D \Rightarrow f(z)$ satisfy C-R equations.

Proof:- Let $f(z)$ be analytic function in R

$$w = f(z) = u + iv$$

then $f'(z) = \frac{dw}{dz}$ exist at every points of

the domain.

i.e. consider

$$f'(z) = \lim_{\Delta z \rightarrow 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

$$f'(z) = \lim_{\Delta z \rightarrow 0} \frac{(u + \Delta u) + i(v + \Delta v) - (u + iv)}{\Delta z}$$

$$= \lim_{\Delta z \rightarrow 0} \frac{\Delta u + i \Delta v}{\Delta z}$$

$$\text{i.e. } f'(z) = \lim_{\Delta z \rightarrow 0} \left(\frac{\Delta u}{\Delta z} + i \frac{\Delta v}{\Delta z} \right) = \lim_{\substack{\Delta x \rightarrow 0 \\ \Delta y \rightarrow 0}} \frac{\Delta u + i \Delta v}{\Delta x + i \Delta y}$$

(1) let $\Delta z \rightarrow 0$, as first $\Delta y \rightarrow 0$ then $\Delta x \rightarrow 0$.
i.e. (along real axis)

$$f'(z) = \lim_{\Delta x \rightarrow 0} \left(\frac{\Delta u}{\Delta x} + i \frac{\Delta v}{\Delta x} \right)$$

$$f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \quad \text{--- (1)}$$

(2) let $\Delta z \rightarrow 0$ along y axis (i.e. first $\Delta x \rightarrow 0$ then $\Delta y \rightarrow 0$)

then

$$f'(z) = \lim_{\Delta y \rightarrow 0} \frac{\Delta u}{i \Delta y} + \frac{i \Delta v}{i \Delta y}$$

$$= \lim_{\Delta y \rightarrow 0} \frac{\Delta v}{\Delta y} - i \lim_{\Delta y \rightarrow 0} \frac{\Delta u}{\Delta y}$$

$$f'(z) = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y} \quad \text{--- (2)}$$

but $f(z)$ is analytic i.e. differentiable at all point R
i.e. $f'(z)$ exist and unique

from (1) and (2)

$$f'(z) = u_x + i v_x = v_y - i u_y$$

comparing

$$\begin{aligned} u_x &= v_y \\ u_y &= -v_x \end{aligned}$$

C-R equations

Sufficient conditions:-

Theorem:- The sufficient condⁿ for $f(z)$ to be analytic in Region R are

(i) $u_x = v_y, u_y = -v_x$ ~~ie~~ ~~ie~~

(ii) u_x, u_y, v_x, v_y are continuous function in region R .

ie If a function $f(z)$ satisfy C-R eqⁿ in R and u_x, u_y, v_x, v_y are continuous then $f(z)$ is analytic in R .

Note:-

1) If a function $f(z) = u + iv$ is analytic in a domain D , then u and v satisfy C-R equations at all point in D .

2) C-R eqⁿ are necessary but not sufficient for analytic function.

3) C-R eqⁿ are sufficient if u_x, v_x, u_y, v_y are continuous.

(1) Determine whether $\frac{1}{z}$ is analytic or not.

Soln:- $f(z) = \frac{1}{z} = \frac{1}{x+iy}$

$$f(z) = \frac{x-iy}{(x+iy)(x-iy)}$$

$$f(z) = \frac{x-iy}{x^2+y^2} = u+iv$$

$$u = \frac{x}{x^2+y^2}, \quad v = \frac{-y}{x^2+y^2}$$

* function will be analytic if it satisfy C-R eqⁿ and has continuous partial derivatives u_x, u_y, v_x, v_y .

$$u_x = \frac{y^2 - x^2}{(x^2+y^2)^2}, \quad u_y = \frac{-2xy}{(x^2+y^2)^2}$$

$$v_x = \frac{2xy}{(x^2+y^2)^2}, \quad v_y = \frac{y^2 - x^2}{(x^2+y^2)^2}$$

thus C-R eqⁿs are satisfied. we see that partial derivatives are continuous everywhere except at $(0,0)$.

therefor $f(z)$ is analytic everywhere except $(0,0)$.

Also ~~from~~ $f'(z) = -\frac{1}{z^2}$ we can see that $f'(z)$ ~~is~~ exists everywhere except at $z=0$. Hence $f(z)$ is analytic everywhere except at 0 .

Q:- Discuss the analyticity of $z\bar{z}$.

Solⁿ $f(z) = z\bar{z} = (x+iy)(x-iy)$

$$f(z) = x^2 + y^2 = u + i^0v$$

$$u = x^2 + y^2, \quad v = 0$$

$$\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial u}{\partial y} = 2y, \quad \frac{\partial v}{\partial x} = 0, \quad \frac{\partial v}{\partial y} = 0$$

C-R eqⁿs are

$$u_x = v_y$$

$$u_y = -v_x$$

$$2x = 0$$

$$2y = 0$$

C-R eqⁿs are satisfied at $(0,0)$ only.

i.e. function is not analytic anywhere.
check derivative at $(0,0)$
at $(0,0)$

$$f'(0) = \lim_{z \rightarrow 0} \frac{f(z) - f(0)}{z}$$

$$f'(0) = \lim_{z \rightarrow 0} \frac{x^2 + y^2}{x + iy} = \lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} \frac{x^2 + y^2}{x + iy}$$

Now let $x, y \rightarrow 0$ along $y = mx$
i.e.

$$f'(0) = \lim_{x \rightarrow 0} \frac{x^2 + m^2 x^2}{x + imx} = \lim_{x \rightarrow 0} \frac{x(1+m^2)}{1+im}$$

which is independent⁼⁰ of m .

i.e. ~~lim~~ i.e. limit exist and independent of path. $z \rightarrow 0$ therefore ~~f(x)~~ function is differentiable at $z=0$ & $f'(0)=0$.

Q:- Show that the function $f(z) = u + iv$ where

$$f(z) = \begin{cases} \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$

satisfies C-R eqⁿ at $z=0$. Is the function differentiable at $z=0$.

Solⁿ:- need to check $\left. \begin{matrix} u_x = v_y \\ u_y = -v_x \end{matrix} \right\}$ at $z=0$

$$u = \frac{x^3 - y^3}{x^2 + y^2}, \quad v = \frac{x^3 + y^3}{x^2 + y^2}$$

$$\frac{\partial u}{\partial x} = \lim_{x \rightarrow 0} \frac{u(x,0) - u(0,0)}{x} = \lim_{x \rightarrow 0} \frac{\frac{x^3}{x^2} - 0}{x}$$

$$\frac{\partial u}{\partial x} = 1$$

$$\frac{\partial u}{\partial y} = \lim_{y \rightarrow 0} \frac{u(0,y) - u(0,0)}{y} = \lim_{y \rightarrow 0} \frac{\frac{-y^3}{y^2} - 0}{y} = -1$$

$$\frac{\partial v}{\partial x} = \lim_{x \rightarrow 0} \frac{v(x,0) - v(0,0)}{x} = 1$$

$$\frac{\partial v}{\partial y} = \lim_{y \rightarrow 0} \frac{v(0,y) - v(0,0)}{y} = 1$$

we see that $\left. \begin{matrix} u_x = v_y \\ u_y = -v_x \end{matrix} \right\}$ i.e. C-R eqⁿ are satisfied at $z=0$.

Now

$$f'(0) = \lim_{z \rightarrow 0} \frac{f(z) - f(0)}{z - 0}$$

$$= \lim_{z \rightarrow 0} \frac{x^3 - y^3 + i(x^3 + y^3)}{(x^2 + y^2)(x + iy)}$$

$$f'(0) = \lim_{\substack{x \rightarrow 0, \\ y \rightarrow 0}} \frac{(x^3 - y^3) + i(x^3 + y^3)}{(x^2 + y^2)(x + iy)}$$

case Ist

Let $z \rightarrow 0$ along $y = mx$

$$f'(0) = \lim_{x \rightarrow 0} \frac{x^3 - m^3 x^3 + i(x^3 + m^3 x^3)}{(x^2 + m^2 x^2)(x + imx)}$$

$$f'(0) = \frac{1}{x} \frac{(1 - m^3) + i(1 + m^3)}{(1 + m^2)(1 + im)} = \frac{x^3(1 - m^3) + i x^3(1 + m^3)}{x^3(1 + m^2)(1 + im)}$$

$$= \frac{(1 - m^3) + i(1 + m^3)}{(1 + m^2)(1 + im)} = \frac{i(1 + m^3)}{(1 + m^2)(1 + im)}$$

depends on m

case IInd:-

let $z \rightarrow 0$ along x axis i.e. first $y \rightarrow 0$ then $x \rightarrow 0$

$$f'(0) = \lim_{x \rightarrow 0} \frac{x^3 + i x^3}{x^3} = (1 + i)$$

optional

by taking two different paths we saw that limit is not unique i.e. function is not differentiable at $z = 0$.

Date _____
Page _____

Q:- Show that the function $e^x(\cos y + i \sin y)$ is an analytic function, find its derivative

Hint:- show $u_x = v_y$ & $u_y = -v_x$ and u_x, v_y, u_y, v_x are continuous.
find $f'(z) = u_x + i v_x$ or $v_y + i u_y$

Q:- Show that $\bar{z}$ is not analytic anywhere.

Q. show that $f(z) = xy + iy$ is everywhere continuous but is nowhere analytic.

Q. find a, b, c if $f(z) = x + ay + i(bx + cy)$ is analytic. Hence find $f'(z)$.

Q. check the analyticity of function $f(z) = |z|^2$

Q. Determine which of following functions are analytic

(1) $\frac{x - iy}{x^2 + y^2}$ (2) $2xy + i(x^2 - y^2)$ (3) $x^2 + iy^2$

Ans:- (NO)

(NO)

(at all pts except $y=x$)

4) $xy + iy^2$

Q. show that $f(z) = \sqrt{|xy|}$ is not differentiable at origin even though eqn are satisfied at origin.

C-R eq's in polar form:- $f(z) = u(r, \theta) + i v(r, \theta)$
 $w = u + i v$

$$\left. \begin{aligned} u_r &= \frac{1}{r} v_\theta \\ u_\theta &= -r v_r \end{aligned} \right\} \text{ is polar form of C-R equations.}$$

derivative in polar coordinates:-

$$\frac{dw}{dz} = f'(z) = (\cos \theta - i \sin \theta) \frac{\partial w}{\partial r} \quad \text{or} \quad f'(z) = -\frac{i}{r} (\cos \theta - i \sin \theta) \frac{\partial w}{\partial \theta}$$

ORTHOGONAL SYSTEM:-

Every analytic function $f(z) = u + i v$ defines two families of curves $u(x, y) = c_1$ and $v(x, y) = c_2$, which form an orthogonal system.

consider

$$u(x, y) = c_1 \quad \text{--- (i)}$$

$$v(x, y) = c_2 \quad \text{--- (ii)}$$

differentiating (i) with r. to x:-

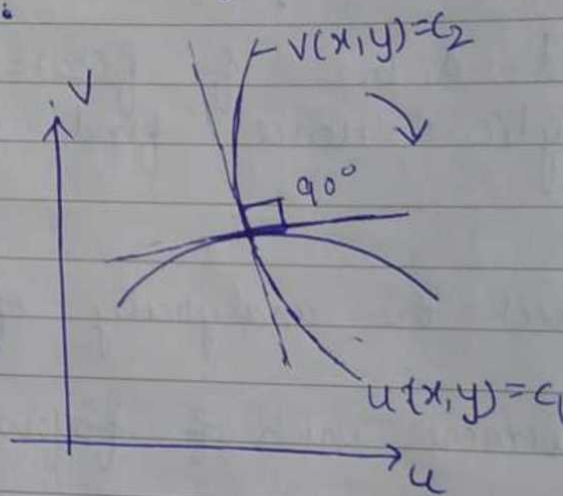
$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = - \frac{\left(\frac{\partial u}{\partial x} \right)}{\left(\frac{\partial u}{\partial y} \right)} = m_1 \text{ (say)}$$

diff. (2) w. r. to x

we get

$$\frac{dy}{dx} = - \frac{\left(\frac{\partial v}{\partial x} \right)}{\left(\frac{\partial v}{\partial y} \right)} = m_2 \text{ (say)}$$



$f(z)$ is analytic therefor $u_x = v_y, u_y = -v_x$

$$m_1 \cdot m_2 = \frac{v_y \cdot v_x}{-v_x v_y}$$

$$m_1, m_2 = -1$$

i.e. curves intersect at right angle i.e. they form orthogonal system.

$f(z) = z^n$ is analytic in $\mathbb{C}$.

(2) $f(z) = e^z$ is " " " " " "

(2) $f(z) = e^z$ is " " " " " "

(3) $f(z) = 1/z$ is analytic everywhere except $z=0$.

(4) Polynomial rational fun. $f(z) = \frac{p(z)}{q(z)}$ - where

P and Q are ~~analytic~~ polynomials, ~~are~~ is analytic everywhere in $\mathbb{C}$ except the points where $Q(z) = 0$.

(5) If $f(z)$ and $g(z)$ are analytic then $f(z) \pm g(z)$, $f(z) \cdot g(z)$, $\frac{f(z)}{g(z)}$ (except where $g(z)=0$), all are analytic functions of z .

Q:- find P such that $f(z) = r^2 \cos(2\theta) + ir^2 \sin(2\theta)$ is analytic.

Ans ()

Q:- Determine P such that $f(z) = \frac{1}{2} \log(x^2 + y^2) + i \tan^{-1}\left(\frac{y}{x}\right)$ be analytic function.

Ans ()

Q:- Show that real and imaginary part of $w = \log(z)$ satisfy C-R equations, when z is not zero. find derivative.

Hint:-

$$f(z) = \log(z)$$

$$\text{let } z = r \cos \theta + ir \sin \theta$$

$$f(z) = \log(r(\cos \theta + i \sin \theta))$$

$$= \log r + \log e^{i\theta} = \log r + i\theta$$

$$u + iv = \log r + i\theta$$

$$u = \log r = \log \sqrt{x^2 + y^2}, \quad v = \theta = \tan^{-1}(y/x)$$

Q: It is given that $f(z)$ and its conjugate $\overline{f(z)}$ are both analytic. Determine the function $f(z)$.

soln $f(z) = u + iv, \quad \overline{f(z)} = u - iv$

$f(z)$ is analytic i.e. $\left. \begin{array}{l} u_x = v_y \\ v_x = -u_y \end{array} \right\} \text{--- (1)}$ $\left| \right.$ $\overline{f(z)}$ is also analytic

$\left. \begin{array}{l} u_x = -v_y \\ u_y = v_x \end{array} \right\} \text{--- (2)}$

from (1) & (2) $u_x = u_y = v_x = v_y = 0$

i.e. $\left. \begin{array}{l} u_x = 0 \Rightarrow u = \text{constant} \\ v_x = 0 \Rightarrow v = \text{constant} \end{array} \right\} \Rightarrow f(z) = u + iv \text{ is Constant function.}$