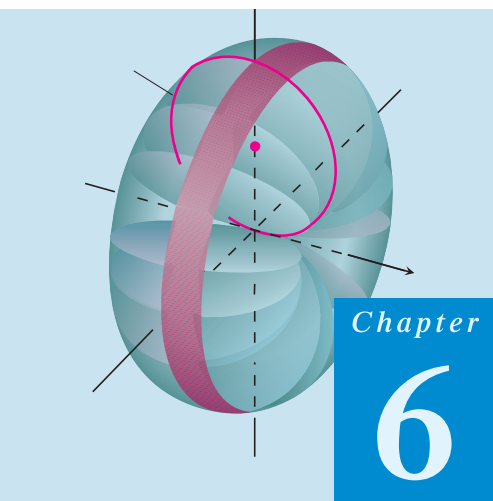




APPLICATIONS OF DEFINITE INTEGRALS

OVERVIEW In Chapter 5 we discovered the connection between Riemann sums

$$S_P = \sum_{k=1}^n f(c_k) \Delta x_k$$

associated with a partition P of the finite closed interval $[a, b]$ and the process of integration. We found that for a continuous function f on $[a, b]$, the limit of S_P as the norm of the partition $\|P\|$ approaches zero is the number

$$\int_a^b f(x) dx = F(b) - F(a)$$

where F is any antiderivative of f . We applied this to the problems of computing the area between the x -axis and the graph of $y = f(x)$ for $a \leq x \leq b$, and to finding the area between two curves.

In this chapter we extend the applications to finding volumes, lengths of plane curves, centers of mass, areas of surfaces of revolution, work, and fluid forces against planar walls. We define all these as limits of Riemann sums of continuous functions on closed intervals—that is, as definite integrals which can be evaluated using the Fundamental Theorem of Calculus.

6.1

Volumes by Slicing and Rotation About an Axis

In this section we define volumes of solids whose cross-sections are plane regions. A **cross-section** of a solid S is the plane region formed by intersecting S with a plane (Figure 6.1).

Suppose we want to find the volume of a solid S like the one in Figure 6.1. We begin by extending the definition of a cylinder from classical geometry to cylindrical solids with arbitrary bases (Figure 6.2). If the cylindrical solid has a known base area A and height h , then the volume of the cylindrical solid is

$$\text{Volume} = \text{area} \times \text{height} = A \cdot h.$$

This equation forms the basis for defining the volumes of many solids that are not cylindrical by the *method of slicing*.

If the cross-section of the solid S at each point x in the interval $[a, b]$ is a region $R(x)$ of area $A(x)$, and A is a continuous function of x , we can define and calculate the volume of the solid S as a definite integral in the following way.

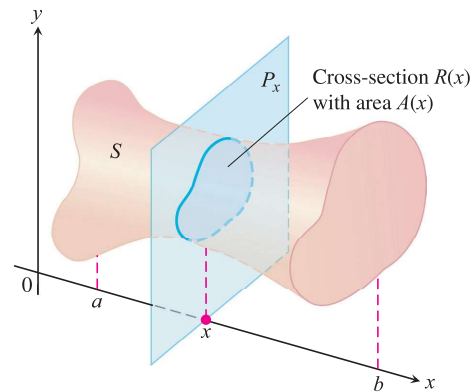


FIGURE 6.1 A cross-section of the solid S formed by intersecting S with a plane P_x perpendicular to the x -axis through the point x in the interval $[a, b]$.

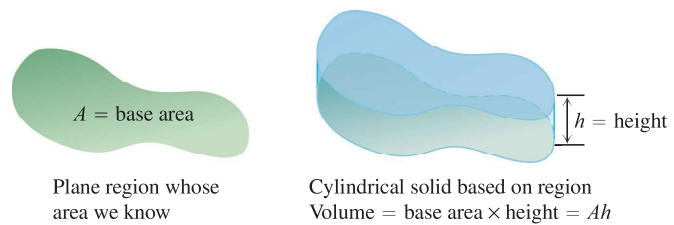


FIGURE 6.2 The volume of a cylindrical solid is always defined to be its base area times its height.

We partition $[a, b]$ into subintervals of width (length) Δx_k and slice the solid, as we would a loaf of bread, by planes perpendicular to the x -axis at the partition points $a = x_0 < x_1 < \cdots < x_n = b$. The planes P_{x_k} , perpendicular to the x -axis at the partition points, slice S into thin “slabs” (like thin slices of a loaf of bread). A typical slab is shown in Figure 6.3. We approximate the slab between the plane at x_{k-1} and the plane at x_k by a cylindrical solid with base area $A(x_k)$ and height $\Delta x_k = x_k - x_{k-1}$ (Figure 6.4). The volume V_k of this cylindrical solid is $A(x_k) \cdot \Delta x_k$, which is approximately the same volume as that of the slab:

$$\text{Volume of the } k\text{th slab} \approx V_k = A(x_k) \Delta x_k.$$

The volume V of the entire solid S is therefore approximated by the sum of these cylindrical volumes,

$$V \approx \sum_{k=1}^n V_k = \sum_{k=1}^n A(x_k) \Delta x_k.$$

This is a Riemann sum for the function $A(x)$ on $[a, b]$. We expect the approximations from these sums to improve as the norm of the partition of $[a, b]$ goes to zero, so we define their limiting definite integral to be the volume of the solid S .

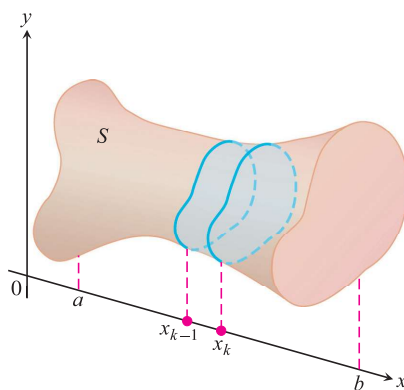


FIGURE 6.3 A typical thin slab in the solid S .

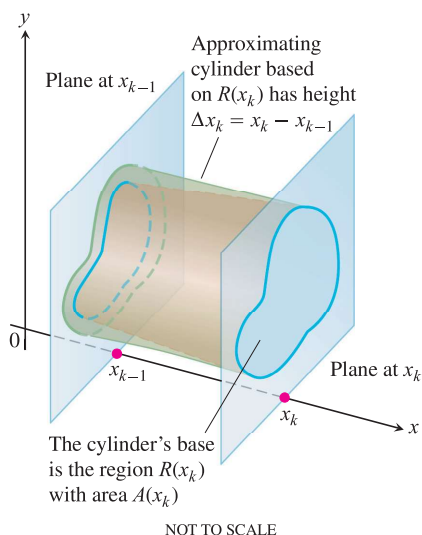


FIGURE 6.4 The solid thin slab in Figure 6.3 is approximated by the cylindrical solid with base $R(x_k)$ having area $A(x_k)$ and height $\Delta x_k = x_k - x_{k-1}$.

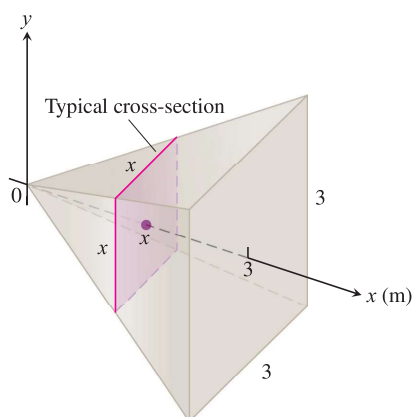


FIGURE 6.5 The cross-sections of the pyramid in Example 1 are squares.

DEFINITION Volume

The **volume** of a solid of known integrable cross-sectional area $A(x)$ from $x = a$ to $x = b$ is the integral of A from a to b ,

$$V = \int_a^b A(x) \, dx.$$

This definition applies whenever $A(x)$ is continuous, or more generally, when it is integrable. To apply the formula in the definition to calculate the volume of a solid, take the following steps:

Calculating the Volume of a Solid

1. Sketch the solid and a typical cross-section.
2. Find a formula for $A(x)$, the area of a typical cross-section.
3. Find the limits of integration.
4. Integrate $A(x)$ using the Fundamental Theorem.

EXAMPLE 1 Volume of a Pyramid

A pyramid 3 m high has a square base that is 3 m on a side. The cross-section of the pyramid perpendicular to the altitude x m down from the vertex is a square x m on a side. Find the volume of the pyramid.

Solution

1. *A sketch.* We draw the pyramid with its altitude along the x -axis and its vertex at the origin and include a typical cross-section (Figure 6.5).
2. *A formula for $A(x)$.* The cross-section at x is a square x meters on a side, so its area is

$$A(x) = x^2.$$
3. *The limits of integration.* The squares lie on the planes from $x = 0$ to $x = 3$.
4. *Integrate to find the volume.*

$$V = \int_0^3 A(x) \, dx = \int_0^3 x^2 \, dx = \left. \frac{x^3}{3} \right|_0^3 = 9 \, \text{m}^3$$

EXAMPLE 2 Cavalieri's Principle

Cavalieri's principle says that solids with equal altitudes and identical cross-sectional areas at each height have the same volume (Figure 6.6). This follows immediately from the definition of volume, because the cross-sectional area function $A(x)$ and the interval $[a, b]$ are the same for both solids.

HISTORICAL BIOGRAPHY

Bonaventura Cavalieri
(1598–1647)

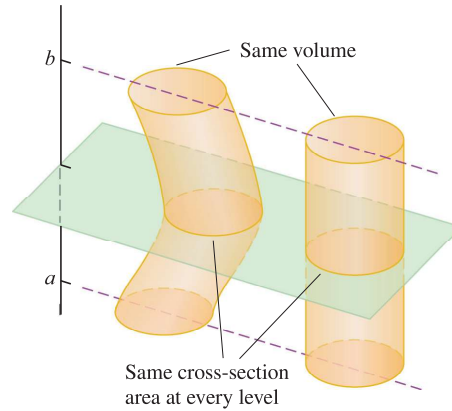


FIGURE 6.6 Cavalieri's Principle: These solids have the same volume, which can be illustrated with stacks of coins (Example 2).

EXAMPLE 3 Volume of a Wedge

A curved wedge is cut from a cylinder of radius 3 by two planes. One plane is perpendicular to the axis of the cylinder. The second plane crosses the first plane at a 45° angle at the center of the cylinder. Find the volume of the wedge.

Solution We draw the wedge and sketch a typical cross-section perpendicular to the x -axis (Figure 6.7). The cross-section at x is a rectangle of area

$$\begin{aligned} A(x) &= (\text{height})(\text{width}) = (x)(2\sqrt{9-x^2}) \\ &= 2x\sqrt{9-x^2}. \end{aligned}$$

The rectangles run from $x = 0$ to $x = 3$, so we have

$$\begin{aligned} V &= \int_a^b A(x) \, dx = \int_0^3 2x\sqrt{9-x^2} \, dx \\ &= -\frac{2}{3}(9-x^2)^{3/2} \Big|_0^3 \\ &= 0 + \frac{2}{3}(9)^{3/2} \\ &= 18. \end{aligned}$$

Let $u = 9 - x^2$,
 $du = -2x \, dx$, integrate,
and substitute back.

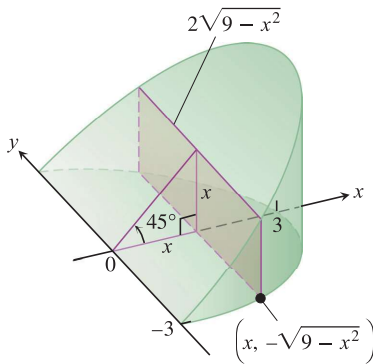


FIGURE 6.7 The wedge of Example 3, sliced perpendicular to the x -axis. The cross-sections are rectangles.

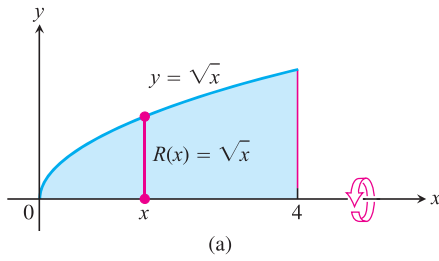
Solids of Revolution: The Disk Method

The solid generated by rotating a plane region about an axis in its plane is called a **solid of revolution**. To find the volume of a solid like the one shown in Figure 6.8, we need only observe that the cross-sectional area $A(x)$ is the area of a disk of radius $R(x)$, the distance of the planar region's boundary from the axis of revolution. The area is then

$$A(x) = \pi(\text{radius})^2 = \pi[R(x)]^2.$$

So the definition of volume gives

$$V = \int_a^b A(x) \, dx = \int_a^b \pi[R(x)]^2 \, dx.$$



This method for calculating the volume of a solid of revolution is often called the **disk method** because a cross-section is a circular disk of radius $R(x)$.

EXAMPLE 4 A Solid of Revolution (Rotation About the x -Axis)

The region between the curve $y = \sqrt{x}$, $0 \leq x \leq 4$, and the x -axis is revolved about the x -axis to generate a solid. Find its volume.

Solution We draw figures showing the region, a typical radius, and the generated solid (Figure 6.8). The volume is

$$\begin{aligned} V &= \int_a^b \pi [R(x)]^2 dx \\ &= \int_0^4 \pi \boxed{\phantom{R(x) = \sqrt{x}}} \quad R(x) = \sqrt{x} \\ &= \pi \int_0^4 x dx = \pi \left[\frac{x^2}{2} \right]_0^4 = \pi \frac{(4)^2}{2} = 8\pi. \end{aligned}$$

EXAMPLE 5 Volume of a Sphere

The circle

$$x^2 + y^2 = a^2$$

is rotated about the x -axis to generate a sphere. Find its volume.

Solution We imagine the sphere cut into thin slices by planes perpendicular to the x -axis (Figure 6.9). The cross-sectional area at a typical point x between $-a$ and a is

$$A(x) = \pi y^2 = \pi(a^2 - x^2).$$

Therefore, the volume is

$$V = \int_{-a}^a A(x) dx = \int_{-a}^a \pi(a^2 - x^2) dx = \pi \left[a^2x - \frac{x^3}{3} \right]_{-a}^a = \frac{4}{3} \pi a^3.$$

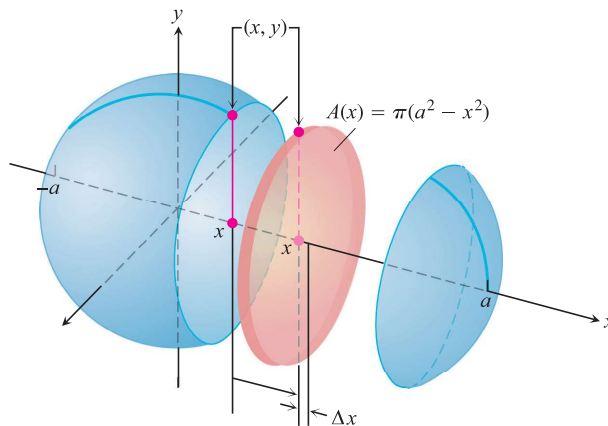


FIGURE 6.9 The sphere generated by rotating the circle $x^2 + y^2 = a^2$ about the x -axis. The radius is $R(x) = y = \sqrt{a^2 - x^2}$ (Example 5).

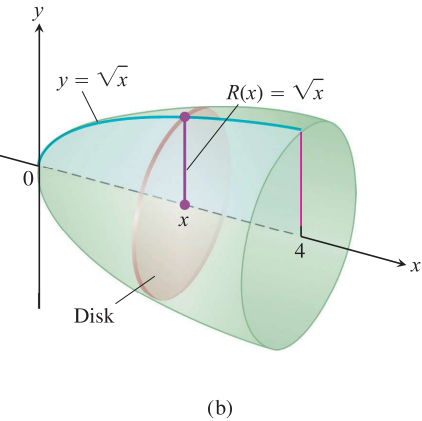


FIGURE 6.8 The region (a) and solid of revolution (b) in Example 4.

The axis of revolution in the next example is not the x -axis, but the rule for calculating the volume is the same: Integrate $\pi(\text{radius})^2$ between appropriate limits.

EXAMPLE 6 A Solid of Revolution (Rotation About the Line $y = 1$)

Find the volume of the solid generated by revolving the region bounded by $y = \sqrt{x}$ and the lines $y = 1$, $x = 4$ about the line $y = 1$.

Solution We draw figures showing the region, a typical radius, and the generated solid (Figure 6.10). The volume is

$$\begin{aligned} V &= \int_1^4 \pi [R(x)]^2 dx \\ &= \int_1^4 \pi \text{ [radius]}^2 dx \\ &= \pi \int_1^4 \text{ [radius]}^2 dx \\ &= \pi \left[\frac{x^2}{2} - 2 \cdot \frac{2}{3} x^{3/2} + x \right]_1^4 = \frac{7\pi}{6}. \end{aligned}$$

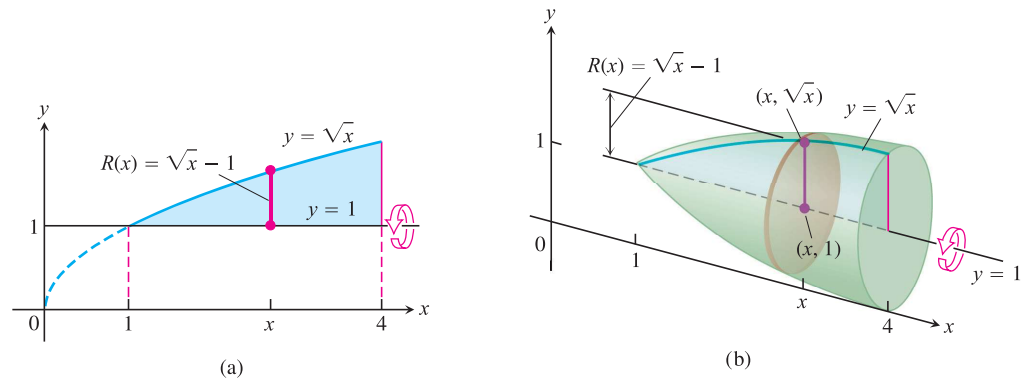


FIGURE 6.10 The region (a) and solid of revolution (b) in Example 6.

To find the volume of a solid generated by revolving a region between the y -axis and a curve $x = R(y)$, $c \leq y \leq d$, about the y -axis, we use the same method with x replaced by y . In this case, the circular cross-section is

$$A(y) = \pi[\text{radius}]^2 = \pi[R(y)]^2.$$

EXAMPLE 7 Rotation About the y -Axis

Find the volume of the solid generated by revolving the region between the y -axis and the curve $x = 2/y$, $1 \leq y \leq 4$, about the y -axis.

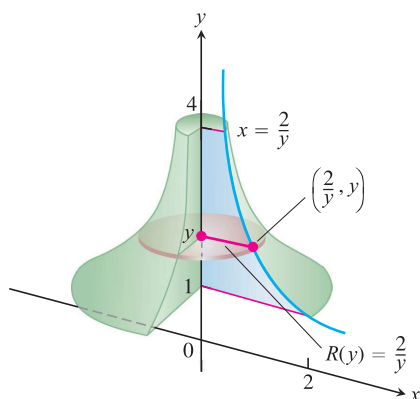
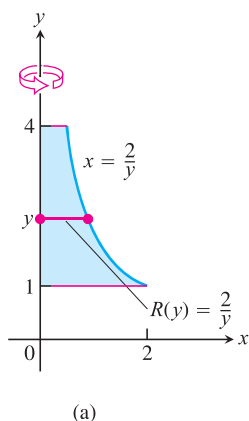


FIGURE 6.11 The region (a) and part of the solid of revolution (b) in Example 7.

Solution We draw figures showing the region, a typical radius, and the generated solid (Figure 6.11). The volume is

$$\begin{aligned} V &= \int_1^4 \pi [R(y)]^2 dy \\ &= \int_1^4 \pi \left(\frac{2}{y}\right)^2 dy \\ &= \pi \int_1^4 \frac{4}{y^2} dy = 4\pi \left[-\frac{1}{y}\right]_1^4 = 4\pi \left[\frac{3}{4}\right] \\ &= 3\pi. \end{aligned}$$

EXAMPLE 8 Rotation About a Vertical Axis

Find the volume of the solid generated by revolving the region between the parabola $x = y^2 + 1$ and the line $x = 3$ about the line $x = 3$.

Solution We draw figures showing the region, a typical radius, and the generated solid (Figure 6.12). Note that the cross-sections are perpendicular to the line $x = 3$. The volume is

$$\begin{aligned} V &= \int_{-\sqrt{2}}^{\sqrt{2}} \pi [R(y)]^2 dy \\ &= \int_{-\sqrt{2}}^{\sqrt{2}} \pi [2 - y^2]^2 dy \quad \begin{array}{l} R(y) = 3 - (y^2 + 1) \\ \quad = 2 - y^2 \end{array} \\ &= \pi \int_{-\sqrt{2}}^{\sqrt{2}} [4 - 4y^2 + y^4] dy \\ &= \pi \left[4y - \frac{4}{3}y^3 + \frac{y^5}{5} \right]_{-\sqrt{2}}^{\sqrt{2}} \\ &= \frac{64\pi\sqrt{2}}{15}. \end{aligned}$$

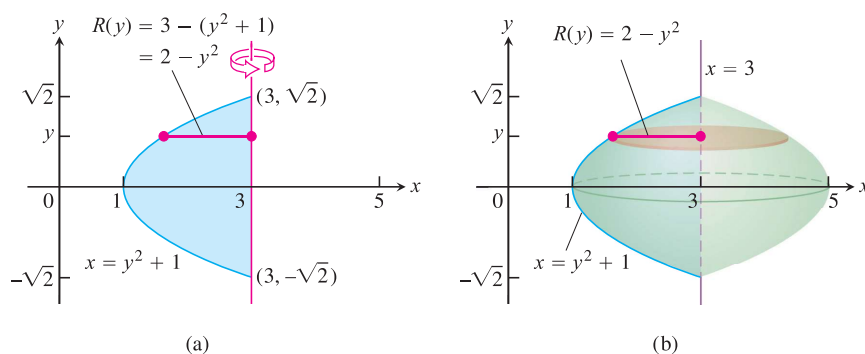


FIGURE 6.12 The region (a) and solid of revolution (b) in Example 8.

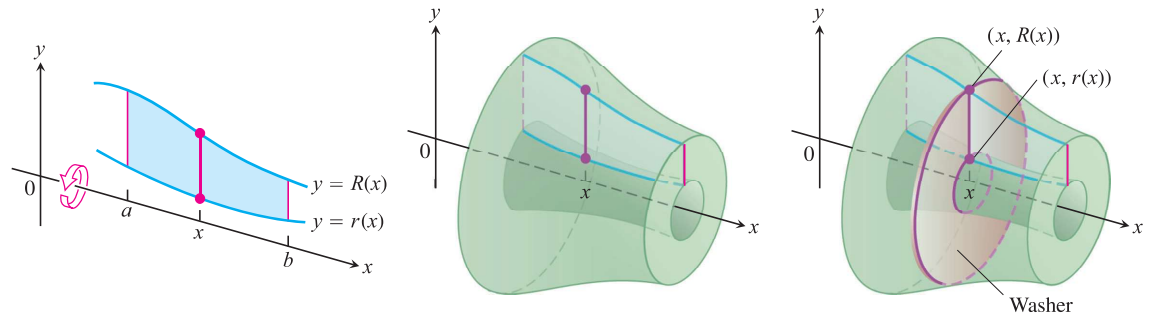


FIGURE 6.13 The cross-sections of the solid of revolution generated here are washers, not disks, so the integral $\int_a^b A(x) dx$ leads to a slightly different formula.

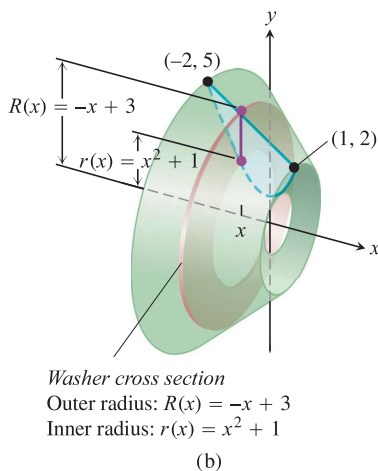
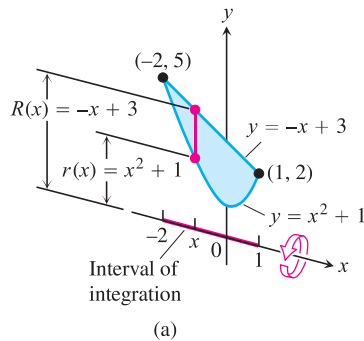


FIGURE 6.14 (a) The region in Example 9 spanned by a line segment perpendicular to the axis of revolution. (b) When the region is revolved about the x -axis, the line segment generates a washer.

Solids of Revolution: The Washer Method

If the region we revolve to generate a solid does not border on or cross the axis of revolution, the solid has a hole in it (Figure 6.13). The cross-sections perpendicular to the axis of revolution are washers (the purplish circular surface in Figure 6.13) instead of disks. The dimensions of a typical washer are

$$\text{Outer radius: } R(x)$$

$$\text{Inner radius: } r(x)$$

The washer's area is

$$A(x) = \pi[R(x)]^2 - \pi[r(x)]^2 = \pi([R(x)]^2 - [r(x)]^2).$$

Consequently, the definition of volume gives

$$V = \int_a^b A(x) dx = \int_a^b \pi([R(x)]^2 - [r(x)]^2) dx.$$

This method for calculating the volume of a solid of revolution is called the **washer method** because a slab is a circular washer of outer radius $R(x)$ and inner radius $r(x)$.

EXAMPLE 9 A Washer Cross-Section (Rotation About the x -Axis)

The region bounded by the curve $y = x^2 + 1$ and the line $y = -x + 3$ is revolved about the x -axis to generate a solid. Find the volume of the solid.

Solution

1. Draw the region and sketch a line segment across it perpendicular to the axis of revolution (the red segment in Figure 6.14).
2. Find the outer and inner radii of the washer that would be swept out by the line segment if it were revolved about the x -axis along with the region.

These radii are the distances of the ends of the line segment from the axis of revolution (Figure 6.14).

$$\text{Outer radius: } R(x) = -x + 3$$

$$\text{Inner radius: } r(x) = x^2 + 1$$

3. Find the limits of integration by finding the x -coordinates of the intersection points of the curve and line in Figure 6.14a.

$$x^2 + 1 = -x + 3$$

$$x^2 + x - 2 = 0$$

$$(x + 2)(x - 1) = 0$$

$$x = -2, \quad x = 1$$

4. Evaluate the volume integral.

$$\begin{aligned} V &= \int_a^b \pi([R(x)]^2 - [r(x)]^2) dx \\ &= \int_{-2}^1 \pi((-x + 3)^2 - (x^2 + 1)^2) dx && \text{Values from Steps 2 and 3} \\ &= \int_{-2}^1 \pi(8 - 6x - x^2 - x^4) dx \\ &= \pi \left[8x - 3x^2 - \frac{x^3}{3} - \frac{x^5}{5} \right]_{-2}^1 = \frac{117\pi}{5} \end{aligned}$$

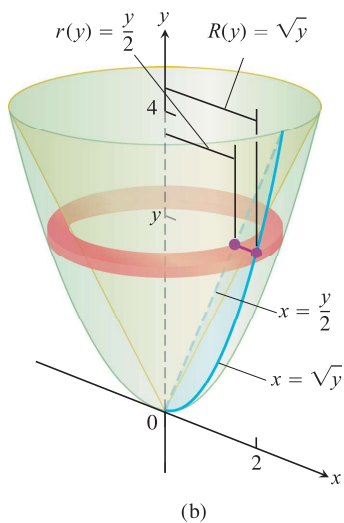
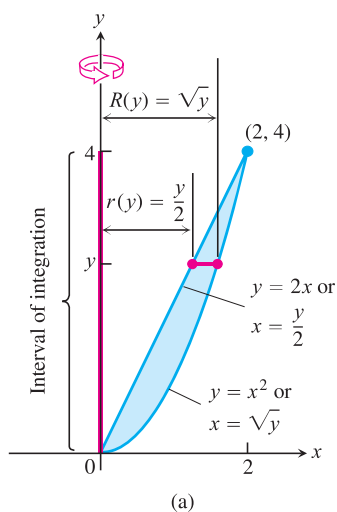


FIGURE 6.15 (a) The region being rotated about the y -axis, the washer radii, and limits of integration in Example 10. (b) The washer swept out by the line segment in part (a).

To find the volume of a solid formed by revolving a region about the y -axis, we use the same procedure as in Example 9, but integrate with respect to y instead of x . In this situation the line segment sweeping out a typical washer is perpendicular to the y -axis (the axis of revolution), and the outer and inner radii of the washer are functions of y .

EXAMPLE 10 A Washer Cross-Section (Rotation About the y -Axis)

The region bounded by the parabola $y = x^2$ and the line $y = 2x$ in the first quadrant is revolved about the y -axis to generate a solid. Find the volume of the solid.

Solution First we sketch the region and draw a line segment across it perpendicular to the axis of revolution (the y -axis). See Figure 6.15a.

The radii of the washer swept out by the line segment are $R(y) = \sqrt{y}$, $r(y) = y/2$ (Figure 6.15).

The line and parabola intersect at $y = 0$ and $y = 4$, so the limits of integration are $c = 0$ and $d = 4$. We integrate to find the volume:

$$\begin{aligned} V &= \int_c^d \pi([R(y)]^2 - [r(y)]^2) dy \\ &= \int_0^4 \pi \left(\left[\sqrt{y} \right]^2 - \left[\frac{y}{2} \right]^2 \right) dy \\ &= \pi \int_0^4 \left(y - \frac{y^2}{4} \right) dy = \pi \left[\frac{y^2}{2} - \frac{y^3}{12} \right]_0^4 = \frac{8}{3} \pi. \end{aligned}$$

Summary

In all of our volume examples, no matter how the cross-sectional area $A(x)$ of a typical slab is determined, the definition of volume as the definite integral $V = \int_a^b A(x) \, dx$ is the heart of the calculations we made.

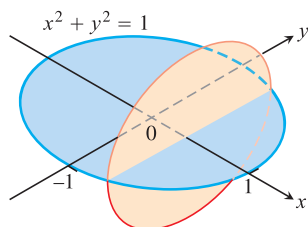
EXERCISES 6.1

Cross-Sectional Areas

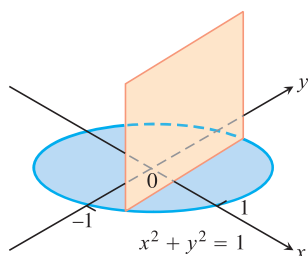
In Exercises 1 and 2, find a formula for the area $A(x)$ of the cross-sections of the solid perpendicular to the x -axis.

1. The solid lies between planes perpendicular to the x -axis at $x = -1$ and $x = 1$. In each case, the cross-sections perpendicular to the x -axis between these planes run from the semicircle $y = -\sqrt{1 - x^2}$ to the semicircle $y = \sqrt{1 - x^2}$.

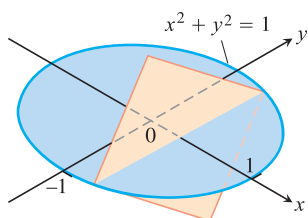
- a. The cross-sections are circular disks with diameters in the xy -plane.



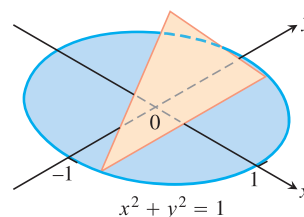
- b. The cross-sections are squares with bases in the xy -plane.



- c. The cross-sections are squares with diagonals in the xy -plane. (The length of a square's diagonal is $\sqrt{2}$ times the length of its sides.)

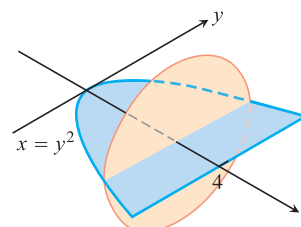


- d. The cross-sections are equilateral triangles with bases in the xy -plane.

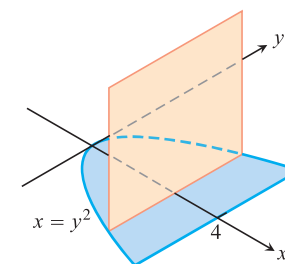


2. The solid lies between planes perpendicular to the x -axis at $x = 0$ and $x = 4$. The cross-sections perpendicular to the x -axis between these planes run from the parabola $y = -\sqrt{x}$ to the parabola $y = \sqrt{x}$.

- a. The cross-sections are circular disks with diameters in the xy -plane.



- b. The cross-sections are squares with bases in the xy -plane.



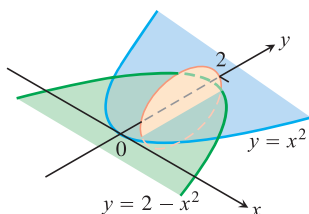
- c. The cross-sections are squares with diagonals in the xy -plane.
d. The cross-sections are equilateral triangles with bases in the xy -plane.

Volumes by Slicing

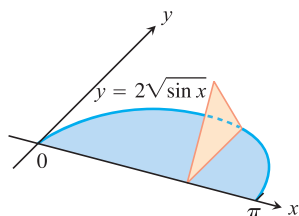
Find the volumes of the solids in Exercises 3–10.



3. The solid lies between planes perpendicular to the x -axis at $x = 0$ and $x = 4$. The cross-sections perpendicular to the axis on the interval $0 \leq x \leq 4$ are squares whose diagonals run from the parabola $y = -\sqrt{x}$ to the parabola $y = \sqrt{x}$.
4. The solid lies between planes perpendicular to the x -axis at $x = -1$ and $x = 1$. The cross-sections perpendicular to the x -axis are circular disks whose diameters run from the parabola $y = x^2$ to the parabola $y = 2 - x^2$.

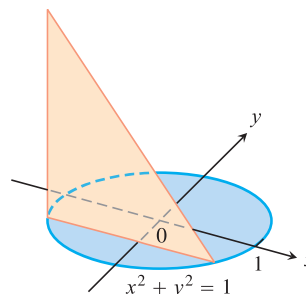


5. The solid lies between planes perpendicular to the x -axis at $x = -1$ and $x = 1$. The cross-sections perpendicular to the x -axis between these planes are squares whose bases run from the semicircle $y = -\sqrt{1 - x^2}$ to the semicircle $y = \sqrt{1 - x^2}$.
6. The solid lies between planes perpendicular to the x -axis at $x = -1$ and $x = 1$. The cross-sections perpendicular to the x -axis between these planes are squares whose diagonals run from the semicircle $y = -\sqrt{1 - x^2}$ to the semicircle $y = \sqrt{1 - x^2}$.
7. The base of a solid is the region between the curve $y = 2\sqrt{\sin x}$ and the interval $[0, \pi]$ on the x -axis. The cross-sections perpendicular to the x -axis are
 - a. equilateral triangles with bases running from the x -axis to the curve as shown in the figure.

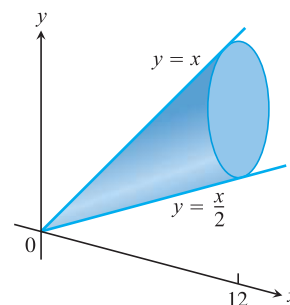


- b. squares with bases running from the x -axis to the curve.
8. The solid lies between planes perpendicular to the x -axis at $x = -\pi/3$ and $x = \pi/3$. The cross-sections perpendicular to the x -axis are
 - a. circular disks with diameters running from the curve $y = \tan x$ to the curve $y = \sec x$.
 - b. squares whose bases run from the curve $y = \tan x$ to the curve $y = \sec x$.
9. The solid lies between planes perpendicular to the y -axis at $y = 0$ and $y = 2$. The cross-sections perpendicular to the y -axis are circular disks with diameters running from the y -axis to the parabola $x = \sqrt{5}y^2$.

10. The base of the solid is the disk $x^2 + y^2 \leq 1$. The cross-sections by planes perpendicular to the y -axis between $y = -1$ and $y = 1$ are isosceles right triangles with one leg in the disk.



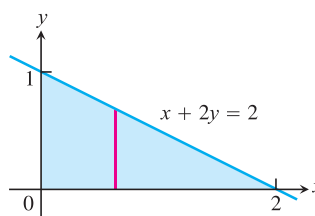
11. **A twisted solid** A square of side length s lies in a plane perpendicular to a line L . One vertex of the square lies on L . As this square moves a distance h along L , the square turns one revolution about L to generate a corkscrew-like column with square cross-sections.
 - a. Find the volume of the column.
 - b. What will the volume be if the square turns twice instead of once? Give reasons for your answer.
12. **Cavalieri's Principle** A solid lies between planes perpendicular to the x -axis at $x = 0$ and $x = 12$. The cross-sections by planes perpendicular to the x -axis are circular disks whose diameters run from the line $y = x/2$ to the line $y = x$ as shown in the accompanying figure. Explain why the solid has the same volume as a right circular cone with base radius 3 and height 12.



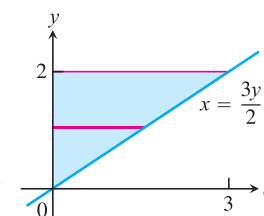
Volumes by the Disk Method

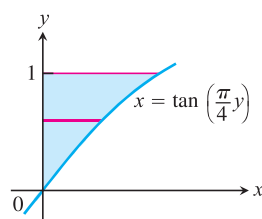
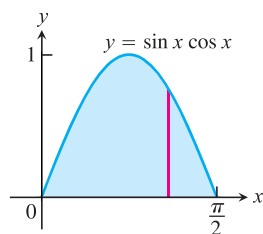
In Exercises 13–16, find the volume of the solid generated by revolving the shaded region about the given axis.

13. About the x -axis



14. About the y -axis



15. About the y -axis16. About the x -axis

Find the volumes of the solids generated by revolving the regions bounded by the lines and curves in Exercises 17–22 about the x -axis.

17. $y = x^2$, $y = 0$, $x = 2$ 18. $y = x^3$, $y = 0$, $x = 2$

19. $y = \sqrt{9 - x^2}$, $y = 0$ 20. $y = x - x^2$, $y = 0$

21. $y = \sqrt{\cos x}$, $0 \leq x \leq \pi/2$, $y = 0$, $x = 0$

22. $y = \sec x$, $y = 0$, $x = -\pi/4$, $x = \pi/4$

In Exercises 23 and 24, find the volume of the solid generated by revolving the region about the given line.

23. The region in the first quadrant bounded above by the line $y = \sqrt{2}$, below by the curve $y = \sec x \tan x$, and on the left by the y -axis, about the line $y = \sqrt{2}$

24. The region in the first quadrant bounded above by the line $y = 2$, below by the curve $y = 2 \sin x$, $0 \leq x \leq \pi/2$, and on the left by the y -axis, about the line $y = 2$

Find the volumes of the solids generated by revolving the regions bounded by the lines and curves in Exercises 25–30 about the y -axis.

25. The region enclosed by $x = \sqrt{5}y^2$, $x = 0$, $y = -1$, $y = 1$

26. The region enclosed by $x = y^{3/2}$, $x = 0$, $y = 2$

27. The region enclosed by $x = \sqrt{2 \sin 2y}$, $0 \leq y \leq \pi/2$, $x = 0$

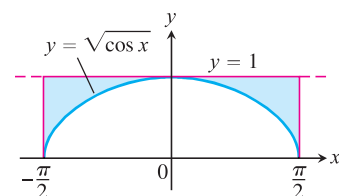
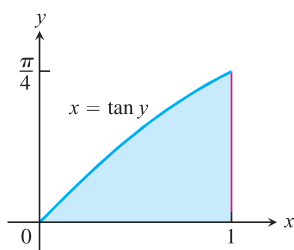
28. The region enclosed by $x = \sqrt{\cos(\pi y/4)}$, $-2 \leq y \leq 0$, $x = 0$

29. $x = 2/(y + 1)$, $x = 0$, $y = 0$, $y = 3$

30. $x = \sqrt{2y/(y^2 + 1)}$, $x = 0$, $y = 1$

Volumes by the Washer Method

Find the volumes of the solids generated by revolving the shaded regions in Exercises 31 and 32 about the indicated axes.

31. The x -axis32. The y -axis

Find the volumes of the solids generated by revolving the regions bounded by the lines and curves in Exercises 33–38 about the x -axis.

33. $y = x$, $y = 1$, $x = 0$ 34. $y = 2\sqrt{x}$, $y = 2$, $x = 0$

35. $y = x^2 + 1$, $y = x + 3$ 36. $y = 4 - x^2$, $y = 2 - x$

37. $y = \sec x$, $y = \sqrt{2}$, $-\pi/4 \leq x \leq \pi/4$

38. $y = \sec x$, $y = \tan x$, $x = 0$, $x = 1$

In Exercises 39–42, find the volume of the solid generated by revolving each region about the y -axis.

39. The region enclosed by the triangle with vertices $(1, 0)$, $(2, 1)$, and $(1, 1)$

40. The region enclosed by the triangle with vertices $(0, 1)$, $(1, 0)$, and $(1, 1)$

41. The region in the first quadrant bounded above by the parabola $y = x^2$, below by the x -axis, and on the right by the line $x = 2$

42. The region in the first quadrant bounded on the left by the circle $x^2 + y^2 = 3$, on the right by the line $x = \sqrt{3}$, and above by the line $y = \sqrt{3}$

In Exercises 43 and 44, find the volume of the solid generated by revolving each region about the given axis.

43. The region in the first quadrant bounded above by the curve $y = x^2$, below by the x -axis, and on the right by the line $x = 1$, about the line $x = -1$

44. The region in the second quadrant bounded above by the curve $y = -x^3$, below by the x -axis, and on the left by the line $x = -1$, about the line $x = -2$

Volumes of Solids of Revolution

45. Find the volume of the solid generated by revolving the region bounded by $y = \sqrt{x}$ and the lines $y = 2$ and $x = 0$ about

- a. the x -axis. b. the y -axis.
c. the line $y = 2$. d. the line $x = 4$.

46. Find the volume of the solid generated by revolving the triangular region bounded by the lines $y = 2x$, $y = 0$, and $x = 1$ about

- a. the line $x = 1$. b. the line $x = 2$.

47. Find the volume of the solid generated by revolving the region bounded by the parabola $y = x^2$ and the line $y = 1$ about

- a. the line $y = 1$. b. the line $y = 2$.
c. the line $y = -1$.

48. By integration, find the volume of the solid generated by revolving the triangular region with vertices $(0, 0)$, $(b, 0)$, $(0, h)$ about

- a. the x -axis. b. the y -axis.

Theory and Applications

49. **The volume of a torus** The disk $x^2 + y^2 \leq a^2$ is revolved about the line $x = b$ ($b > a$) to generate a solid shaped like a doughnut

and called a *torus*. Find its volume. (Hint: $\int_{-a}^a \sqrt{a^2 - y^2} dy = \pi a^2/2$, since it is the area of a semicircle of radius a .)

- 50. Volume of a bowl** A bowl has a shape that can be generated by revolving the graph of $y = x^2/2$ between $y = 0$ and $y = 5$ about the y -axis.

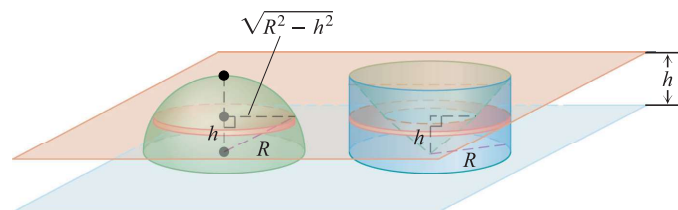
- Find the volume of the bowl.
- Related rates** If we fill the bowl with water at a constant rate of 3 cubic units per second, how fast will the water level in the bowl be rising when the water is 4 units deep?

51. Volume of a bowl

- A hemispherical bowl of radius a contains water to a depth h . Find the volume of water in the bowl.
- Related rates** Water runs into a sunken concrete hemispherical bowl of radius 5 m at the rate of $0.2 \text{ m}^3/\text{sec}$. How fast is the water level in the bowl rising when the water is 4 m deep?

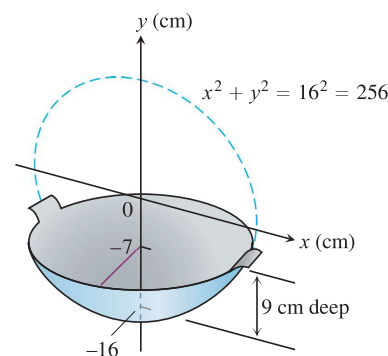
- 52.** Explain how you could estimate the volume of a solid of revolution by measuring the shadow cast on a table parallel to its axis of revolution by a light shining directly above it.

- 53. Volume of a hemisphere** Derive the formula $V = (2/3)\pi R^3$ for the volume of a hemisphere of radius R by comparing its cross-sections with the cross-sections of a solid right circular cylinder of radius R and height R from which a solid right circular cone of base radius R and height R has been removed as suggested by the accompanying figure.

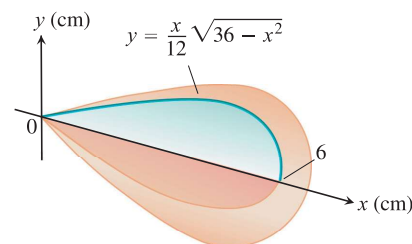


- 54. Volume of a cone** Use calculus to find the volume of a right circular cone of height h and base radius r .

- 55. Designing a wok** You are designing a wok frying pan that will be shaped like a spherical bowl with handles. A bit of experimentation at home persuades you that you can get one that holds about 3 L if you make it 9 cm deep and give the sphere a radius of 16 cm. To be sure, you picture the wok as a solid of revolution, as shown here, and calculate its volume with an integral. To the nearest cubic centimeter, what volume do you really get? (1 L = 1000 cm^3 .)



- 56. Designing a plumb bob** Having been asked to design a brass plumb bob that will weigh in the neighborhood of 190 g, you decide to shape it like the solid of revolution shown here. Find the plumb bob's volume. If you specify a brass that weighs 8.5 g/cm^3 , how much will the plumb bob weigh (to the nearest gram)?



- 57. Max-min** The arch $y = \sin x$, $0 \leq x \leq \pi$, is revolved about the line $y = c$, $0 \leq c \leq 1$, to generate the solid in Figure 6.16.

- Find the value of c that minimizes the volume of the solid. What is the minimum volume?
- What value of c in $[0, 1]$ maximizes the volume of the solid?

- T c.** Graph the solid's volume as a function of c , first for $0 \leq c \leq 1$ and then on a larger domain. What happens to the volume of the solid as c moves away from $[0, 1]$? Does this make sense physically? Give reasons for your answers.

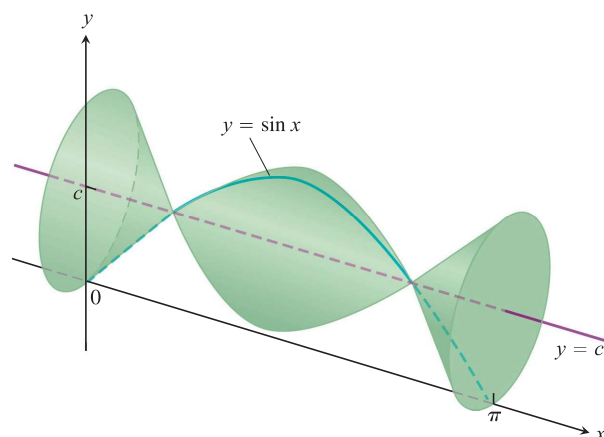


FIGURE 6.16

- 58. An auxiliary fuel tank** You are designing an auxiliary fuel tank that will fit under a helicopter's fuselage to extend its range. After some experimentation at your drawing board, you decide to shape the tank like the surface generated by revolving the curve $y = 1 - (x^2/16)$, $-4 \leq x \leq 4$, about the x -axis (dimensions in feet).
- How many cubic feet of fuel will the tank hold (to the nearest cubic foot)?
 - A cubic foot holds 7.481 gal. If the helicopter gets 2 mi to the gallon, how many additional miles will the helicopter be able to fly once the tank is installed (to the nearest mile)?