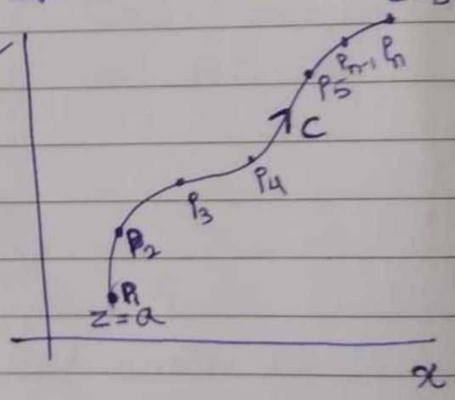
Line Integral :-

In case of real variable, the path of integration $\int_a^b f(x)\,dx$ is always along x axis from $x=a$ to $x=b$, but in case of a complex function $f(z)$ the path of definite integral $\int_{z=a}^{b} f(z)\,dz$ can be along any curve from $z=a$ to $z=b$.

The Integral $\int_C f(z)\,dz$ is called line integral of $f(z)$ along the curve C.



Here $z = x + iy$

$$dz = dx + i\,dy$$

(a) If $y=0$, $dy=0$

$$\boxed{dz = dx}$$

(b) If $x=0$, $dx=0$

$$\boxed{dz = i\,dy}$$

Contour integral

* In case of initial point and final point coincide so that c is a close curve, then this integral is called contour integral and denoted by $\oint_C f(z)\, dz$.

If $f(z) = u + iv$, $dz = dx + idy$

then
$$\oint_C f(z)\, dz = \oint_C (u+iv)(dx+idy)$$

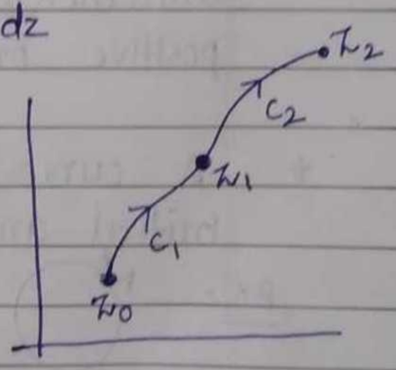
$$\oint_C f(z)\, dz = \oint_C (u\,dx - v\,dy) + i \oint_C (v\,dx + u\,dy)$$

this shows that evaluation of line integral of complex function can be reduced to the evaluation of two line integrals of real function.

Properties of integration

(1) $$\int_C (f(z) + g(z))\, dz = \int_C f(z)\, dz + \int_C g(z)\, dz$$

(2) If curve c, consist of ~~so~~ smooth curves c_1 and c_2 joined end to end then
$$\int_C f(z)\, dz = \int_{C_1} f(z)\, dz + \int_{C_2} f(z)\, dz$$

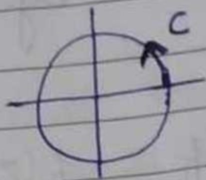


3) $$\int_{-C} f(z)\, dz = -\int_C f(z)\, dz$$

or $$\int_a^b f(z)\, dz = -\int_b^a f(z)\, dz$$

Line integral around ~~unit~~ circle :-

$$\oint_C f(z)\,dz$$



Let $C : |z| = r$ → circle with radius r and center c

We see that on C, any point $z = re^{i\theta} \Rightarrow dz = r i e^{i\theta} d\theta$

therefor $$\oint_C f(z)\,dz = \oint_{|z|=r} f(re^{i\theta})\, r i e^{i\theta}\, d\theta$$

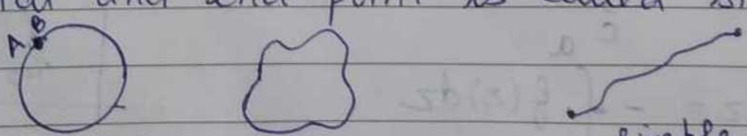
$$\boxed{\oint_C f(z)\,dz = \int_{\theta=0}^{2\pi} f(re^{i\theta})\, r i e^{i\theta}\, d\theta}$$

Note :- The value of complex integral depends upon the path of integration unless the integrand is analytic.

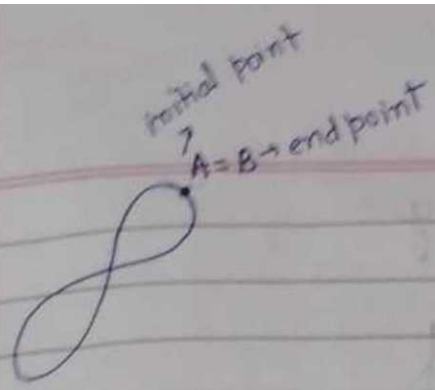
* In case of simple closed curve C, the anticlockwise direction is regarded as positive orientation.

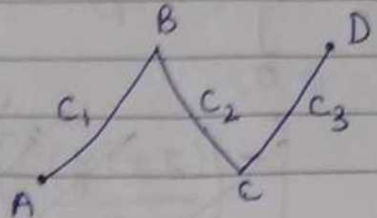
* A curve which does not cross itself except initial and end point is called simple curve.

Ex :-



Simple closed curve — Simple closed curve — simple curve



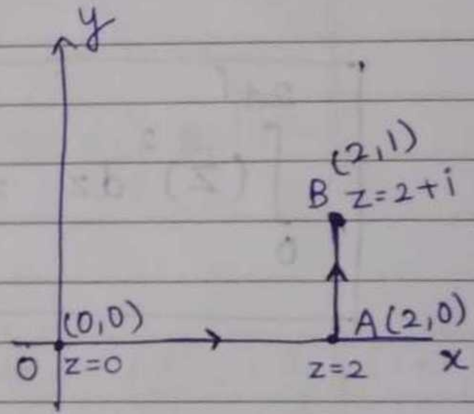


closed but not simple

piecewise smooth simple curve

Ex1 Evaluate $\int_0^{2+i} (\bar{z})^2\, dz$ along the real axis from $z=0$ to $z=2$ and then along a parallel line to y axis from $z=2$ to $z=2+i$.

Soln:-



$$\int_0^{2+i} (\bar{z})^2\, dz = \int_0^{2+i} (x-iy)^2 (dx+idy)$$

$$= \int_0^{2+i} (x^2+y^2-2ixy)(dx+idy)$$

$$= \int_{OA} (x-iy)^2(dx+idy) + \int_{AB} (x-iy)^2 (dx+idy)$$

along OA
$y=0$
$\Rightarrow dy=0$

along AB
$x=2$
$dx=0$

$$= \int_0^2 x^2\,dx + \int_0^1 (2-iy)^2\, i\,dy$$

$$= \cancel{\left(\frac{x^3}{3}\right)_0^2} + \cancel{\left[-\left(\frac{(2-iy)^3}{3}\right)_0\right]_0^1}$$

$$= \frac{8}{3} = \left(\frac{x^3}{3}\right)_0^2 + \int_0^1 (4-y^2-4iy)\, i\,dy$$

$$\Rightarrow \frac{8}{3} + i\left(4y - \frac{y^3}{3} - 4i\frac{y^2}{2}\right)_0^1$$

$$= \frac{8}{3} + i\left(4 - \frac{1}{3} - 2i\right) = \frac{8}{3} + 2 + \frac{11}{3}i$$

$$= \frac{1}{3}(14 + 11i)$$

$$\boxed{\int_0^{2+i} (\bar{z})^2\,dz = \frac{1}{3}(14+11i)}$$

Note:- If $f(z)$ is analytic function, then $\int_{z_0}^{z_1} f(z)\,dz$ is independent of path, joining the points z_0 and z_1 in region R.

Q:- Verify above result by evaluating $\int_C (z^2+3z)\,dz$

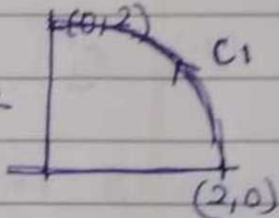
(i) along $|z|=2$ from $(2,0)$ to $(0,2)$ in anticlock wise

(ii) straight line from $(2,0)$ to $(0,2)$

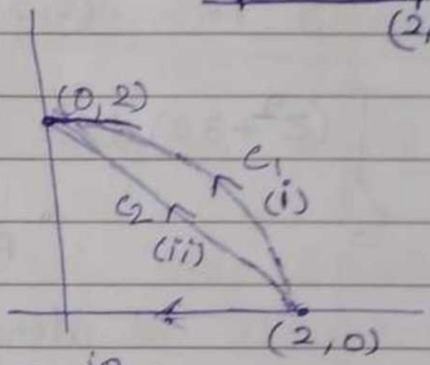
(iii) straight line from $(2,0)$ to $(2,2)$ then $(2,2)$ to $(0,2)$

Sol^n:- $\int_C (z^2+3z)\,dz$

C_1 is arc of circle $|z|=2$



$f(z) = z^2+3z$ is analytic function



(i) $C_1: |z|=2$ along $(2,0)$ to $(0,2)$

$z = 2e^{i\theta}$ $\quad |z|=2$

$$\int_C (z^2+3z)\,dz = \int_0^{\pi/2} (4e^{2i\theta} + 6e^{i\theta})\, 2i\, e^{i\theta}\, d\theta$$

$dz = 2ie^{i\theta}\,d\theta$

$$= \left[8i \frac{e^{3i\theta}}{3i} + 12i \frac{e^{2i\theta}}{2i} \right]_0^{\pi/2}$$

$$= \frac{8}{3}\left(e^{\frac{3\pi}{2}i} - 1\right) + 6\left(e^{i\pi} - 1\right)$$

$$= \frac{8}{3}(-i-1) + 6(-1-1) = -\frac{8}{3}i - \frac{44}{3}$$

(ii) along straight line from $(2,0)$ to $(0,2)$

eq^n of line joining $(2,0)$ to $(0,2)$

$x+y=2$

$C_2 : x + y = 2 \qquad dx + dy = 0$

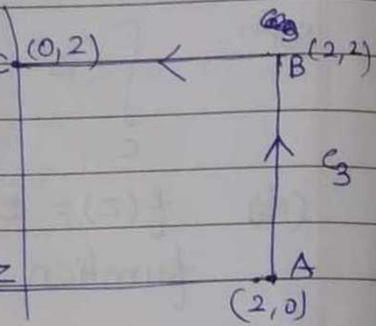
$$\int_{C_2} (z^2 + 3z)\, dz = \int_{C_2} (x^2 - y^2 + 2ixy + 3x + 3iy)(dx + i dy)$$

$$= \int_{x=2}^{0} \left[x^2 - (2-x)^2 + 2ix(2-x) + 3x + 3i(2-x)\right] (dx - i dx) = 0$$

$$= \int_{2}^{0} \left[(-2x^2 + 8x + 2) + i(-2x^2 - 6x + 10)\right] dx$$

$$= \frac{-44}{3} - \frac{8}{3} i$$

(iii) C_3 : straight line from $(2,0)$ to $(2,2)$ then $(2,2)$ to $(0,2)$



$$\int_{C_3} (z^2 + 3z)\, dz = \int_{AB} (z^2 + 3z)\, dz + \int_{BC} (z^2 + 3z)\, dz$$

along AB: $x = 2$, $dx = 0$ $\qquad$ along BC: $y = 2$, $dy = 0$

$$\int_{C_3} (z^2 + 3z)\, dz = \int_{AB} \left[(x+iy)^2 + 3(x+iy)\right](dx + i dy) + \int_{BC} \left[(x+iy)^2 + 3(x+iy)\right][dx + i dy]$$

$$= \int_{y=0}^{2} \left((2+iy)^2 + 3(2+iy)\right) i\, dy + \int_{x=2}^{0} \left[(x+2i)^2 + 3(x+2i)\right] dx$$

$$= \left[\frac{(2+iy)^3}{3} + \frac{3(2+iy)^2}{2}\right]_0^2 + \left[\frac{(x+2i)^3}{3} + \frac{3(x+2i)^2}{2}\right]_2^0$$

$$= \frac{-44}{3} - \frac{8}{3} i$$

The values of three integrals are same, irrespective of curve joining two point, since $f(z)$ is analytic

(2) $\int_0^{1+i} (x^2 - iy)\, dz$ along the path

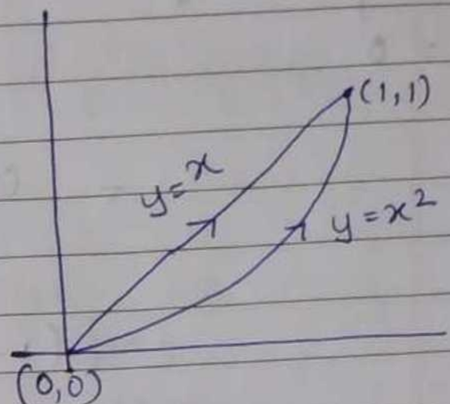
(a) $y = x$ (b) $y = x^2$

Solⁿ → (a) along $y = x$

$$dy = dx$$

$$= \int_{z=0}^{1+i} (x^2 - iy)(dz)$$

$$= \int_{z=0}^{1+i} (x^2 - iy)(dx + i\,dy)$$



$$y = x \Rightarrow dy = dx$$

$$= \int_{x=0}^{1} (x^2 - ix)(dx + i\,dx) = \int_{x=0}^{1} (x^2 - ix)(1+i)\, dx$$

$$= x\int_0^1 x\, dx = \left(\frac{x^3}{3} - i\frac{x^2}{2}\right)_0^1 (1+i)$$

$$= \left(\frac{1}{3} - \frac{i}{2}\right)(1+i)$$

$$= \frac{(2-3i)(1+i)}{6}$$

$$\int_0^{1+i} (x^2 - ix)\, dz = \frac{2 + 2i - 3i + 3}{6} = \frac{5-i}{6}$$

(b) along $y = x^2$

$dy = 2x\,dx$, $dz = dx + i\,2x\,dx$

$dz = (1 + 2ix)\,dx$

$$\int_{z=0}^{1+i} (x^2 - iy)\,dz = \int_{x=0}^{1} (x^2 - ix^2)\,(dx + i\,2x\,dx)$$

$$= (1-i) \int_0^1 x^2\,(1+2ix)\,dx$$

$$= (1-i) \left[\frac{x^3}{3} + 2i\frac{x^4}{4} \right]_0^1$$

$$= (1-i) \left[\frac{1}{3} + \frac{2i}{4} \right] = \frac{(1-i)(2+3i)}{6}$$

$$\boxed{\int_{z=0}^{1+i} (x^2 - iy)\,dz = \frac{5}{6} + \frac{1}{6}i}$$

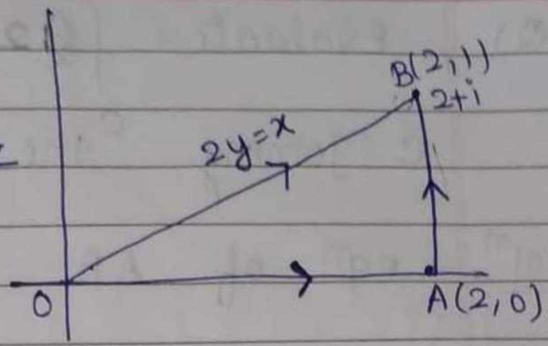
EX3 Evaluate $\int_0^{2+i} (\bar{z})^2\,dz$ along

(a) along real axis to 2 and vertically to $(2+i)$

(b) along line $y = 2x$

(a) along OAB

$$\int_0^{2+i} (\bar{z})^2 \, dz = \int_{z=0}^{2+i} (x^2 - y^2 - 2ixy) \, dz$$



$$= \int_{OA} (x^2 - y^2 - 2ixy) \, dz + \int_{AB} (x^2 - y^2 - 2ixy) \, dz$$

along OA

$y = 0$

$dy = 0$

$dz = dx + i\,dy$

$\Rightarrow \boxed{dz = dx}$

along AB

$x = 2$

$dx = 0$

$\boxed{dz = i\,dy}$

$$\int_0^{2+i} (\bar{z})^2 \, dz = \frac{14}{3} + \frac{11i}{3}$$

(b) along OB $\rightarrow$ $2y = x$

$2\,dy = dx$

$dz = dx + i\,dy$

$dz = 2\,dy + i\,dy = (2+i)\,dy$

$$\int_{z=0}^{2+i} (\bar{z})^2 \, dz = \int_{y=0}^{1} (4y^2 - y^2 - 4iy^2)(2+i) \, dy.$$

$$= (2+i)(3-4i) \int_0^1 y^2 \, dy = \frac{10}{3} - \frac{5i}{3}$$

classmate

Date ____

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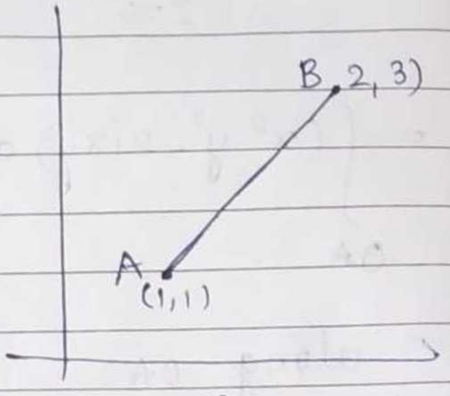
Q) Evaluate $\int_C (12z^2 - \cancel{4iz})\, dz$ along curve c joining the points $(1,1)$ and $(2,3)$

Soln eqn of AB



$y = 2x - 1$

$dy = 2\,dx$

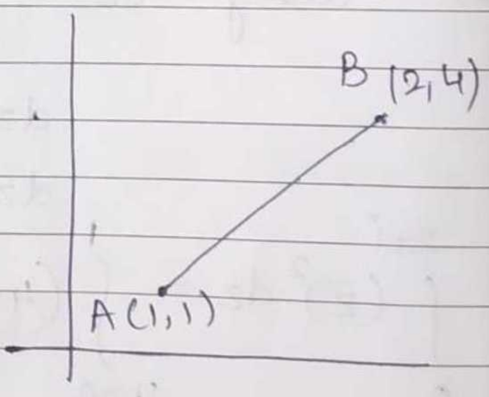
$$\int_{1+i}^{2+3i} (12z^2 - 4iz)\, dz = \int_{x=1}^{2} 12[x + i(2x-1)]^2 [dx + 2i\,dx]$$

Q:- Integrale $f(z) = x^2 + ixy$, from $A(1,1)$ to $B(2,4)$ along $x = t$, $y = t^2$.

Soln $x = t$, $y = t^2$

$dx = dt$ $dy = 2t\,dt$

At A, $t = 1$ and at B $t = 2$



$$\int_C (x^2 + ixy)\, dz$$

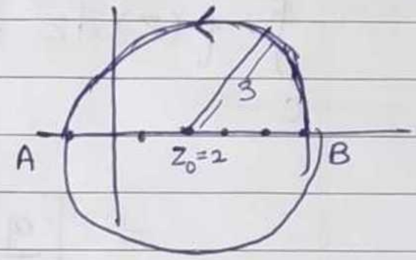
$$= \int_C (x^2 + ixy)(dx + i\,dy)$$

$$= \int_{t=1}^{2} (t^2 + i t^3)\,(dt + i2t\,dt)$$

$$\boxed{\int_C f(z)\,dz = \frac{-151}{15} + \frac{45}{4}\, i}$$

Q:- evaluate $\int_C (z-z^2)\,dz$, where c is upper half of circle $|z-2|=3$.

soln $\int_C (z-z^2)\,dz$



$c: |z-2| = 3$

$$z = r e^{i\theta} = 3e^{i\theta}$$

$$dz = 3i\,e^{i\theta}\,d\theta$$

$$\int_C (z-z^2)\,dz = \int_{\theta=0}^{\pi} \left(3e^{i\theta} - 9e^{2i\theta}\right) 3i\,e^{i\theta}\,d\theta$$

$$= \int_0^{\pi} \left(9i\,e^{2i\theta} - 27i\,e^{3i\theta}\right) d\theta$$

$$= \left(\frac{9i\,e^{2i\theta}}{2i} - \cancel{27}^{\,9}\,i\,\frac{e^{3i\theta}}{\cancel{3}i}\right)_0^{\pi}$$

$$= \left(\frac{9}{2}e^{2i\pi} - 9e^{3i\pi}\right) - \left(\frac{9}{2} - 9\right)$$

$\Rightarrow \frac{9}{2}(1+2+1)$

$= 18$

$e^{i\theta} = \cos\theta + i\sin\theta$

$e^{2\pi i} = \cos 2\pi + i\sin 2\pi$

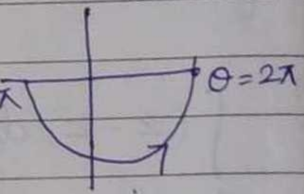
$\boxed{e^{2\pi i} = 1}$

$\boxed{e^{3\pi i} = -1}$

$$\boxed{\int_c (z - z^2)\,dz = 18}$$

Q:- what is value of integral if c is lower half of above circle.

$$\int_c f(z)\,dz = \int_{\theta=\pi}^{2\pi} (9i e^{2i\theta} - 27i e^{3i\theta})\,d\theta$$



$$= \left[\frac{9}{2} e^{2i\theta} - 9 e^{3i\theta}\right]_{\pi}^{2\pi}$$

$$= \left(\frac{9}{2} e^{4\pi i} - 9 e^{6\pi i}\right) - \left(\frac{9}{2} e^{2\pi i} - 9 e^{3\pi i}\right)$$

$$= \left(\frac{9}{2} - 9\right) - \left(\frac{9}{2} + 9\right)$$

$$= -18$$

$$\boxed{\int_c f(z)\,dz = -18}$$

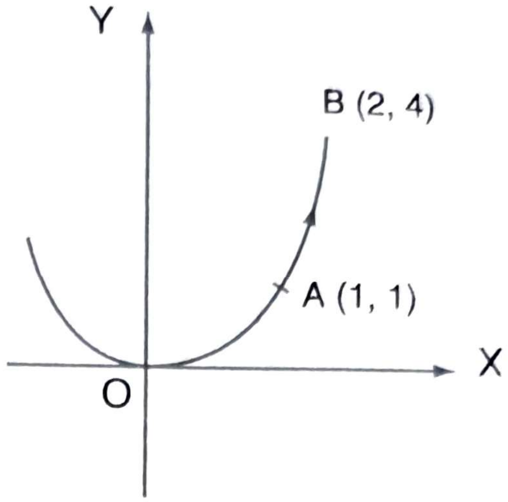
c is lower half of $|z-2| = 3$

Q evaluate $\int_0^{1+i} (x - y + ix^2)\,dz$ along (i) line joining $z=0$ and $1+i$ (2) the parabola $y = x^2$ (3) along $x = t$, $y = 2t - t^2$

Example 3 Evaluate $\int_C z^2 \, dz$ where the ends of C are A (1, 1) and B (2, 4) given that

(i) C is the curve $y = x^2$,

(ii) C is the line $y = 3x - 2$.



Solution : (i) Along $y = x^2$, $dy = 2x\,dx$, x varies from 1 to 2.

$$\therefore \int_C z^2 \, dz$$

$$= \int_{AB} (x + iy)^2 (dx + i\,dy)$$

$$= \int_{AB} (x^2 - y^2 + 2i\,xy)(dx + i\,dy)$$

$$= \int_1^2 \{(x^2 - x^4)\,dx - 2x^3 (2x\,dx)\} + i \int_1^2 \{2x(x^2)\,dx + (x^2 - x^4)\,2x\,dx\}$$

$$= \left[\frac{x^3}{3} - x^5\right]_1^2 + i\left[x^4 - \frac{x^6}{3}\right]_1^2$$

$$= \left(\frac{8}{3} - 32 - \frac{1}{3} + 1\right) + i\left(16 - \frac{64}{3} - 1 + \frac{1}{3}\right)$$

$$= \frac{8 - 96 - 1 + 3}{3} + i\left(\frac{48 - 64 - 3 + 1}{3}\right)$$

$$= \frac{-86}{3} - 6i$$

(ii) Along

$$y = 3x - 2$$

$$dy = 3\,dx$$

$$\therefore \int_C z^2 \, dz = \int_{AB} (x + iy)^2 (dx + i\,dy)$$

$$= \int_{AB} (x^2 - y^2 + 2ixy)(dx + i\,dy)$$

$$= \int_1^2 \{x^2 - (3x - 2)^2 + 2ix(3x - 2)\}(dx + i\,3\,dx)$$

$$= \int_1^2 (x^2 - 9x^2 - 4 + 12x + 6i\,x^2 - 4ix)(1 + 3i)\,dx$$

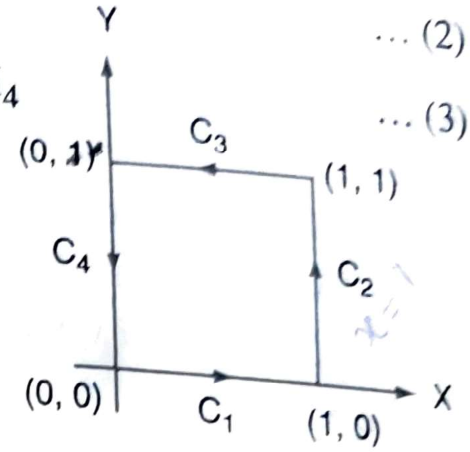
$$= (1 + 3i) \int_1^2 [(-8x^2 + 12x - 4) + i(6x^2 - 4x)]\,dx = \frac{-86}{3} - 6i$$

Example 4 Show that $\int_C (z+1)\, dz = 0$ where C is the boundary of the square whose vertices are at the points $z = 0$, $z = 1$, $z = 1 + i$ and $z = i$.

Solution : Let

$$z = x + iy \qquad \dots (1)$$

$$dz = dx + i\, dy \qquad \dots (2)$$



Let

$$\int_C (z+1)\, dz = I_1 + I_2 + I_3 + I_4 \qquad \dots (3)$$

$$I_1 = \int_{C_1} (z+1)\, dz$$

$$= \int_{C_1} (x + iy + 1)\,(dx + i\, dy)$$

Along C_1, $y = 0$, $dy = 0$

$$dz = dx \qquad \text{[putting } dy = 0 \text{ in (2)]}$$

Also along C_1, x varies from 0 to 1.

$$= \int_0^1 (x+1)\, dx$$

$$= \left[\frac{(x+1)^2}{2}\right]_0^1 = \frac{4}{2} - \frac{1}{2} = \frac{3}{2} \qquad \dots (4)$$

$$I_2 = \int_{C_2} (z+1)\, dz$$

$$= \int_{C_2} (x + iy + 1)\,(dx + i\, dy) = \int_0^1 (1 + iy + 1)\; i\, dy$$

[Along C_2, $x = 1$; $dx = 0$ $\therefore$ $dz = i\, dy$ and y varies from 0 to 1]

$$= i \int_0^1 (2 + iy)\, dy = i\left[\frac{(iy+2)^2}{2i}\right]_0^1$$

$$= \frac{(2+i)^2}{2} - 2 = \frac{4 - 1 + 4i - 4}{2}$$

$$= \frac{-1 + 4i}{2} \qquad \dots (5)$$

$$I_3 = \int_{C_3} (z+1)\ dz$$

$$= \int_{C_3} (x+iy+1)\,(dx + i\,dy)$$

$$= \int_1^0 (x+i+1)\ dx$$

[Along C_3, $y = 1$; $dy = 0$, *i.e.*, $dz = dx$ and x varies from 1 to 0]

$$= \left[\frac{x^2}{2} + (i+1)\,x\right]_1^0$$

$$= 0 - \left[\frac{1}{2} + (i+1)\right]$$

$$= -\frac{1}{2} - i - 1$$

$$= \frac{-3}{2} - i \qquad \ldots (6)$$

$$I_4 = \int_{C_4} (z+1)\ dz$$

$$= \int_{C_4} (x+iy+1)\,(dx + i\,dy)$$

$$= \int_1^0 (iy+1)\ i\,dy$$

[Along C_4, $x = 0$, $dx = 0$; $dz = i\,dy$ and y varies from 1 to 0]

$$= \int_1^0 (-y+i)\ dy = \left[\frac{-y^2}{2} + iy\right]_1^0$$

$$= -\left[\frac{-1}{2} + i\right] = \frac{1}{2} - i \qquad \ldots (7)$$

Adding (4), (5), (6), (7) we get,

$$I = I_1 + I_2 + I_3 + I_4$$

$$= \frac{3}{2} - \frac{1}{2} + 2i - \frac{3}{2} - i + \frac{1}{2} - i$$

$$= 0$$

Example 5 **Evaluate $\int_C \sin z \, dz$ along the line $z = 0$ to $z = i$.**

Solution : Given $z = 0$ to $z = i$.

i.e., $x + iy = 0 + 0i$ to $x + iy = 0 + i$

i.e., $x = 0, y = 0$ to $x = 0, y = 1$. *i.e.,* $(0, 0)$ to $(0, 1)$

Now $\quad \int_C \sin z \, dz = \int_{OA} \sin (x + iy)\,(dx + i\,dy)$

$[z = x + iy; \therefore dz = dx + i\,dy]$

Along OA, $x = 0$, $dx = 0$ and y varies from 0 to 1.

$$= \int_0^1 \sin iy \cdot i\,dy = i\left[\frac{-\cos iy}{i}\right]_0^1$$

$$= -\cos i + 1 = 1 - \cos i$$

Example 6 **Evaluate $\int_C e^z \, dz$, C is $|z| = 1$.**

Solution : Put

$$z = e^{i\theta}$$

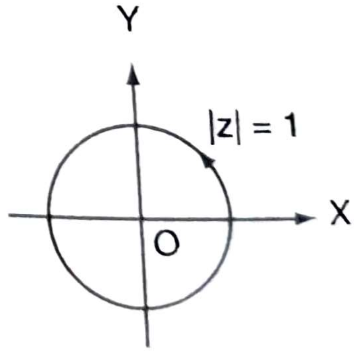
$$\therefore dz = i\,e^{i\theta}\,d\theta$$

$$\therefore \int_C e^z\,dz = \int_0^{2\pi} e^{e^{i\theta}}\, i\,e^{i\theta}\,d\theta$$

Put $\quad e^{i\theta} = x$ t $\qquad$ when $\theta = 0$, $\qquad x = 1$

$\therefore i\,e^{i\theta}\,d\theta = dx \qquad$ when $\theta = 2\pi$, $\qquad x = 1$

$$= \int_1^1 e^x\,dx$$



$$= [e^x]_1^1$$

$$= e^1 - e^1 = 0$$

Example 7 **Prove that $\int_C \frac{dz}{z-a} = 2\pi i$, where C is the circle $|z - a| = r$.**

Solution : The equation of the circle $|z - a| = r$ can be written as

$$z - a = re^{i\theta}, \theta \text{ varies from 0 to } 2\pi$$

$$dz = ri\,e^{i\theta}\,d\theta$$

$$\therefore \int_C \frac{dz}{z-a} = \int_0^{2\pi} \frac{ri\,e^{i\theta}\,d\theta}{re^{i\theta}} = i\,[\theta]_0^{2\pi}$$

$$= 2\pi i$$

Example 8 **Prove that $\int_C (z-a)^n\,dz = 0$ [$n \neq -1$] where C is the circle $|z-a| = r$.**

Solution : Given $|z-a| = r$

i.e.,
$$z - a = re^{i\theta}$$

$$dz = ri\,e^{i\theta}\,d\theta$$

$$\int_C (z-a)^n\,dz = \int_0^{2\pi} r^n e^{in\theta} \cdot ir\,e^{i\theta}\,d\theta$$

$$= ir^{(n+1)} \int_0^{2\pi} e^{i(n+1)\theta}\,d\theta$$

$$= ir^{(n+1)} \left[\frac{e^{i(n+1)\theta}}{i(n+1)}\right]_0^{2\pi}$$

$$= \frac{r^{n+1}}{n+1}\,[e^{i2(n+1)\pi} - 1]$$

$$= \frac{r^{n+1}}{n+1}\,[\cos 2(n+1)\pi + i \sin 2(n+1)\pi - 1]$$

$$= \frac{r^{n+1}}{n+1}\,[1 + 0i - 1] = 0$$

Example 9 **Evaluate $\int_C \log z\,dz$ where C is the unit circle $|z| = 1$.**

Solution : $|z| = 1 \Rightarrow z = e^{i\theta}$, $\therefore\ dz = ie^{i\theta}\,d\theta$

$$[\because |z| = |e^{i\theta}|$$
$$= |\cos\theta + i\sin\theta|$$
$$= \sqrt{\cos^2\theta + \sin^2\theta} = 1$$

Here θ varies from 0 to 2π.

$$\therefore \int_C \log z\,dz = \int_0^{2\pi} \log(e^{i\theta}) \cdot ie^{i\theta}\,d\theta$$

$$= \int_0^{2\pi} i\theta \cdot i\, e^{i\theta}\, d\theta \qquad [\because \log e^x = x]$$

$$= -\int_0^{2\pi} \theta\, e^{i\theta}\, d\theta$$

$$= -\left[\theta\left(\frac{e^{i\theta}}{i}\right) - 1\left(\frac{e^{i\theta}}{i^2}\right)\right]_0^{2\pi}$$

[Using Bernoulli's formula]

$$= -\left[\frac{2\pi\, e^{i2\pi}}{i} + e^{i2\pi} - 1\right]$$

$$= -\left[e^{2\pi i}\left(\frac{2\pi}{i} + 1\right) - 1\right]$$

$$= -\left[\frac{2\pi}{i} + 1 - 1\right]$$

$$[\because e^{i2\pi} = \cos 2\pi + i \sin 2\pi = 1]$$

$$= -\frac{2\pi}{i} = -\frac{2\pi}{i} \times \frac{i}{i} = 2\pi i$$

Example 10 **Evaluate** $\int_0^{2+i} (\overline{z})^2\, dz$ **along the line** $y = \frac{x}{2}$.

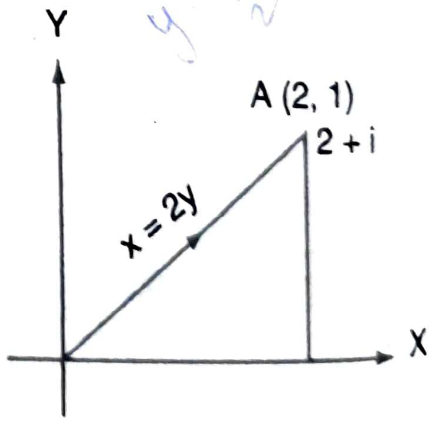
Solution : Let $z = x + iy$

$$\therefore dz = dx + i\, dy$$

$$\overline{z} = x - iy$$

$$(\overline{z})^2 = (x - iy)^2$$

$$= x^2 - y^2 - 2i\, xy$$



$$\therefore \int_0^{2+i} (\overline{z})^2\, dz$$

$$= \int_0^{A} (x^2 - y^2 - 2i\, xy)\,(dx + i\, dy) \qquad \dots (1)$$

Along OA, $y = \frac{x}{2}$ $\therefore dy = \frac{1}{2}\, dx$

$\therefore$ Replacing y by $\frac{x}{2}$ in (1)

$$= \int_0^2 \left(x^2 - \frac{x^2}{4} - 2i\,x \cdot \frac{x}{2}\right)\left(dx + i\,\frac{dx}{2}\right)$$

[$\because x$ varies from 0 to 2 along OA]

$$= \int_0^2 \left(\frac{3x^2}{4} - ix^2\right)\left(1 + \frac{i}{2}\right) dx$$

$$= \left(1 + \frac{i}{2}\right)\left(\frac{3}{4} - i\right)\int_0^2 x^2\,dx$$

$$= \left[\frac{3}{4} - i + \frac{3i}{8} + \frac{1}{2}\right]\left[\frac{x^3}{3}\right]_0^2$$

$$= \left(\frac{5}{4} - \frac{5i}{8}\right)\left(\frac{8}{3}\right) = \frac{5}{3}\,(2 - i)$$

EXERCISES

1. Evaluate $\int_{(0,0)}^{(1,1)} [(x^2 + y^2)\,dx - 2xy\,dy]$ along (*i*) $y = x$ (*ii*) $x = y^2$ (*iii*) $y = x^2$.

   [Ans. (i) 0; (ii) $\frac{1}{3}$; (iii) $\frac{-4}{15}$]

2. Evaluate $\int_{1-i}^{2+3i} (z^2 + z)\,dz$ along the line joining the points $(1, -1)$ and $(2, 3)$.

   [Ans. $\frac{64i - 103}{6}$]

3. Evaluate $\int_{(0,2)}^{(4,0)} (x^2 + y^2)\,dx$ along the path $y^2 = 4 - x$.

   [Ans. $\frac{88}{3}$]

4. Evaluate $\int_{(1,1)}^{(2,4)} [(2x^2 + 4xy)\,dx + (2x^2 - y^2)\,dy]$ along (*i*) $y = x^2$ (*ii*) $y = 3x - 2$.

   [Ans. (i) $\frac{41}{3}$; (ii) $\frac{41}{3}$]

5. Prove that $\int_{-2}^{-2+i} (2 + z)^2\,dz = \frac{-i}{3}$.

6. Evaluate $\int_{(0,0)}^{(1,3)} [x^2y\,dx + (x^2 - y^2)\,dy]$ along (*i*) $y = 3x^2$ (*ii*) $y = 3x$.

   [Ans. (i) $\frac{-69}{10}$; (ii) $\frac{-29}{4}$]