



VIT

Vellore Institute of Technology
(Deemed to be University under section 3 of UGC Act, 1956)

REG.NO.:

SCHOOL OF ADVANCED SCIENCES
DEPARTMENT OF MATHEMATICS
CONTINUOUS ASSESSMENT TEST - II
FALL SEMESTER 2024-2025

SLOT: A2 + TA2 +
TAA2

Programme Name & Branch : BTech (Information Technology/AI and ML/Data Science)

Course Code and Course Name : BMAT205L Discrete Mathematics and Graph Theory

Faculty Name(s) : Dr. Anjaneyulu G.S.G.N., Dr. Pallavi Mishra, Dr. Gayatri S. Panicker

Class Number(s) : VL2024250102592/2591/2590

Date of Examination : 13.10.2024

Exam Duration : 90 minutes **Maximum Marks: 50**

General instruction(s):

- Answer All Questions
- M - Max mark; CO - Course Outcome; BL - Blooms Taxonomy Level (1 - Remember, 2 - Understand, 3 - Apply, 4 - Analyse, 5 - Evaluate, 6 - Create)
- Course Outcomes:
 - CO2 - Use algebraic structures in applications
 - CO3 - Counting techniques in engineering problems
 - CO4 - Use lattice and Boolean algebra properties in digital circuits

Q. No	Question	M	CO	BL
1.	Let $G = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$ be the generator matrix of a (3,6) encoding function. What is the maximum number of errors this encoding function can detect? What is the maximum number of errors it can correct? Decode 110010.	10	2	3
2.	a) Find the number of VTOP passwords, if each password is to be of length 4, and must contain at least one capital letter, at least one small letter, and at least one digit. Note that other symbols or special characters are NOT allowed in a password.	5	3	6
	b) Suppose that 10 distinct numbers are chosen at random from 1 to 99, and let this be called set X . Prove that there exist $Y, Z \subseteq X$, such that both Y and Z are non-empty, disjoint and proper, and $\sum_{y \in Y} y = \sum_{z \in Z} z$.	5		
3.	Let $n \in \mathbb{N}_0$, where \mathbb{N}_0 depicts the set of whole numbers. Consider $X_n = \{x_1 x_2 \dots x_n : \text{for each } i \leq n, x_i \in \{0, 1, 2, \dots, 9\}\}$, that is, X_n is the set of n -digit strings with digits from 0 to 9. Let $A_n = \{x \in X_n : x \text{ has at least one } 0\}$. If $a_n = A_n $, find a recurrence relation for a_n . Solve this recurrence relation using the method of generating functions.	10	3	4
4.	Let $A = \mathbb{R}$. Define a binary relation R on A by aRb iff $\exists k \in \mathbb{R}$ such that $ka = b$. Is R a partial order? Justify.	10	4	4
5.	Let B be a Boolean algebra, and let $a, b \in B$. Prove that $a \wedge \neg b = 0 \Leftrightarrow a \wedge b = a$.	10	4	3

BMAT205L - Discrete Mathematics and Graph Theory

CAT - 2 Model Solutions

Slot: A2 + TA2 + TAA2

Fall Semester 2024 - 25

VIT - Vellore

Prepared by Dr. Gayatri S. Panicker

Total Marks: 50

Time: 90 minutes

Answer all the questions.

1. Let $G = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$ be the generator matrix of a $(3,6)$ encoding function.

What is the maximum number of errors this encoding function can detect? What is the maximum number of errors it can correct? Decode 110010. (10 marks)

Solution: There are 8 possible input strings, with corresponding codewords given as follows (using $E(w) = w \times G$):

$$E(000) = 000000$$

$$E(001) = 001011$$

$$E(010) = 010101$$

$$E(011) = 011110$$

$$E(100) = 100110$$

$$E(101) = 101101$$

$$E(110) = 110011$$

$$E(111) = 111000.$$

On computation, the minimum Hamming distance amongst the codewords can be found to be 3. Taking $k + 1 = 3$, we see that this encoding function can detect at most 2 errors. Along similar lines, we see that it can correct at most 1 error.

We can decode the given string using the parity check matrix. It can be seen that the parity check matrix is given by:

$$H = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Let $r = [1 \ 1 \ 0 \ 0 \ 1 \ 0]$. Therefore $Hr^T = [0 \ 0 \ 1]$. Note that this is the 6th column of H . It follows that there has been exactly one error in transmission, which

has occurred in the 6th position of r . Therefore the corrected codeword is $r = 110011$, and the decoded word is $w = 110$.

2. (a) Find the number of VTOP passwords, if each password is to be of length 4, and must contain at least one capital letter, at least one small letter, and at least one digit. Note that other symbols or special characters are NOT allowed in a password. (5 marks)

Solution: Let A be the set of all 4-length passwords that contain NO capital letter, let B be the set of all 4-length passwords that contain NO small letter, and let C be the set of all 4-length passwords that contain NO digits.

We must find $|A^c \cap B^c \cap C^c|$. Note that, by De Morgan's law, $|A^c \cap B^c \cap C^c| = |(A \cup B \cup C)^c| = |X| - |A \cup B \cup C|$, where X is the set of all strings of length 4 made using capital and small letters and digits. Therefore, $|X| = (26 + 26 + 10)^4 = (62)^4$.

On applying the Inclusion-Exclusion Principle we see that $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$.

Since A contains all strings that have NO capital letters, that is, all the entries in such a string must be small letters or digits, we must have $|A| = (26 + 10)^4 = (36)^4$. Along similar lines, $|B| = (26 + 10)^4 = (36)^4$, while $|C| = (26 + 26)^4 = (52)^4$.

Further, $|A \cap B| = (10)^4$, since any string in this set must contain only digits. Similarly, $|A \cap C| = (26)^4$, and $|B \cap C| = (26)^4$. Finally, $|A \cap B \cap C| = |\emptyset| = 0$.

It follows that the number of VTOP passwords is $(62)^4 - ((36)^4 + (36)^4 + (52)^4 - (10)^4 - (26)^4 - (26)^4 + 0)$.

- (b) Suppose that 10 distinct numbers are chosen at random from 1 to 99, and let this be called set X . Prove that there exist $Y, Z \subseteq X$, such that both Y and Z are non-empty, disjoint and proper, and $\sum_{y \in Y} y = \sum_{z \in Z} z$. (5 marks)

Solution: Suppose that the 10 distinct numbers chosen at random are $x_1, x_2, x_3, \dots, x_{10}$, where $x_i < x_{i+1}$ for each $1 \leq i \leq 9$. The number of distinct non-empty and proper subsets of this set X are $2^{10} - 2 = 1,022$. The possible values for sums of these subsets range from x_1 (which is the minimum possible sum)

to $\sum_{i=2}^{10} x_i$ (which is the maximum possible sum). Since the numbers are chosen

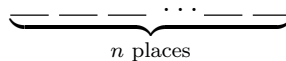
from 1 to 99, the range of possible sums is bounded by 1 and $\sum_{j=91}^{99} j = 855$.

Consider the pigeons as the non-empty proper subsets of X (of which there are 1022), and the boxes as the sum of the given subset (of which there are 855), then by Pigeonhole Principle, we see that there must be one box with at least two pigeons, that is, there must be one sum which is shared by two different subsets of X . Let these subsets be A and B . Therefore, $\sum_{a \in A} a = \sum_{b \in B} b$. We know that A and B are distinct subsets, therefore $A \neq B$. Without loss of generality, consider the case where $A \subsetneq B$. Then $B \setminus A \neq \emptyset$ and $\sum_{a \in A} a = \sum_{b \in B} b = \sum_{a \in A} a + \sum_{x \in B \setminus A} x$, that is, $\sum_{x \in B \setminus A} x = 0$, a contradiction. Therefore, we conclude that $A \not\subseteq B$ and $B \not\subseteq A$.

Since A and B may have elements in common, we consider $Y = A \setminus B$ and $Z = B \setminus A$. It follows that $\sum_{y \in Y} y = \sum_{z \in Z} z$, with Y and Z being non-empty, disjoint and proper subsets of X .

3. Let $n \in \mathbb{N}_0$, where \mathbb{N}_0 depicts the set of whole numbers. Consider $X_n = \{x_1 x_2 \cdots x_n : \text{for each } i \leq n, x_i \in \{0, 1, 2, \dots, 9\}\}$, that is, X_n is the set of n -digit strings with digits from 0 to 9. Let $A_n = \{x \in X_n : x \text{ has at least one } 0\}$. If $a_n = |A_n|$, find a recurrence relation for a_n . Solve this recurrence relation using the method of generating functions. (10 marks)

Solution: We first note that $a_0 = 0$ as there are no strings of length 0 with at least one 0. Consider the following element of A_n , where $n \geq 1$:



Consider the last position in this string. This may be 0 or a non-zero digit.

In the case that the last position is non-zero, we must have at least one of the previous positions to contain 0. In other words, the number of n -digit strings where the last position is k (where $k \neq 0$), is the same as the number of elements in A_{n-1} . Since there are 9 non-zero digits, there will be $9a_{n-1}$ strings of this form.

Consider the case where the last digit is 0. Then there are two possibilities. Either the first $n - 1$ digits have *no* zero, or they have *at least one* zero. The number of strings ending in 0 where the first $n - 1$ digits have no zero is 9^{n-1} . If the preceding $n - 1$ digits have at least one 0, then the number of strings of this form are a_{n-1} .

It follows that the desired recurrence relation where $n \geq 1$ is

$$a_n = 9a_{n-1} + (9^{n-1} + a_{n-1}),$$

that is,

$$a_n = 10a_{n-1} + 9^{n-1}.$$

We assume that the generating function associated with the sequence (a_n) is $G(x) = \sum_{n=0}^{\infty} a_n x^n$. From the recurrence relation above, we obtain the following:

$$\begin{aligned} \sum_{n=1}^{\infty} a_n x^n &= 10x \sum_{n=1}^{\infty} a_{n-1} x^{n-1} + x \sum_{n=1}^{\infty} 9^{n-1} x^{n-1} \\ (G(x) - a_0) &= 10x(G(x)) + x \left(\frac{1}{1-9x} \right) \\ G(x) - 0 &= 10xG(x) + \frac{x}{1-9x} \\ G(x)(1-10x) &= \frac{x}{1-9x} \\ G(x) &= \frac{x}{(1-9x)(1-10x)}. \end{aligned}$$

Therefore

$$G(x) = \frac{1}{1-10x} - \frac{1}{1-9x}$$

Finally, writing each entity in terms of a power series, we have:

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} 10^n x^n - \sum_{n=0}^{\infty} 9^n x^n$$

In other words,

$$a_n = 10^n - 9^n.$$

4. Let $A = \mathbb{R}$. Define a binary relation R on A by aRb iff $\exists k \in \mathbb{R}$ such that $ka = b$. Is R a partial order? Justify. (10 marks)

Solution: This relation is not antisymmetric. We see that $1R2$ since there exists $2 \in \mathbb{R}$ such that $2 \times 1 = 2$. Along similar lines, we also have $2R1$, since there exists $\frac{1}{2} \in \mathbb{R}$ such that $\frac{1}{2} \times 2 = 1$. Since $1R2$ and $2R1$ but $1 \neq 2$, it follows that R is not antisymmetric, and therefore not a partial order on A .

5. Let B be a Boolean algebra, and let $a, b \in B$. Prove that $a \wedge \neg b = 0 \Leftrightarrow a \wedge b = a$. (10 marks)

Solution: Let $a, b \in B$ such that $a \wedge \neg b = 0$. We have the following:

$$\begin{aligned} a \wedge b &= (a \wedge b) \vee 0 \\ &= (a \wedge b) \vee (a \wedge \neg b) \\ &= a \wedge (b \vee \neg b) \\ &= a \wedge 1 \\ &= a. \end{aligned}$$

On the other hand, we assume that $a \wedge b = a$. Consider

$$\begin{aligned} a \wedge \neg b &= (a \wedge b) \wedge \neg b \\ &= a \wedge (b \wedge \neg b) \\ &= a \wedge 0 \\ &= 0. \end{aligned}$$