



# VIT

Vellore Institute of Technology  
(Deemed to be University under section 3 of UGC Act, 1956)

REG.NO.:

SLOT: C2 + TC2 +  
TCC2

**SCHOOL OF ADVANCED SCIENCES  
CONTINUOUS ASSESSMENT TEST - II  
FALL SEMESTER 2025-2026**

**Programme Name & Branch** : BTech (SCOPE)  
**Course Code and Course Name** : BMAT205L Discrete Mathematics and Graph Theory  
**Faculty Name(s)** : Common slot  
**Class Number(s)** : Common slot  
**Date of Examination** : 07.10.2025  
**Exam Duration** : 90 minutes **Maximum Marks: 50**

**General instruction(s):**

- Answer All Questions
  - M - Max mark; CO – Course Outcome; BL – Blooms Taxonomy Level (1 – Remember, 2 – Understand, 3 – Apply, 4 – Analyse, 5 – Evaluate, 6 – Create)
- Course Outcomes:  
 CO2 – Use algebraic structures in applications  
 CO3 – Counting techniques in engineering problems  
 CO4 – Use lattice and Boolean algebra properties in digital circuits

Q. No	Question	M	CO	BL
1.	a) Show that there are exactly two group homomorphisms from $(\mathbb{Z}_2, +_2)$ to $(\mathbb{Z}_6, +_6)$ .	5	2	2
	b) Find the generator matrix corresponding to the (2,3) even parity check encoding function. Justify.	5		
2.	Let $n \in \mathbb{N}$ , and $S = \{1, 2, \dots, 2n\}$ be given. Show that $n$ is the largest possible cardinality of a subset $A \subseteq S$ satisfying the condition: given $a, b \in A$ such that $a \neq b$ , then $a$ does not divide $b$ and $b$ does not divide $a$ .	10	3	3
3.	How many non-negative integer solutions exist for the equation $x_1 + x_2 + x_3 + x_4 = 23$ , where $x_1, x_2, x_3 \geq 0$ and $x_4 \geq 3$ ?	10	3	3
4.	Solve the recurrence relation $a_n = 3a_{n-1} + 1$ , for $n \geq 1, a_0 = 2$ .	10	3	2
5.	Let $A$ be the set of all 3-bit strings. Let $x_1x_2x_3, y_1y_2y_3 \in A$ . Define a binary relation $R$ on $A$ by: $x_1x_2x_3Ry_1y_2y_3 \Leftrightarrow \forall i \in \{1, 2, 3\}, x_i \leq y_i$ . Is $(A, R)$ a poset? Justify. Does it form a lattice? Justify.	10	4	2

# BMAT205L - Discrete Mathematics and Graph Theory

CAT - 2 Model Solutions  
Slot: C2 + TC2 + TCC2  
Fall Semester 2025 - 26  
VIT - Vellore

Total Marks: 50

Time: 90 minutes

Answer all the questions.

1. (a) Show that there are exactly two group homomorphisms from  $(\mathbb{Z}_2, +_2)$  to  $(\mathbb{Z}_6, +_6)$ . (5 marks)

**Solution:** Let  $f : \mathbb{Z}_2 \rightarrow \mathbb{Z}_6$  be a group homomorphism. Since it is a homomorphism, we must have  $f(\bar{0}) = \bar{0}$ . It only remains to be seen what possibilities  $f(\bar{1})$  has.

Since  $f$  is a homomorphism, we must have  $f(\bar{1} +_2 \bar{1}) = f(\bar{1}) +_6 f(\bar{1})$ . In other words,  $f(\bar{1}) +_6 f(\bar{1}) = f(\bar{0}) = \bar{0}$ .

Since  $f(\bar{1}) \in \{\bar{0}, \bar{1}, \dots, \bar{5}\}$ , by observation, we conclude that the only possibility for  $f(\bar{1})$  is  $\bar{0}$  or  $\bar{3}$  (as these are the only two elements that satisfy  $\bar{x} +_6 \bar{x} = \bar{0}$  in  $\mathbb{Z}_6$ ).

Therefore, the only two possibilities for  $f$  are  $f_0$  and  $f_1$ , where  $f_0$  is the trivial homomorphism satisfying  $f_0(\bar{x}) = \bar{0}$  for each  $\bar{x} \in \mathbb{Z}_2$ , and  $f_1$  is the map where  $f_1(\bar{0}) = \bar{0}$  and  $f_1(\bar{1}) = \bar{3}$ .

- (b) Find the generator matrix corresponding to the  $(2, 3)$  even parity check encoding function. Justify. (5 marks)

**Solution:** We know that the encoding function is the even parity check  $E : B^2 \rightarrow B^3$ . From the properties of the generator matrix, we see that  $G$  must be of size  $2 \times 3$ , where the initial  $2 \times 2$  block is the identity matrix  $I_2$ . Therefore, the form of  $G$  is as follows:

$$G_{2 \times 3} = \begin{bmatrix} 1 & 0 & x \\ 0 & 1 & y \end{bmatrix}$$

The unknown entries  $x$  and  $y$  form the matrix  $A$ , which has to be determined. The relation between the encoding function  $E$  and the generator matrix  $G$  is the following for each word  $w \in B^2$ :

$$E(w) = w_{1 \times 2} G_{2 \times 3}.$$

Also, since  $E$  is the even parity check encoding function, we have the following:

$$E(00) = 000,$$

$$E(01) = 011,$$

$$E(10) = 101,$$

$$E(11) = 110.$$

Therefore, from matrix multiplication, we have the following:

$$0x + 0y = 0,$$

$$0x + 1y = 1,$$

$$1x + 0y = 1,$$

$$1x + 1y = 0.$$

It is clear that we have  $x = 1, y = 1$ . Therefore the generator matrix is given by:

$$G_{2 \times 3} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

2. Let  $n \in \mathbb{N}$  and  $S = \{1, 2, \dots, 2n\}$  be given. Show that  $n$  is the largest possible cardinality of a subset  $A \subseteq S$  satisfying the condition: given  $a, b \in A$  such that  $a \neq b$ , then  $a$  does not divide  $b$  and  $b$  does not divide  $a$ . (10 marks)

**Solution:** There are two aspects to the solution to this problem - first, that there exists a subset  $A \subseteq S$  of cardinality  $n$  with the desired property, and second, that no other subset of  $S$  with larger cardinality could have the required property.

We first claim that the set  $A = \{n+1, n+2, \dots, 2n\}$  satisfies the required property. If not, then there exist  $a, b \in A$  where  $a \neq b$  such that  $a|b$  or  $b|a$ . Without loss of generality, we may assume that  $a|b$ . The proof in the case that  $b|a$  will follow along similar lines.

Since  $a, b \in A$ , where  $a|b$ , we have  $a = n+i$  and  $b = n+j$  where  $i < j \in \{1, 2, \dots, n\}$  and  $(n+i)|(n+j)$ . The minimum value of  $n+j$  is  $2(n+i)$ . It follows that:

$$\begin{aligned} n+j &\geq 2(n+i) \\ &= 2n+2i \\ &> 2n. \end{aligned}$$

This is a contradiction. Therefore the given set  $A$  has the required property.

In order to prove the second part of the problem, we use Pigeonhole Principle. Consider  $B \subseteq S$  where  $|B| = k \geq n+1$ . Write each element  $b \in B$  as a power of 2 times

an odd number, that is  $b = 2^{m_b} \times \mathcal{O}_b$ . Consider the pigeons as the elements of  $B$ , and the holes as the odd numbers  $1, 3, 5, \dots, 2n - 1$ . There are  $k$  pigeons and  $n$  holes, where  $k > n$ . From Pigeonhole Principle, it follows that there is a hole with at least two pigeons. In other words, there exist two elements of  $B$ , say  $a, b$  where  $a \neq b$ , with the same odd number factor. It follows that the smaller of the two divides the larger.

This completes the proof.

3. How many non-negative integer solutions exist for the equation  $x_1 + x_2 + x_3 + x_4 = 23$ , where  $x_1, x_2, x_3 \geq 0$  and  $x_4 \geq 3$ ? (10 marks)

**Solution:** One possible solution would be to consider a new set of variables,  $y_i$ , where  $1 \leq i \leq 4$ , satisfying the equation given by the following:

$$y_1 + y_2 + y_3 + y_4 = 20,$$

where  $y_i \geq 0$  and  $y_i = x_i$  for  $1 \leq i \leq 3$ , while  $y_4 = x_4 - 3$ , as there is a one-one correspondence between the solution sets of the two equations.

This would be akin to the problem of picking 20 balls from a bag with balls of 4 colours, which can be done in  $\binom{23}{20} = 1,771$  ways.

4. Solve the recurrence relation  $a_n = 3a_{n-1} + 1$ , for  $n \geq 1$ ,  $a_0 = 2$ . (10 marks)

**Solution:** Using the method of generating functions, we consider the power series  $G(x) = \sum_{n=0}^{\infty} a_n x^n$ . On multiplying both sides of each equation in the recurrence

relation by  $x^n$  and taking summation, we have the following:

$$\begin{aligned}\sum_{n=1}^{\infty} a_n x^n &= 3 \sum_{n=1}^{\infty} a_{n-1} x^n + \sum_{n=1}^{\infty} x^n \\ \sum_{n=1}^{\infty} a_n x^n &= 3x \sum_{n=1}^{\infty} a_{n-1} x^{n-1} + \sum_{n=1}^{\infty} x^n \\ \sum_{n=1}^{\infty} a_n x^n &= 3x \sum_{n=0}^{\infty} a_n x^n + \sum_{n=1}^{\infty} x^n \\ (G(x) - a_0) &= 3xG(x) + \left( \frac{1}{1-x} - 1 \right) \\ G(x) - 2 &= 3xG(x) + \frac{x}{1-x}.\end{aligned}$$

Taking the  $G(x)$  terms to one side, we have:

$$G(x) = \frac{2-x}{(1-x)(1-3x)}$$

On finding the partial fraction decomposition of the right-hand side, we have:

$$\begin{aligned}G(x) &= \frac{-\left(\frac{1}{2}\right)}{1-x} + \frac{\left(\frac{5}{2}\right)}{1-3x} \\ G(x) &= -\frac{1}{2} \sum_{n=0}^{\infty} x^n + \frac{5}{2} \sum_{n=0}^{\infty} 3^n x^n\end{aligned}$$

Therefore, for each  $n \geq 0$ , we have

$$a_n = -\frac{1}{2} + \frac{5}{2} 3^n.$$

5. Let  $A$  be the set of all 3-bit strings. Let  $x_1x_2x_3, y_1y_2y_3 \in A$ . Define a binary relation  $R$  on  $A$  by:  $x_1x_2x_3Ry_1y_2y_3 \iff \forall i \in \{1, 2, 3\}, x_i \leq y_i$ . Is  $(A, R)$  a poset? Justify. Does it form a lattice? Justify. (10 marks)

**Solution:** In order to get an idea of the relation, we can see that, for instance,  $100R101$ , while  $101 \not R 110$ . To check if  $R$  is a partial order, we have the following:

- (i) Let  $x_1x_2x_3 \in A$ . Since  $x_i \leq x_i$  for each  $i \in \{1, 2, 3\}$ , we have  $x_1x_2x_3Rx_1x_2x_3$ , that is,  $R$  is reflexive.

- (ii) Let  $x_1x_2x_3, y_1y_2y_3 \in A$  such that  $x_1x_2x_3Ry_1y_2y_3$  and  $y_1y_2y_3Rx_1x_2x_3$ . From the definition of  $R$ , we have  $x_i \leq y_i$  and  $y_i \leq x_i$  for each  $i \in \{1, 2, 3\}$ , from which it follows that  $x_i = y_i$  for each  $i$ , that is,  $x_1x_2x_3 = y_1y_2y_3$ . Therefore,  $R$  is antisymmetric.
- (iii) Let  $x_1x_2x_3, y_1y_2y_3, z_1z_2z_3 \in A$  such that  $x_1x_2x_3Ry_1y_2y_3$  and  $y_1y_2y_3Rz_1z_2z_3$ . It follows that, for each  $1 \leq i \leq 3$ , we have  $x_i \leq y_i$  and  $y_i \leq z_i$ . From the transitivity of  $\leq$  on  $\mathbb{R}$ , we have  $x_i \leq z_i$ , that is,  $x_1x_2x_3Rz_1z_2z_3$ . Therefore  $R$  is transitive.

It follows that  $(A, R)$  is a poset. In order to ascertain if it forms a lattice, we have to check that for each  $a, b \in A$ , the  $\text{lub}\{a, b\}$  and  $\text{glb}\{a, b\}$  should exist in  $A$ . It is straightforward to see that for  $x_1x_2x_3, y_1y_2y_3 \in A$ , we have  $\text{lub}\{x_1x_2x_3, y_1y_2y_3\} = z_1z_2z_3$  where  $z_i = \max\{x_i, y_i\}$ , while  $\text{glb}\{x_1x_2x_3, y_1y_2y_3\} = w_1w_2w_3$  where  $w_i = \min\{x_i, y_i\}$ , which exist in  $A$ , as  $z_i, w_i \in \{0, 1\}$ . Therefore  $A$  does form a lattice with respect to the partial order  $R$ .