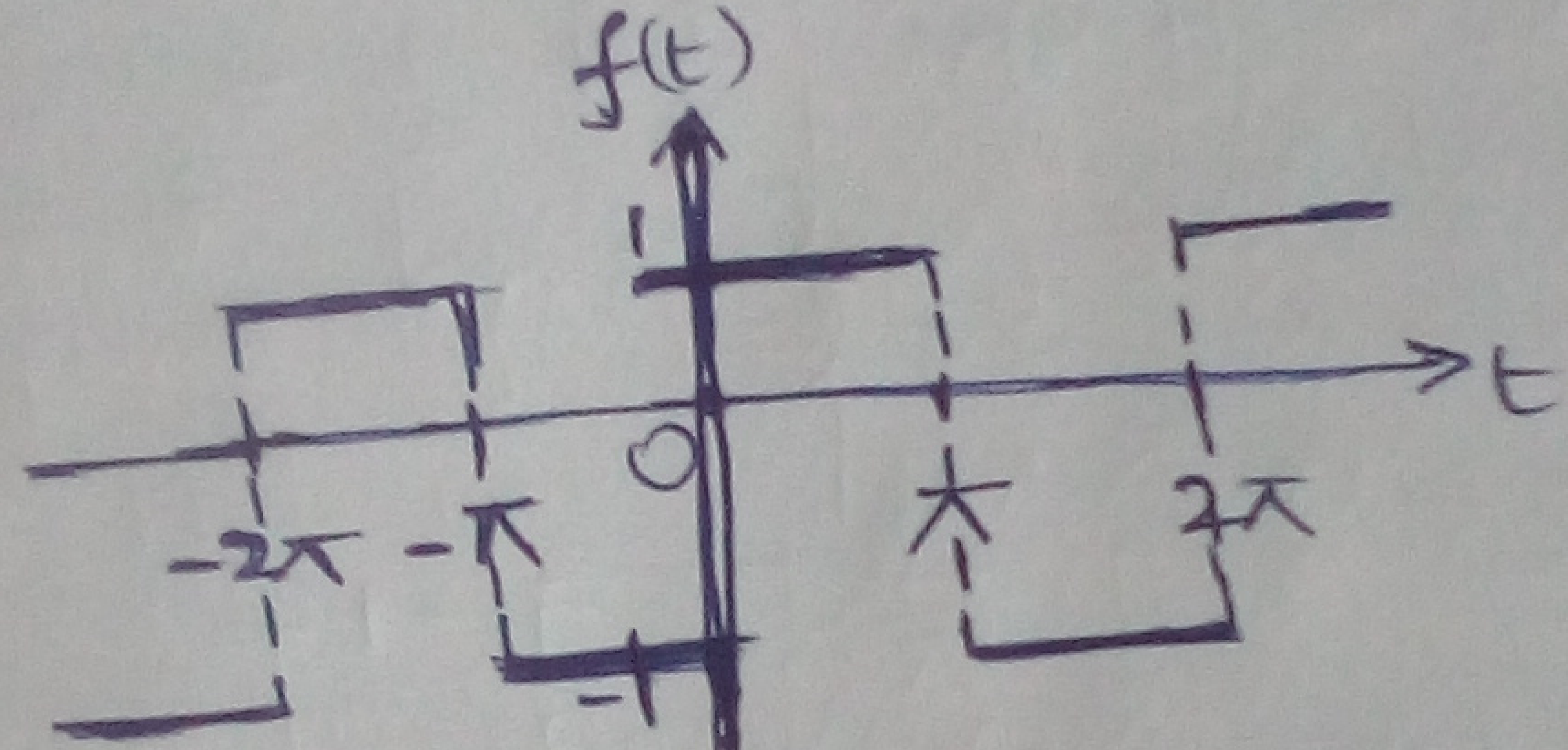
## Module 5.2:

## Fourier series for the functions having points of discontinuity:

1. Find the Fourier serier for the following periodic signal in the time domain and hence deduce that $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots=\frac{\pi}{4}$.




Sol: From the given figure, we have

$$f(t)=\begin{cases}-1, & -\pi<t<0\\ 1, & 0<t<\pi\end{cases}$$

Fourier serier for $f(t)$ in the interval $(-\pi,\pi)$ is given by

$$f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nt+\sum_{n=1}^{\infty}b_n\sin nt.$$

Now, $$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)\,dt=\frac{1}{\pi}\left(\int_{-\pi}^{0}(-1)\,dt+\int_{0}^{1}1\,dt\right)=0$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)\cos nt\,dt$$

$$=\frac{1}{\pi}\left(\int_{-\pi}^{0}(-1)\cos nt\,dt+\int_{0}^{\pi}(1)\cos nt\,dt\right)$$

$$= \frac{1}{\pi}\left(\left(-\frac{\sin nt}{n}\right)_{-\pi}^{0} + \left(\frac{\sin nt}{n}\right)_{0}^{\pi}\right)$$

$$= \frac{1}{\pi}(0+0) = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cdot \sin nt\, dt$$

$$= \frac{1}{\pi}\left(\int_{-\pi}^{0}(-1)\sin nt + \int_{0}^{\pi}(1)\sin nt\, dt\right)$$

$$= \frac{1}{\pi}\left(\left[\frac{\cos nt}{n}\right]_{-\pi}^{0} + \left[-\frac{\cos nt}{n}\right]_{0}^{\pi}\right)$$

$$= \frac{1}{\pi}\left(\frac{1}{n}[1-\cos n\pi] + \left(\frac{-1}{n}\right)[\cos n\pi - 1]\right)$$

$$= \frac{1}{n\pi}[1-\cos n\pi - \cos n\pi + 1]$$

$$= \frac{2}{n\pi}[1-\cos n\pi] = \frac{2}{n\pi}[1-(-1)^n].$$

$$\therefore f(t) = \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{1-(-1)^n}{n}\right]\sin nt$$

or, $$f(t) = \frac{2}{\pi}\sum_{n=1,3,5,\ldots}\left[\frac{1-(-1)^n}{n}\right]\sin nt$$

$$\left(\because \frac{[1-(-1)^n]}{n} = 0 \text{ for } n=2,\ldots\right)$$

Taking $t = \frac{\pi}{2}$, we get

$$1 = \frac{2}{\pi}\left[\frac{2}{1}(1) + \frac{2}{3}(-1) + \frac{2}{5}(1) + \frac{2}{7}(-1) + \cdots\right]$$

Hence $$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}$$

② Find the Fourier Serier for $f(x) = \begin{cases} 1, & 0 < x < \pi \\ 0, & \pi < x < 2\pi \end{cases}$

Answer: $a_0 = 1$; $a_n = 0$ and $b_n = \frac{1}{n\pi}[1-(-1)^n]$

So,
$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1}{n\pi}(1-(-1)^n)\sin(nx)$$

$$= \frac{1}{2} + \frac{2}{\pi}\left[\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right]$$

③ Expand $f(x) = \begin{cases} -\pi, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$

as a Fourier Series and hence deduce that

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$$

Answer:

$a_0 = -\frac{\pi}{2}$; $a_n = \frac{1}{n^2\pi}[(-1)^n - 1] = \begin{cases} 0, & n = 2,4,6,\ldots \\ \frac{-2}{n^2\pi}, & n = 1,3,5,\ldots \end{cases}$

and $b_n = \frac{1}{n}[1-2(-1)^n]$.

Therefore, $f(x) = \frac{-\pi}{4} + \sum_{n=1}^{\infty} \frac{1}{n^2\pi}[(-1)^n - 1]\cos nx$

$$+ \sum_{n=1}^{\infty} \frac{1}{n}[1-2(-1)^n]\sin nx$$

or, $f(x) = \frac{-\pi}{4} - \frac{2}{\pi}\left[\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right]$

$$+\left[3\sin x - \frac{\sin 2x}{2} + \frac{3\sin 3x}{3} - \frac{\sin 4x}{4} + \cdots\right]$$

Taking $x = 0$, we get

$$f(0) = -\frac{\pi}{4} - \frac{2}{\pi}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right)$$

$$\frac{f(0+) + f(0-)}{2} = -\frac{\pi}{4} - \frac{2}{\pi}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right)$$

$$\left[\lim_{x \to 0^+} f(x) = 0 \text{ and } \lim_{x \to 0^-} f(x) = -\pi\right]$$

$$-\frac{\pi}{2} = -\frac{\pi}{4} - \frac{2}{\pi}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right)$$

Hence $1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$

3. Expand $f(x) = \begin{cases} 0, & -2 < x < 0 \\ 1, & 0 < x < 2 \end{cases}$ as a Fourier Series

## Even and Odd functions:

A function $f(x)$ is said to be even if $f(-x) = f(x)$ and odd if $f(-x) = -f(x)$.

we have $\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & \text{if } f(x) \text{ is even} \\ 0 & \text{if } f(x) \text{ is odd} \end{cases}$

(i) Fourier series for an even function $f(x)$ in the interval $(-l, l)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

where $a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx = \frac{2}{l}\int_{0}^{l} f(x)\,dx$

and $a_n = \frac{1}{l}\int_{-l}^{l} \underbrace{\underbrace{f(x)}_{\text{even}} \underbrace{\cos\left(\frac{n\pi x}{l}\right)}_{\text{even}}}_{\text{even}}\,dx$

$$= \frac{2}{l}\int_{0}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

(ii) Fourier Series for an odd function $f(x)$ in the interval $(-l, l)$ is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

where $b_n = \frac{1}{l}\int_{-l}^{l} \underbrace{\underbrace{f(x)}_{\text{odd}} \cdot \underbrace{\sin\left(\frac{n\pi x}{l}\right)}_{\text{odd}}}_{\text{even}}\,dx$

$$= \frac{2}{l}\int_{0}^{l} f(x)\cdot\sin\left(\frac{n\pi x}{l}\right)dx.$$

Problems:

1. Expand $f(x) = x$ as a Fourier Series in the interval $(-\pi, \pi)$.

Sol: Clearly $f(x)=x$ is an odd function in the interval $(-\pi,\pi)$.

So, the Fourier Series for $f(x)=x$ in the interval $(-\pi,\pi)$ is given by

$$f(x)=\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) \quad [\text{here } l=\pi]$$

Now, $$b_n=\frac{2}{l}\int_0^l f(x).\sin\left(\frac{n\pi x}{l}\right)dx$$

$$=\frac{2}{\pi}\int_0^\pi x\sin(nx)\,dx$$

$$=\frac{2}{\pi}\left[\left[x\left(-\frac{\cos nx}{n}\right)\right]_0^\pi-\left[1\left(-\frac{\sin(nx)}{n^2}\right)\right]_0^\pi\right]$$

$$=\frac{2}{\pi}\left\{-\frac{1}{n}(\pi\cos n\pi-0)+\frac{1}{n^2}(0)\right\}$$

$$=-\frac{2}{n}\cos n\pi=-\frac{2}{n}(-1)^n$$

$$=\frac{2}{n}(-1)^{n+1}$$

Therefore,

$$f(x)=\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

i.e., $$x=2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(nx)$$

② Expand $f(x) = \begin{cases} 1+\frac{2x}{\pi}, & -\pi \le x \le 0 \\ 1-\frac{2x}{\pi}, & 0 \le x \le \pi \end{cases}$

as a Fourier series and hence deduce

that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$.

Sol: Here,

$$f(-x) = \begin{cases} 1-\frac{2x}{\pi}, & -\pi \le -x \le 0 \\ 1+\frac{2x}{\pi}, & 0 \le -x \le \pi \end{cases}$$

$$= \begin{cases} 1-\frac{2x}{\pi}, & 0 \le x \le \pi \\ 1+\frac{2x}{\pi}, & -\pi \le x \le 0 \end{cases}$$

$$= f(x)$$

So, $f(x)$ is an even function in $(-\pi, \pi)$.

Therefore, the Fourier series expansion for $f(x)$ in the interval $(-\pi, \pi)$ is given by $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$,

where $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,dx$

$$= \frac{2}{\pi} \int_0^{\pi} f(x)\,dx$$

$$= \frac{2}{\pi} \int^{\pi} \left(1 - \frac{2x}{\pi}\right) dx = 0$$

and $a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$

$$= \frac{2}{\pi}\int_0^{\pi}\left(1-\frac{2x}{\pi}\right)\cos nx\,dx$$

$$= \frac{2}{\pi}\left\{\left[\left(1-\frac{2x}{\pi}\right)\left(\frac{\sin nx}{n}\right)\right]_0^{\pi} - \left[\left(-\frac{2}{\pi}\right)\left(\frac{-\cos nx}{n^2}\right)\right]_0^{\pi}\right\}$$

$$= \frac{2}{\pi}\left[\frac{-2(-1)^n}{n^2\pi} + \frac{2}{n^2\pi}\right]$$

$$= \frac{4}{n^2\pi^2}\left[1-(-1)^n\right]$$

Hence,

$$f(x) = \frac{0}{2} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(1-(-1)^n)}{n^2}\cos nx$$

i.e., $f(x) = \frac{4}{\pi^2}\left[\frac{2}{1^2}\cos x + \frac{2}{3^2}\cos 3x + \frac{2}{5^2}\cos 5x + \cdots\right]$

$$= \frac{8}{\pi^2}\left[\frac{1}{1^2}\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \cdots\right]$$

Taking $x=0$, we get

$$1 = \frac{8}{\pi^2}\left[\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\right]$$

Thus, $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots = \frac{\pi^2}{8}$

③ Expand $f(x) = x^2$ as a Fourier Series in the interval $(-1, 1)$.

Sol: Clearly $f(x) = x^2$ is an even function in the interval $(-1, 1)$.

Therefore, the Fourier Series for $f(x)$ in the interval $(-1, 1)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right).$$

Here $l = 1$.

Now, $a_0 = \frac{2}{l}\int_0^l f(x)\,dx = 2\int_0^1 x^2\,dx = \frac{2}{3}$

and $a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$

$$= 2\int_0^1 x^2 \cos n\pi x\,dx$$

$$= 2\left\{\left[x^2 \frac{\sin(n\pi x)}{n\pi}\right]_0^1 - \left[2x\left(-\frac{\cos(n\pi x)}{n^2\pi^2}\right)\right]_0^1 + \left[2\cdot\left(-\frac{\sin(n\pi x)}{n^3\pi^3}\right)\right]_0^1\right\}$$

$$= 2\left\{\frac{1}{n\pi}(0) + \frac{2}{n^2\pi^2}(\cos n\pi - 0) - \frac{2}{n^3\pi^3}(0)\right\}$$

$$= \frac{4}{n^2\pi^2}(-1)^n.$$

Hence

$$\boxed{f(x) = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(n\pi x)}$$